The matrix approach to systems of ODEs (linear)

We consider the linear system of ODEs

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (1)$$

where A is a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Indicating $\quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

we can write Eq. (1) as $\quad \dot{Y} = AY \quad (2)$

In this week and in the next we will use algebra to solve (1) or (2).

Equation (1) can be written in matrix form as

$$\dot{Y} = AY \quad (2) \quad \text{where} \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Let us show that the general solution to this system of linear ODEs can be written as

$$Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

where D_1, D_2 ~~eith~~ are arbitrary constants

and $u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$

with $Au_1 = \lambda_1 u_1$ and $Au_2 = \lambda_2 u_2$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = D_1 e^{\lambda_1 t} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + D_2 e^{\lambda_2 t} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$$

<u>Derivation</u> Given ~~A~~ that A has two distinct eigenvalues $\lambda_1 \neq \lambda_2$

then the corresponding eigen vectors u_1 and u_2

are <u>linearly independent</u>

- This implies (see previous lesson) that any 2-dimensional vector can ~~be~~ be written in a <u>unique way</u> as a linear combination of u_1 and u_2.

- It follows that $Y(t)$ can be expressed in a unique way as

$$Y(t) = c_1(t)\, u_1 + c_2(t)\, u_2 \qquad (3)$$

Now since $\dot{Y} = AY$ substituting (3) into (2)

we get

LHS: $\dot{Y} = \dot{c}_1(t)\, u_1 + \dot{c}_2(t)\, u_2$

RHS: $AY = A\left[c_1(t)\, u_1 + c_2(t)\, u_2\right] =$

$$= c_1(t) \underbrace{A u_1}_{Au_1 = \lambda_1 u_1} + c_2(t) \underbrace{A u_2}_{Au_2 = \lambda_2 u_2} = c_1(t)\, \lambda_1 u_1 + c_2(t)\, \lambda_2 u_2$$

Therefore $\dot{Y} = AY$ implies

$$\dot{c}_1(t)\, u_1 + \dot{c}_2(t)\, u_2 = c_1(t)\, \lambda_1\, u_1 + c_2(t)\, \lambda_2\, u_2$$

Rearranging the terms

$$\underbrace{\left[\dot{c}_1(t) - \lambda_1 c_1(t)\right]}_{=0} u_1 + \underbrace{\left[\dot{c}_2(t) - \lambda_2 c_2(t)\right]}_{=0} u_2 = 0$$

Since the eigenvectors u_1 and u_2 are independent this equation implies

$$\begin{cases} \dot{c}_1(t) - \lambda_1 c_1(t) = 0 \\ \dot{c}_2(t) - \lambda_2 c_2(t) = 0 \end{cases} \qquad \begin{cases} \dot{c}_1 = \lambda_1 c_1 \\ \dot{c}_2 = \lambda_2 c_2 \end{cases}$$

$\dot{c}_1 = \lambda_1 c_1$ " $\dot{y} = \lambda y$ "

solve by separation of variables

$$\int \frac{dc_1}{c_1} = \lambda_1 \int dt \quad \Rightarrow \quad \ln|c_1| = \lambda_1 t + C'$$

$$\boxed{c_1(t) = D_1\, e^{\lambda_1 t}}$$

where D_1 is an arbitrary constant

Therefore we set

$$\begin{cases} c_1(t) = D_1 e^{\lambda_1 t} \\ c_2(t) = D_2 e^{\lambda_2 t} \end{cases}$$

Since $Y(t) = c_1(t)\, u_1 + c_2(t)\, u_2$ we set

$$\boxed{Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2}$$

where D_1, D_2 are arbitrary constants

General solution to $\dot{Y} = AY$.

<u>Example</u> Find the general solution of $\dot{Y} = AY$ with

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

<u>Solution</u> A has eigenvalues and eigenvectors given by

$$\lambda_1 = 2 \qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $\lambda_1 \neq \lambda_2$

$$\lambda_2 = -1 \qquad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore the general solution is

$$Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

$$Y(t) = D_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$Y(t) = \begin{pmatrix} D_1 e^{2t} + D_2 e^{-t} \cdot 2 \\ D_1 e^{2t} + D_2 e^{-t} \end{pmatrix}$$

or equivalenty
$$\begin{cases} y_1(t) = D_1 e^{2t} + 2 D_2 e^{-t} \\ y_2(t) = D_1 e^{2t} + D_2 e^{-t} \end{cases}$$

Provide the solution for the IVP.

$\dot{Y} = AY$ with I.C. $Y(0) = k u_1$ where $k \in \mathbb{R}$

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution

$$Y(t) = D_1 e^{2t} u_1 + D_2 e^{-t} u_2$$

Imposing the I.C.

$$Y(0) = D_1 u_1 + D_2 u_2 = k u_1$$

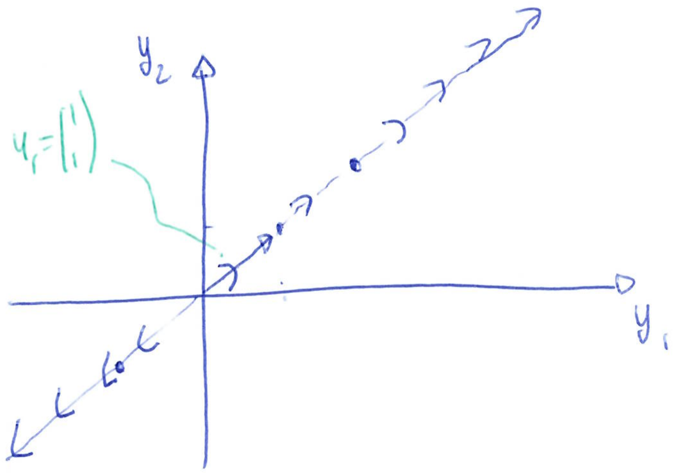
Rearranging

$$\underbrace{(D_1 - k)}_{=0} u_1 + \underbrace{D_2}_{=0} u_2 = 0$$

$$\begin{cases} D_1 = k \\ D_2 = 0 \end{cases}$$

The solution to the IVP

$$Y(t) = k e^{2t} u_1 = k e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



The trajectories with IC.

$Y(0) = k u_1$ move along the direction of vector u_1

u_1 ~~is an~~ defines an INVARIANT MANIFOLD.

- If $k > 0$

$$\begin{cases} y_1(t) = k e^{2t} \\ y_2(t) = k e^{2t} \end{cases} \qquad \frac{y_1}{y_2} = 1$$

If $t \to \infty$ $\quad y_1(t) \to +\infty \quad y_2(t) \to +\infty$

- If $k < 0$

$$\begin{cases} y_1(t) = k e^{2t} \\ y_2(t) = k e^{2t} \end{cases} \qquad y_1/y_2 = 1$$

If $t \to \infty$ $\quad y_1(t) \to -\infty \quad y_2(t) \to -\infty$

In this case we say that u_1 defines a INVARIANT MANIFOLD that is UNSTABLE.

All trajectories move away from the origin on this manifold.