A review of algebra

Consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- the trace of A indicated as Tr A is given by

$$\mathrm{Tr}\, A = a_{11} + a_{22} \quad \text{(sum of diagonal terms)}$$

- the determinant of a 2x2 matrix A is given by

$$\det A = a_{11} a_{22} - a_{12} a_{21}$$

If the $\det A \neq 0$ we can define the inverse of A indicated as A^{-1} and satisfying

$$A A^{-1} = A^{-1} A = Id \qquad \text{where} \qquad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A^{-1} can be expressed as

$$A^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \frac{1}{\det A}$$

If A^{-1} exists the equation

$$A\, Y = b$$

has solution
$$Y = A^{-1} b.$$

Indeed
$$\underbrace{A^{-1}A}_{Id} Y = A^{-1} b \quad \Rightarrow \quad Y = A^{-1} b. \quad \square$$

Eigenvalues and eigenvectors of A.

Let $\quad u = \begin{pmatrix} p \\ q \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad$ if $\quad Au = \lambda u \qquad \lambda \in \mathbb{R}$

λ is called the <u>eigenvalue</u> of A

u is called the <u>eigenvector</u> of A corresponding to the eigenvalue λ.

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① <u>Theorem</u> For any 2×2 matrix A there are two eigenvalues which are roots of the quadratic equation

$$\det(A - \lambda\, Id) = 0 \qquad (3)$$

or equivalently.

$$\lambda^2 - (\mathrm{Tr}\, A)\lambda + \det A = 0 \qquad (4)$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(A - \lambda Id) = \det \begin{pmatrix} a_{11} - \lambda \cdot 1 & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} = 0$$

$$\lambda^2 - (\mathrm{Tr}\, A)\lambda + \det A = 0 \qquad \text{[check at home]}$$

It follows that there are 3 possibilities:

a) λ_1, λ_2 are distinct and real roots $\lambda_1, \lambda_2 \in \mathbb{R}$

b) $\lambda_1 = \lambda_2$ are indentical real roots $\lambda_1 = \lambda_2 \in \mathbb{R}$

c) $\lambda_1 = \alpha + i\beta \quad \lambda_2 = \alpha - i\beta \qquad \alpha, \beta \in \mathbb{R} \quad \beta \neq 0$

λ_1, λ_2 are complex conjugate.

(2) If $\lambda_1 \neq \lambda_2$ (cases (a) and (c)) the two eigenvectors u_1 and u_2 each determined up to a constant factor.

are linearly independent

This means that $u_2 \neq k u_1$ with $k \in \mathbb{R}$

If u_1 and u_2 are linearly independent, any 2-dimensional vector y can be written in a unique way as

$$y = c_1 u_1 + c_2 u_2 \qquad \text{where } c_1, c_2 \in \mathbb{R}$$

Therefore if $y = c_1 u_1 + c_2 u_2 = 0$ *

$$\Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

$$* \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + c_2 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{where } u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$$

③ If $a_{12} = a_{21}$ then either

a) $\lambda_1 = \lambda_2$

or

b) the eigenvectors u_1 and u_2 are <u>orthogonal</u>

$$u_1 u_2^T = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} (p_2, q_2) = p_1 p_2 + q_1 q_2 = 0$$

Example of calculation of eigenvalues and eigenvector of a matrix.

Calculate the eigenvalues and eigenvector of the matrix A given by

$$A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

<u>Eigenvalues</u> To find all eigenvalues we impose

$$\det(A - \lambda Id) = 0$$

$$\det(A - \lambda Id) = \det \begin{pmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{pmatrix} = 0$$

$$(-4-\lambda)(5-\lambda) - 6(-3) = 0 \quad \Rightarrow -20 - 5\lambda + 4\lambda + \lambda^2 + 18 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

Let us find the roots λ_1, λ_2

$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

$$\boxed{\lambda_1 = 2}$$

$$\boxed{\lambda_2 = -1}$$

λ_1, λ_2 are real and distinct.

<u>Eigenvectors</u> For eigenvalue $\lambda_1 = 2$ we will have the eigenvector

$$u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \quad \text{satisfying} \quad A u_1 = \lambda_1 u_1$$

$$\begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$\begin{cases} -4p_1 + 6q_1 = \lambda_1 p_1 = 2p_1 \\ -3p_1 + 5q_1 = \lambda_1 q_1 = 2q_1 \end{cases}$$

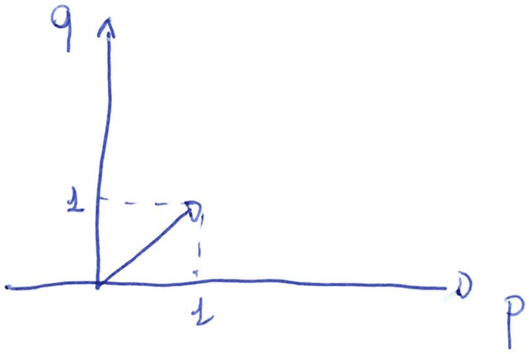
$$\begin{cases} -\cancel{6}p_1 + \cancel{6}q_1 = 0 \\ -\cancel{3}p_1 + \cancel{3}q_1 = 0 \end{cases} \qquad \begin{cases} p_1 = q_1 \\ p_1 = q_1 \end{cases}$$

If $p_1 = 1 \Rightarrow q_1 = 1$

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenvectors are always determined up to a non-zero factor.

We have chosen $p_1 = 1 \Rightarrow p_1 = q_1 = 1 \Rightarrow q_1 = 1$



With a similar procedure you can find the eigenvector u_2 corresponding to the eigenvalue $\lambda_2 = -1$

$$u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ or any vector differing by a constant factor.