Feedback on Coursework #1

a) $\underbrace{2x^2 y}\,y' + \underbrace{2xy^2 + 3} = 0$ 1st-order

Is this exact?

$P(x,y) + Q(x,y)\,y' = 0$

$P(x,y) = 2xy^2 + 3$

$Q(x,y) = 2x^2 y$

This ODE is exact iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(2xy^2 + 3\right) = 2x \cdot 2y \quad \checkmark$$

$\parallel$ Yes! It is exact

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(2x^2 y\right) = 2y \cdot 2x \quad \checkmark$$

b) $\underbrace{y e^{xy} - 2\frac{y^2}{x^3}}_{\substack{\parallel \\ P(x,y)}} + y'\underbrace{\left(x e^{xy} + 2\frac{y}{x^2}\right)}_{\substack{\parallel \\ Q(x,y)}} = 0$ Is this exact?

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$?

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(y e^{xy} - 2\frac{y^2}{x^3}\right) = e^{xy} + yx e^{xy} - \frac{2}{x^3}(2y)$$

$$= e^{xy} + yx e^{xy} - \frac{4}{x^3}y \quad \checkmark$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy} + \frac{2y}{x^2}\right) = \underline{xy\,e^{xy}} + \underline{e^{xy}} + 2y\underline{\left(\frac{-2}{x^3}\right)}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \text{The ODE is exact.}$$

$y'' = 5 + y$ $\quad$ $y'' - y = 5$ $\quad$ 2^{nd}-order linear inhomogeneous

$y' + y'' = 2x^2 y$ $\quad$ $y'' + y' - 2x^2 y = 0$ $\quad$ 2^{nd}-order linear homogeneous.

$x^2 y'' = 5y$ $\quad$ $x^2 y'' - 5y = 0$ $\quad$ Euler.

$xy' = (-5y + x)\sin\left(\frac{y}{x}\right)$ $\quad$ $y' = \left(-\frac{5y}{x} + 1\right)\sin\frac{y}{x}$ $\quad$ 1^{st}-order scale invariant

$y'' = \frac{x}{y}$ $\quad$ $y' = \frac{x}{y}$ $\quad$ None of the above

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$y \to by$

$x \to bx$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \to \frac{d}{d(bx)}\,\frac{d(\cancel{b}y)}{d(\cancel{b}x)} = \frac{1}{b}\frac{d^2y}{dx^2}$$

$y''' = \frac{x}{y}$ $\quad$ None of the above.

Solving the IVP $y' = y^{3/4}/x$ with $y(1)=0$

implies finding a solution passing through the point

$$(x_0, y_0) = (1, 0) \checkmark$$

The IVP has a unique solution if

b) there exist a rectangular region of the two dimensional space (x,y) whose center is the point (x_0, y_0) where the IVP has a unique solution.

Does the IVP satisfy the hypotheses of the Picard-Lindelöf theorem?

$$y' = f(x,y) = y^{3/4} \frac{1}{x}$$

D: $|x-1| \le A$ $\quad |y| \le B$ with $A>0$ $B>0$

$1-A \le x \le 1+A$ $\quad -B \le y \le B$

(1) $f(x,y) = y^{3/4} \frac{1}{x}$ is continuous in D

D cannot include any point $(0,y)$

Since $x \in D$ satisfies

$$0 < 1-A \le x \le 1+A$$

$$\Rightarrow \boxed{A < 1}$$

(2) $\frac{\partial f}{\partial y}$ is bounded in D

$$\int \frac{dy}{y^{3/4}} = \int \frac{dx}{x}$$

$$H(y) = \int \frac{dy}{y^{3/4}} = 4\,y^{1/4}$$

$$F(x) = \int \frac{dx}{x} = \ln|x|$$

Implicit solution $\quad H(y) = F(x) + C$

$$4y^{1/4} = (\ln|x| + C)$$

$$y = \left[\frac{1}{4}(\ln|x| + C)\right]^4$$

where $C \in \mathbb{R}$ is an arbitrary constant.

$$0 = y(1) = \left[\frac{1}{4}(\ln 1 + C)\right]^4 = \left(\frac{C}{4}\right)^4 \Rightarrow C = 0$$

$$\begin{cases} y = \left(\frac{1}{4}\ln|x|\right)^4 \\ y = 0 \end{cases}$$

More than one.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y^{3/4}\,\frac{1}{x}\right) = \frac{1}{x}\,\frac{3}{4}\,y^{-1/4}$$ diverging for

$x = 0$

$y = 0$

in D $\quad -B \leq y \leq B \quad$ there is no $B > 0$ such that

$\frac{\partial f}{\partial y}$ is NOT bounded in D

$\Rightarrow$ The hypotheses of the Picard - Lindelöf theorem are not satisfied

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How many solutions the IVP has?

$$y' = y^{3/4}\,\frac{1}{x} = g(y)f(x) \qquad y(1) = 0$$

where $g(y) = y^{3/4}$

$f(x) = \frac{1}{x}$

Since $g(0) = 0$ ✓ then $y(x) = 0$ is a solution to the ODE

This is a separable ODE.

$$\int \frac{dy}{y^{3/4}} = \int \frac{dx}{x}$$

$$H(y) = \int \frac{dy}{y^{3/4}} = 4\,y^{1/4}$$

$$F(x) = \int \frac{dx}{x} = \ln|x|$$

The general solution is $H(y) = F(x) + C$ with C arbitrary constant

$$4y^{1/4} = \ln|x| + C'$$

$$y^{1/4} = \frac{1}{4}\left(\ln|x| + C\right)$$

$$y = \left[\frac{1}{4}\left(\ln|x| + C\right)\right]^4$$

Imposing $y(1) = 0$

$$0 = y(1) = \left[\frac{1}{4}(\ln 1 + C')\right]^4 = \left(\frac{C}{4}\right)^4 \implies C = 0$$

$$y = \left(\frac{1}{4}\ln|x|\right)^4$$

More than one solution.

Find the general solution of the 1st-order ODE

$\cancel{y' = x^2}$ $\qquad y' x^2 = -y^2$

$$y' = -\frac{y^2}{x^2} = g(y)\,f(x) \qquad g(y) = y^2$$

$$f(x) = -\frac{1}{x^2}$$

$$\int \frac{dy}{y^2} = \int -\frac{dx}{x^2}$$

$$H(y) = \int \frac{dy}{y^2} = -\frac{1}{y}$$

$$F(x) = -\int \frac{dx}{x^2} = \frac{1}{x}$$

Implicit solution $H(y) = F(x) + C$

$$-\frac{1}{y} = \frac{1}{x} + C$$

$$\int x^{-\alpha} dx = \frac{1}{1-\alpha} x^{-\alpha+1}$$

$$y = -\frac{1}{\frac{1}{x} + C} = -\frac{1}{\frac{1+Cx}{x}} = -\frac{x}{1+Cx} \quad \checkmark$$

$$y = -\frac{2x}{2\cancel{x} + C'x} = -\frac{x}{1+\frac{C'}{2}x} \quad \checkmark$$

$$\frac{C'}{2} = C$$