# Linear and non-linear autonomous systems of $1^{st}$-order ODEs

A linear autonomous system of $1^{st}$-order ODE with two dependent variables $y_1, y_2$ reads

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where $A$ is a $2\times 2$ matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{with } a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$$

This is equivalent to the system

$$\begin{cases} \dot{y}_1 = f_1(y_1, y_2) = a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 = f_2(y_1, y_2) = a_{21} y_1 + a_{22} y_2 \end{cases}$$

A dynamical autonomous system of $1^{st}$-order ODE

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \quad \text{with} \quad \begin{matrix} f_1(y_1, y_2) \neq a_{11} y_1 + a_{12} y_2 \\ f_2(y_1, y_2) \neq a_{21} y_1 + a_{22} y_2 \end{matrix}$$

is NON-LINEAR.

Example

$$\begin{cases} \dot{y}_1 = e^{e^{y_1}} + y_1 y_2^3 - \tan y_1 \\ \dot{y}_2 = 3 y_1 \end{cases}$$

Non linear.

$$\begin{cases} \dot{y}_1 = 4 y_1 y_2 - 2 \\ \ddot{y}_2 = (y_1 - 2)(y_1 - 2y_2) \end{cases}$$

Non linear.

---

## Revision of Taylor series (calculus)

① Consider a function $f(y)$ that is infinitely differentiable at point $y = y^*$. The function can be expressed as the power series

$$f(y) = \underbrace{f(y^*) + \frac{1}{1!} f'(y^*)(y - y^*)}_{\text{linear approximation}} + \frac{1}{2!} f''(y^*)(y - y^*)^2 + \dots$$

② Consider a ~~f~~ two variable function $f(y_1, y_2)$ that is infinitely differentiable at point $(y_1, y_2) = (y_1^*, y_2^*)$. The function can be expressed as

$$f(y_1, y_2) = \underbrace{f(y_1^*, y_2^*) + \left.\frac{\partial f}{\partial y_1}\right|_{(y_1, y_2) = (y_1^*, y_2^*)} (y_1 - y_1^*) + \left.\frac{\partial f}{\partial y_2}\right|_{(y_1, y_2) = (y_1^*, y_2^*)} (y_2 - y_2^*)}_{\text{linear approximation.}} + \dots$$

Linearisation of a non-linear system of ODEs around the equilibrium point $(y_1^*, y_2^*)$

We consider the non-linear autonomous system of ODEs

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

We assume that $(y_1^*, y_2^*)$ is an equilibrium point.

So by definition we have

$$f_1(y_1^*, y_2^*) = 0$$

$$f_2(y_1^*, y_2^*) = 0$$

We consider $(y_1, y_2)$ close to the fixed point $(y_1^*, y_2^*)$ and we linearize $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$

$$f_1(y_1, y_2) = \overset{\overset{0}{\shortparallel}}{f_1(y_1^*, y_2^*)} + \left.\frac{\partial f_1}{\partial y_1}\right|_{(y_1, y_2) = (y_1^*, y_2^*)} (y_1 - y_1^*) + \left.\frac{\partial f_1}{\partial y_2}\right|_{(y_1, y_2) = (y_1^*, y_2^*)} (y_2 - y_2^*)$$

$$f_2(y_1, y_2) = \overset{\overset{0}{\shortparallel}}{f_2(y_1^*, y_2^*)} + \left.\frac{\partial f_2}{\partial y_1}\right|_{(y_1, y_2) = (y_1^*, y_2^*)} (y_1 - y_1^*) + \left.\frac{\partial f_2}{\partial y_2}\right|_{(y_1, y_2) = (y_1^*, y_2^*)} (y_2 - y_2^*)$$

We truncated the expansion at the linear order

By setting

$$A = \begin{pmatrix} \overset{= a_{11}}{\left.\dfrac{\partial f_1}{\partial y_1}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}} & \overset{= a_{12}}{\left.\dfrac{\partial f_1}{\partial y_2}\right|_{(y_1,y_2)=(y_1^*,y_2^*)}} \\ \overset{= a_{21}}{\left.\dfrac{\partial f_2}{\partial y_1}\right|_{(y_1,y_1)=(y_1^*,y_2^*)}} & \overset{= a_{22}}{\left.\dfrac{\partial f_2}{\partial y_2}\right|_{(y_1,y_2)=(y_1^*,y_2^*}} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We obtain in this way

$$f_1(y_1,y_2) = a_{11}\,(y_1-y_1^*) + a_{12}\,(y_2-y_2^*)$$

$$f_2(y_1,y_2) = a_{21}\,(y_1-y_1^*) + a_{22}\,(y_2-y_1^*)$$

or equivalently we can express the system of ODEs as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 - y_1^* \\ y_2 - y_2^* \end{pmatrix}$$

Now we make the change of variables

$$\tilde{y}_1 = y_1 - y_1^* \qquad \dot{\tilde{y}}_1 = \dot{y}_1$$

$$\tilde{y}_2 = y_2 - y_2^* \qquad \dot{\tilde{y}}_2 = \dot{y}_2$$

Therefore we obtain

$$\begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = A \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$$

Therefore this is the linearised system close to the equilibrium point $(y_1^*, y_2^*)$.

$$\boxed{\text{The dynamical behaviour of the linearised system around the equilibrium point is much more easy to study than the original non-linear system.}}$$

---

Example: Linearise the system of ODEs close to the equilibrium point $(0,0)$.

$$\begin{cases} \dot{y}_1 = \tan y_1 + e^{y_2} - 1 \\ \dot{y}_2 = 3 \sin y_1 \end{cases}$$

(1) Check that $(0,0)$ is an equilibrium point.

$$\dot{y}_1 = \tan y_1 + e^{y_2} - 1 = f_1(y_1, y_2)$$

$$\dot{y}_2 = 3 \sin y_1 = f_2(y_1, y_2)$$

The dynamical behaviour of the linearised system around the equilibrium point is much more easy to study than the original non-linear system.

---

<u>Example</u> Linearize the system of ODEs close to the equilibrium point $(0,0)$.

$$\begin{cases} \dot{y}_1 = \tan y_1 + e^{y_2} - 1 \\ \dot{y}_2 = 3 \sin y_1 \end{cases}$$

(A) Let us check that $(0,0)$ is an equilibrium point. Let us define

$$f_1(y_1, y_2) = \tan y_1 + e^{y_2} - 1$$

$$f_2(y_1, y_2) = 3 \sin y_1$$

$\cancel{f_1(0,0)}$ $(0,0)$ is an equilibrium point if and only if

$f_1(0,0) = 0$ ✓ $\quad f_1(0,0) = \tan 0 + e^0 - 1 = 0 + 1 - 1 = 0$ ✓

$f_2(0,0) = 0 \quad f_2(0,0) = 3 \sin 0 = 0$

$(0,0)$ is an equilibrium point.

(B) Let us linearise the non linear system around the point

$$(0,0) = (y_1^*, y_2^*) \qquad \begin{cases} y_1^* = 0 \\ y_2^* = 0 \end{cases}$$

The linearised system of ODE reads $\tilde{y}_1 = y_1$ $\tilde{y}_2 = y_2$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

with
$$A = \begin{pmatrix} \left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} & \left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} \\ \left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} & \left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} \end{pmatrix}$$

$$\left.\frac{\partial f_1}{\partial y_1}\right|_{(0,0)} = \frac{\partial}{\partial y_1}\left(\tan y_1 + e^{y_2} - 1\right)\Big|_{0,0} = \left.\frac{1}{\cos^2 y_1}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f_1}{\partial y_2}\right|_{(0,0)} = \frac{\partial}{\partial y_2}\left(\tan y_1 + e^{y_2} - 1\right)\Big|_{0,0} = \left. e^{y_2}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f_2}{\partial y_1}\right|_{(0,0)} = \frac{\partial}{\partial y_1}\left(3 \sin y_1\right)\Big|_{(0,0)} = \left. 3\cos y_1\right|_{(0,0)} = 3$$

$$\left.\frac{\partial f_2}{\partial y_2}\right|_{(0,0)} = \frac{\partial}{\partial y_2}\left(3 \sin y_1\right)\Big|_{(0,0)} = 0 \qquad \Rightarrow A = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$ Linearised system!