Queen Mary
University of London

MTH5123: Differential Equations

Duration: 2 hours

Date and time: 4th May 2016, 10:00 am–12:00pm

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Y. V. Fyodorov

Question 1.

(a) **(i)** Find all functions $f(y)$ for which the following differential equation becomes exact:

$$e^x\, f(y) + x^2 + (e^x \cos y + y)\,\frac{dy}{dx} = 0 \tag{1}$$

[4]

(ii) Suppose, $f(y)$ is chosen so that the equation (1) is exact and $f\,(\pi) = 0$. Solve (1) in implicit form. **[8]**

(b) Consider the Initial Value Problem

$$(x+2)\frac{dy}{dx} + (y+2)^{2/3} = 0, \quad y(0) = b\,. \tag{2}$$

where b is a real parameter and we assume $b \geqslant -2$.

(i) Find the value of the parameter b such that the corresponding Initial Value Problem may have more than one solution and explain your choice. Confirm your choice by giving explicitly at least two different solutions of (2) for such a value of the parameter. **[8]**

(ii) Use the Picard-Lindelöf theorem to verify that the existence and uniqueness of the solution for (2) with $b = 0$ is guaranteed in the rectangular domain $\mathcal{D} := \{|x| < A,\ |y| < B\}$ with $A = 1/2$ and $B = 1$. **[5]**

Question 2.

Write down the solution to the following Boundary Value Problem for the second order non-homogeneous differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = f(x), \quad y(1) = 0,\ y'(3) = 0 \tag{3}$$

by using the Green's function method along the following lines:

(a) Using that the left-hand side of the ordinary differential equations is in the form of an Euler-type equation determine the general solution of the associated homogeneous ordinary differential equation.

[6]

(b) Formulate the corresponding left-end and right-end initial value problems and use their solutions to construct the Green's function $G(x, s)$.

[14]

(c) Write down the solution to the Boundary Value Problem (3) in terms of $G(x, s)$ and $f(x)$ and use it to find the explicit form of the solution for $f(x) = x^2$.

[5]

Question 3.

(a) Consider a system of two linear first-order ordinary differential equations:

$$\dot{x} = -2x + y, \quad \dot{y} = -5x + 4y\,.$$

(i) Determine eigenvalues and eigenvectors associated with the system, find equations for stable and unstable invariant manifolds and sketch the phase portrait.

[11]

(ii) For the nonlinear system

$$\dot{x} = f_1(x,y), \quad \dot{y} = f_2(x,y)$$

with

$$f_1(x,y) = (1-y)(2x-y), \quad f_2(x,y) = (2+x)(x-2y)$$

show that there exists an equilibrium point with $y = -4$ and determine its $x-$coordinate in the (x,y) plane. Linearize the system around such an equilibrium and determine its nature (stable vs. unstable) and type (saddle, focus, or node). Describe in words the shape of trajectories close to the point.

[9]

(b) Consider a system of two nonlinear first-order ordinary differential equations:

$$\dot{x} = -y + axy^2, \quad \dot{y} = x - bx^2y\,.$$

where a, b are real constants. Find a relation between a and b such that the function $V(x,y) = \frac{1}{2}(x^2 + y^2)$ can be used as a Lyapunov function ensuring the stability of such a system in the whole (x,y) plane.

[5]

Question 4.

(a) Find the general solution of the homogeneous ordinary differential equation

$$y'' - 4y' + 13y = 0.$$

[6]

(b) Find the general solution of the non-homogeneous ordinary differential equation

$$y'' - 4y' + 13y = 18e^{2x}.$$

[12]

(c) Find the explicit solution to the following Initial Value Problem

$$y' = \frac{y+x}{\ln(y+x)} - 1\,, \quad y(0) = e.$$

[7]

End of Paper—An appendix of 2 pages follows.

Useful Facts.

- **Useful integrals**:

$$\int x^a\,dx = \frac{1}{a+1}\,x^{a+1}, \quad \forall a \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| \quad \text{for } a=-1; \quad \int \ln x\,\frac{dx}{x} = \frac{1}{2}\ln^2|x|$$

$$\int \cos x\,dx = \sin x, \quad \int \sin x\,dx = -\cos x,$$

$$\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x, \quad \int \tan x\,dx = -\ln|\cos x|$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right), \quad a \neq \pm ib$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right), \quad a \neq \pm ib$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{|x-a|}{|x+a|},$$

- **Useful trigonometric formulae**:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2\sin x\cos x$$

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B, \quad \cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$$

- **Reminder on Ordinary Differential Equations:**

If the equation $P(x,y)+Q(x,y)\,\frac{dy}{dx} = 0$ is exact, its solution can be found in the form $F(x,y) = Const.$ where

$$P = \frac{\partial F}{\partial x} \quad \text{and} \quad Q = \frac{\partial F}{\partial y}$$

- The Euler type equation $ax^2y'' + bxy' + cy = 0$ is solved by replacing $x = e^t$ and introducing the new function $z(t)$ by the relations

$$z(t) = y(e^t), \quad \Rightarrow \quad \frac{dz}{dt} = e^t y', \qquad \frac{d^2z}{dt^2} = e^t y' + e^{2t} y''$$

- If there exists a unique solution $y(x)$ to a non-homogeneous **boundary value** problem for ordinary differential equations
$\mathcal{L}(y) = a_2(x)y'' + a_1(x)y' + a_0(x) = f(x)$ in an interval $x \in [x_1, x_2]$ with linear homogeneous boundary conditions

$$\alpha y'(x_1) + \beta y(x_1) = 0, \quad \gamma y'(x_2) + \delta y(x_2) = 0$$

it can be found by the Green's function method:

$$y(x) = \int_{x_1}^{x_2} G(x,s)\, f(s)\, ds, \quad G(x,s) = \begin{cases} A(s)\, y_L(x), & x_1 \leqslant x \leqslant s \\ B(s)\, y_R(x), & s \leqslant x \leqslant x_2 \end{cases}$$

where

$$A(s) = \frac{y_R(s)}{a_2(s)W(s)}, \quad B(s) = \frac{y_L(s)}{a_2(s)W(s)}, \quad W(s) = y_L(s)y'_R(s) - y_R(s)y'_L(s)$$

and $y_L(x), y_R(x)$ are solutions to the left/right initial value problems:

$$\mathcal{L}(y) = 0,\ y(x_1) = \alpha,\ y'(x_1) = -\beta; \quad \text{and} \quad \mathcal{L}(y) = 0,\ y(x_2) = \gamma,\ y'(x_2) = -\delta$$

End of Appendix.