Educated guess method

The variation of parameter method can be applied to any ODE of the type

$$a_2 y'' + a_1 y' + a_0 y = f(x) \qquad (1)$$

The educated guess method can only be applied to the inhomogeneous ODE of the type (1) which have

$$\boxed{f(x) = p(x) e^{ax}} \qquad (2)$$

- where $p(x)$ is a polynomial of degree k
- $a \neq \lambda_1, \lambda_2$ where λ_1 and λ_2 are the roots of the characteristic equation

$$M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 \lambda = 0$$

In these conditions the educated guess method provides one particular solution to (1) that we indicate by $y_p(x)$.

The general solution to (1) will be

$$y_g(x) = y_p(x) + y_h(x)$$

Particular solution $y_p(x)$

The particular solution $y_p(x)$ can be written as

$$y_p(x) = Q(x)\, e^{ax} \qquad (3)$$

with $Q(x)$ being a polynomial of the same degree as $p(x)$ (degree k)

$$Q(x) = d_k x^k + d_{k-1} x^{k-1} + \dots \quad \dots + d_1 x + d_0$$

Where the coefficients $d_k, d_{k-1}, \dots d_1, d_0 \in \mathbb{R}$ can be determined by imposing that (3) is a solution of (1).

We will use an example to illustrate this educated guess method

<u>Example</u>: $\quad y'' + 2y' - 3y = x\, e^{2x}$

This is a linear, 2^{nd}-order, inhomogeneous ODE with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

$a_2 = 1 \quad a_1 = 2 \quad a_0 = -3 \quad f(x) = x\, e^{2x}$

(1) Check that we can use the educated guess method

$f(x) = p(x)\, e^{ax} \qquad a = 2 \quad p(x) = x$

(A) $p(x) \exists x$ polynomial of degree $k = 1$ ✓

(B) $a = 2 \neq \lambda_1, \lambda_2$ roots of $M_2(\lambda) = \lambda^2 + 2\lambda - 3 = 0$ $\begin{cases} \lambda_1 = -3 \\ \lambda_2 = +1 \end{cases}$ ✓

The root of the characteristic equation λ_1, λ_2 should be different than a

$a \neq \lambda_1 \quad a \neq \lambda_2$

$$M_2(\lambda) = \lambda^2 + 2\lambda - 3 = 0 \qquad \lambda_1 = -3 \quad \lambda_2 = +1$$

$a \neq \lambda_1 \quad a \neq \lambda_2 \qquad \checkmark$

We can apply the educated guess method.

(2) Find the general solution of the homogeneous problem

$$a_2 y'' + a_1 y' + a_0 y = 0$$

The roots of the characteristic equation are $\lambda_1 = -3, \lambda_2 = 1$

The general solution to the homogeneous problem

$$y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$ with c_1, c_2 arbitrary constant

$$\boxed{y_h(x) = c_1 e^{-3x} + c_2 e^{x}}$$

(3) We use the educated guess method to find the particular solution $y_p(x)$ of the inhomogeneous ODE

$$y'' + 2y' - 3y = x e^{2x}$$

We look for solutions of the type

$$y_p(x) = Q(x) e^{ax} = Q(x) e^{2x}$$

(4) We determine d_1 and d_0 by imposing that

$$y_p(x) = (d_1 x + d_0)\, e^{2x} \quad \text{is a solution of}$$

$$y'' + 2y' - 3y = x e^{2x} \qquad (1)$$

To this end we calculate $y_p'(x)$ and $y_p''(x)$

$$y_p'(x) = \frac{d}{dx}\left[(d_1 x + d_0)\, e^{2x}\right] = 2(d_1 x + d_0)\, e^{2x} +$$

$$+ d_1 e^{2x}$$

$$\boxed{y_p'(x) = (2d_1 x + 2d_0 + d_1)\, e^{2x}} \quad *$$

$$y_p''(x) = \frac{d}{dx}\, y'(x) = \frac{d}{dx}\left[(2d_1 x + 2d_0 + d_1)\, e^{2x}\right] =$$

$$y_p''(x) = 2(2d_1 x + 2d_0 + d_1)\, e^{2x} + 2d_1 e^{2x}$$

$$\boxed{y_p''(x) = (4d_1 x + 4d_0 + 4d_1)\, e^{2x}} \quad **$$

Inserting * and ** into (1) we set

$$(4d_1 x + 4d_0 + 4d_1)\, \cancel{e^{2x}} + 2(2d_1 x + 2d_0 + d_1)\, \cancel{e^{2x}} - 3(d_1 x + d_0)\, \cancel{e^{2x}} = x\, \cancel{e^{2x}}$$

Rearranging

$$5d_1 x + 6d_1 + 5d_0 = x$$

Matching the coefficients of the polynomials

$$\begin{cases} 5d_1 = 1 \\ 6d_1 + 5d_0 = 0 \end{cases} \qquad \begin{cases} d_1 = 1/5 \\ d_0 = -\frac{6}{5} d_1 = -\frac{6}{25} \end{cases}$$

The particular solution to (1) is given by

$$y_p(x) = \cancel{\tfrac{1}{5}x} \left(d_1 x + d_0\right) e^{2x}$$

$$\boxed{y_p(x) = \frac{1}{5}\left(x - \frac{6}{5}\right) e^{2x}}$$

5 The general solution $y_g(x)$ of the inhomogeneous ODE

$$y'' + 2y' - 3y = 0$$

is given by $\quad y_g(x) = y_h(x) + y_p(x)$

with $\quad y_h(x) = c_1 e^{-3x} + c_2 e^{x}$

$$y_p(x) = \frac{1}{5}\left(x - \frac{6}{5}\right) e^{2x}$$

$$\boxed{y_g(x) = \frac{1}{5}\left(x - \frac{6}{5}\right) e^{2x} + c_1 e^{-3x} + c_2 e^{x}}$$

$\square$

Inserting $*$ and $**$ into the inhomogeneous ODE we get

$$y'' + 2y' - 3y = x e^{2x}$$

$$(4d_1 x + 4d_0 + 4d_1)\cancel{e^{2x}} + 2(2d_1 x + 2d_0 + d_1)\cancel{e^{2x}} - 3(d_1 x + d_0)\cancel{e^{2x}} = x\cancel{e^{2x}}$$

$$5d_1 x + 6d_1 + 5d_0 = x$$

$$\begin{cases} 5d_1 = 1 \\ 6d_1 + 5d_0 = 0 \end{cases} \qquad \begin{cases} d_1 = 1/5 \\ d_0 = -\frac{6}{5} d_1 = -\frac{6}{25} \end{cases}$$

$$y_p(x) = (d_1 x + d_0) e^{2x}$$

$$\boxed{y_p(x) = \frac{1}{5}\left(x - 6/5\right) e^{2x}}$$

(5) The general solution $y_p(x)$ of the inhomogeneous ODE

$$y'' + 2y' - 3y = x e^{2x}$$

is given by $\quad y_g(x) = y_p(x) + y_h(x)$

$$\boxed{y_g(x) = \frac{1}{5}\left(x - \frac{6}{5}\right) e^{2x} + c_1 e^{-3x} + c_2 e^{x}}$$

□

Note: The educated guess method can also be applied to inhomogeneous ODEs of the type

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

with $f(x) = p(x) \cos(a x)$

or $f(x) = p(x) \sin(a x)$

Where $\cos(a x)$ and $\sin(a x)$ are not solutions of the homogeneous problem.

In this case the particular solution $y_p(x)$ will have the form

$$y_p(x) = Q(x) \left[A \cos(a x) + B \sin(a x) \right]$$

where $Q(x)$ is a polynomial of the same degree as $p(x)$

$$Q(x) = d_k x^k + d_{k-1} x^{k-1} + \dots + d_1 x + d_0$$

In this case the application of the educated guess method follows the same steps of the example above.