Variation of parameter method (2^{nd}-order ODEs)

The variation of parameter method solves 2^{nd}-order linear inhomogeneous ODEs with constant coefficient of the type

$$\boxed{a y'' + b y' + c y = f(x)} \qquad (1) \qquad \begin{cases} a_2 = a \\ a_1 = b \\ a_0 = c \end{cases}$$

where $f(x) \neq 0$ and $a, b, c \in \mathbb{R}$ $a \neq 0$

The method provides the general solution to (1) which is of the form

$$\boxed{y_g(x) = y_p(x) + y_h(x)}$$

where $y_p(x)$ is a particular solution to (1)

$y_h(x)$ is the general solution to the homogeneous problem corresponding to (1)

$$a y'' + b y' + c y = 0 \qquad \begin{cases} a_2 = a \\ a_1 = b \\ a_0 = c \end{cases}$$

- $y_h(x)$ is given by the general solution to the homogeneous problem.

Assuming the characteristic equation

$M_2(\lambda) = a\lambda^2 + b\lambda + c = 0$ has two DISTINCT SOLUTIONS $\boxed{\lambda_1 \neq \lambda_2}$

Then $\boxed{y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$ with $c_1, c_2 \in \mathbb{R}$ arbitrary constants.

- The variation of parameter methods provides also a particular solution $y_p(x)$ to (1).

$$\boxed{y_p(x) = \frac{1}{a_2(\lambda_1 - \lambda_2)} \left\{ e^{\lambda_1 x} \int f(x) e^{-\lambda_1 x} dx - e^{\lambda_2 x} \int f(x) e^{-\lambda_2 x} dx \right\}}$$

Note: The usefulness of this method relies on the ability to solve $\int f(x) e^{-\lambda_i x} dx$

Variation of parameter method

Step I Find the general solution to the homogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

We assume that the characteristic equation has two distinct solutions λ_1, λ_2 with $\lambda_1 \neq \lambda_2$

$$M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \quad \text{and } M_2(\lambda_1) = M_2(\lambda_2) = 0$$

The general solution is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad \text{with } c_1, c_2 \text{ arbitrary constants.}$$

Step: Consider the inhomogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = f(x) \qquad (1)$$

We look for solutions of the type

$$y(x) = c_1(x)\, e^{\lambda_1 x} + c_2(x)\, e^{\lambda_2 x}$$ and we consider

under which condition on $c_1(x), c_2(x)$ this is a solution of (1)

Note: $$\frac{d}{dx}\left(c_i(x)\, e^{\lambda_i x}\right) = \left(\frac{d}{dx} c_i(x)\right) e^{\lambda_i x} + c_i(x)\left(\frac{d}{dx} e^{\lambda_i x}\right)$$

$$= c_i'(x)\, e^{\lambda_i x} + c_i(x)\, \lambda_i\, e^{\lambda_i x}$$

$$\boxed{\frac{d}{dx}\left(c_i(x)\, e^{\lambda_i x}\right) = c_i'(x)\, e^{\lambda_i x} + c_i(x)\, \lambda_i\, e^{\lambda_i x}} \qquad (3)$$

Let us calculate $y'(x)$ and $y''(x)$

- $$y'(x) = \frac{d}{dx}\left[c_1(x)\, e^{\lambda_1 x} + c_2(x) e^{\lambda_2 x}\right]$$

Using Eq. (3) we get

$$y'(x) = \underline{c_1'(x)\, e^{\lambda_1 x}} + \lambda_1 c_1(x)\, e^{\lambda_1 x} + \underline{c_2'(x) e^{\lambda_2 x}} + \lambda_2 c_2(x) e^{\lambda_2 x}$$

We impose that

$$\boxed{c_1'(x)\, e^{\lambda_1 x} + c_2'(x)\, e^{\lambda_2 x} = 0} \qquad (4)$$

Therefore we set

$$\boxed{y'(x) = c_1(x)\, \lambda_1 e^{\lambda_1 x} + c_2(x)\, \lambda_2 e^{\lambda_2 x}} \qquad (5)$$

Differentiating Eq. (5) we get

$$y''(x) = \frac{d}{dx}\, y'(x) \underset{\substack{\uparrow \\ (\text{Eq. }5)}}{=} \frac{d}{dx}\left[c_1(x)\,\lambda_1 e^{\lambda_1 x} + c_2(x)\,\lambda_2 e^{\lambda_2 x} \right] =$$

$$y''(x) = \lambda_1 \frac{d}{dx}\left(c_1(x)\, e^{\lambda_1 x} \right) + \lambda_2 \frac{d}{dx}\left(c_2(x)\, e^{\lambda_2 x} \right)$$

Recalling Eq. (3) $\qquad \frac{d}{dx}\left(c_i(x)\, e^{\lambda_i x} \right) = c_i'(x)\, e^{\lambda_i x} + \lambda_i c_i(x)\, e^{\lambda_i x}$

$$\boxed{y'' = \lambda_1 \left[c_1'(x)\, e^{\lambda_1 x} + \lambda_1 c_1(x)\, e^{\lambda_1 x} \right] + \lambda_2 \left[c_2'(x)\, e^{\lambda_2 x} + \lambda_2 c_2(x)\, e^{\lambda_2 x} \right]} \qquad (6)$$

We put $y(x)$, $y'(x)$, $y''(x)$ [Eq. (2), (5), (6)] in the inhomogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

getting

$$a_2 \underbrace{\left(\lambda_1 c_1'(x)\, e^{\lambda_1 x} + \lambda_1^2 c_1(x)\, e^{\lambda_1 x} + \lambda_2 c_2'(x)\, e^{\lambda_2 x} + \lambda_2^2 c_2(x)\, e^{\lambda_2 x} \right)}_{} {}^{= y''} +$$

$$a_1 \underbrace{\left(c_1(x)\,\lambda_1 e^{\lambda_1 x} + c_2(x)\,\lambda_2 e^{\lambda_2 x} \right)}_{= y'} + a_0 \left(c_1(x)\, e^{\lambda_1 x} + c_2(x)\, e^{\lambda_2 x} \right) = f(x)$$

$$c_1(x)\,e^{l_1 x}\underbrace{\cancel{\left(a_2 l_1^2 + a_1 l_1 + a_0\right)}}_{=0} + c_2(x)\,e^{l_2 x}\underbrace{\cancel{\left(a_2 l_2^2 + a_1 l_2 + a_0\right)}}_{=0} +$$

$$+\left(a_2 l_1 c_1'(x)\,e^{l_1 x} + a_2 l_2 c_2'(x)\,e^{l_2 x}\right) = f(x)$$

$$\boxed{a_2\left(l_1 c_1'(x)\,e^{l_1 x} + l_2 c_2'(x)\,e^{l_2 x}\right) = f(x)}$$

$$\boxed{l_1 c_1'(x)\,e^{l_1 x} + l_2 c_2'(x)\,e^{l_2 x} = \frac{f(x)}{a_2}} \qquad \text{Eq. (7)}$$

which is obtained assuming that ~~Eq~~ the following condition applies

$$\boxed{c_1'(x)\,e^{l_1 x} + c_2'(x)\,e^{l_2 x} = 0} \qquad \text{Eq. (8)}$$

Solving for $c_1'(x)$ $c_2'(x)$

$$\begin{cases} c_1'(x) = \dfrac{-1}{a_2(l_2 - l_1)}\, f(x)\, e^{-l_1 x} \\[2ex] c_2'(x) = +\dfrac{1}{a_2(l_2 - l_1)}\, f(x)\, e^{-l_2 x} \end{cases}$$

Integrating with respect to x

$$c_1(x) = \frac{-1}{a_2(\lambda_2-\lambda_1)}\left[\int f(x) e^{-\lambda_1 x} + C_1\right]$$

$$c_2(x) = \frac{+1}{a_2(\lambda_2-\lambda_1)}\left[\int f(x) e^{-\lambda_2 x} + C_2\right]$$

where C_1, C_2 are arbitrary constants.

Inserting this expression into the solution

$$y(x) = c_1(x) e^{\lambda_1 x} + c_2(x) e^{\lambda_2 x}$$

we get the general solution to the inhomogeneous ODE of the type

$$y(x) = y_p(x) + y_h(x)$$

where $\quad y_h(x) = d_1 e^{\lambda_1 x} + d_2 e^{\lambda_2 x}$

with $\quad d_1 = \dfrac{-C_1}{a_2(\lambda_2-\lambda_1)} \qquad d_2 = \dfrac{+C_2}{a_2(\lambda_2-\lambda_1)}$ arbitrary constant

and the particular solution $y_p(x)$ is given by

$$y_p(x) = \frac{1}{a_2(\lambda_1-\lambda_2)} \left\{ e^{\lambda_1 x} \int f(x)\, e^{-\lambda_1 x}\, dx - e^{\lambda_2 x} \int f(x)\, e^{-\lambda_2 x}\, dx \right\}$$