IVP for 2^{nd}-order linear ODEs

The IVP for 2^{nd}-order linear ODEs comprises of

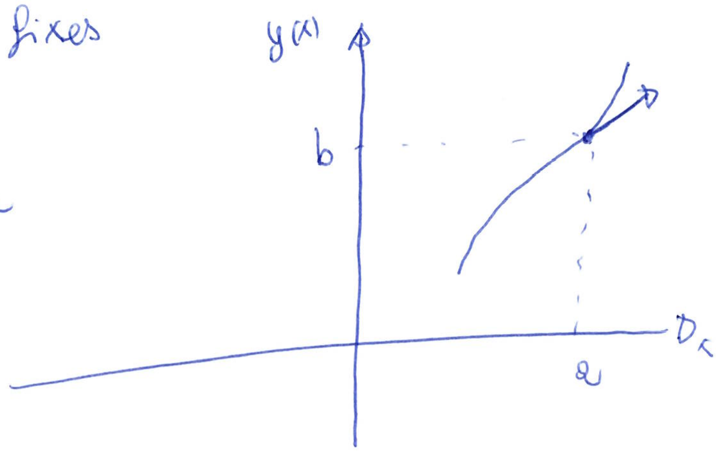
ODE: $a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$

I.C. $\begin{cases} y(a) = b_1 \\ y'(a) = b_2 \end{cases}$

For 1^{st}-order ODE the I.C. was only fixing the value of $y(x)$ for $x = a$

$$y(a) = b$$

For 2^{nd}-order ODEs I.C. fixes



a) the value of $y(x)$ for $x = a$

$$y(a) = b_1$$

b) the value of $y'(x)$ for $x = a$

$$y'(a) = b_2$$

The slope of the tangent vector for $x = a$

Existence and uniqueness of the solution

The generalized Picard-Lindelöf theorem for 2^{nd}-order linear ODEs guarantees that the solution to the IVP exists and is unique as long as $f(x)$, $a_2(x)$, $a_1(x)$, $a_0(x)$ are continuous (see lecture notes for details)

Applications The Newton law for a mass attached to the ceiling by a spring

ODE: $$m\ddot{y} = mg - k(y-\ell) - \gamma\dot{y}$$

where $m, g, k, \ell, \gamma \in \mathbb{R}$

2^{nd}-order linear ODE - inhomogeneous

I.C. $$\begin{cases} y(t_0) = y_0 & \text{initial position} \\ & \text{AND} \\ y'(t_0) = v_0 & \text{initial velocity} \end{cases}$$

The solution exists and is unique

Homogeneous 2nd-order ODEs with constant coefficients

We consider the homogeneous ($f(x)=0$) 2nd-order linear ODE

$$a_2 y'' + a_1 y' + a_0 y = 0 \qquad (1).$$

with $a_2, a_1, a_0 \in \mathbb{R}$ and $a_2 \neq 0$

Examples

$$m\ddot{x} = k(x-\ell) - \gamma \dot{x}$$

$$y'' + 2y' + y = 0$$

$$y'' - y = 0$$

We look for solutions of the type $\quad y(x) = e^{\lambda x} \qquad (2)$

and we want to find the values of λ such that (2) is a solution of (1)

Theorem: Eq. (2) is a solution of the ODE (1) provided that λ satisfies the following CHARACTERISTIC EQUATION

$$\boxed{M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0}$$

__Proof__ Given $y(x) = e^{\lambda x}$

$$y'(x) = \frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x} = \lambda y \qquad \boxed{y' = \lambda y}$$

$$y''(x) = \frac{d}{dx} y' \underset{\substack{\uparrow \\ y' = \lambda y}}{=} \frac{d}{dx} (\lambda y) = \lambda \frac{dy}{dx} \underset{\substack{\uparrow \\ y' = \lambda y}}{=} \lambda (\lambda y) = \lambda^2 y$$

$$\boxed{y'' = \lambda^2 y}$$

Therefore if $y(x) = e^{\lambda x}$ the ODE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

reduces to $(y'' = \lambda^2 y,\ y' = \lambda y)$

$$a_2 \lambda^2 y + a_1 \lambda y + a_0 y = 0$$

$$(a_2 \lambda^2 + a_1 \lambda + a_0) y = 0 \qquad y = e^{\lambda x} \neq 0$$

which occur if and only if

$$\boxed{M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0}$$

Characteristic equation.

$\square$

The characteristic equation can have

(A) two distinct real roots

(B) two complex conjugate roots

(C) two coincident real roots, i.e. one real root with multiplicity two.

(A) Two distinct real roots of $M_2(\lambda)=0$

$\lambda_1, \lambda_2 \in \mathbb{R}$ $\quad \lambda_1 \neq \lambda_2$ $\quad M_2(\lambda_1) = M_2(\lambda_2) = 0$

In this case the general solution to Eq. (1) is given by

$$\boxed{y_g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \qquad (3)$$

with c_1, c_2 arbitrary constants. $\quad c_1, c_2 \in \mathbb{R}$

(B) Two complex conjugate roots of the characteristic equation

$\lambda_1, \lambda_2 \in \mathbb{C}$ $\quad \lambda_1 = \lambda_2^*$ $\quad \lambda_1 = \alpha + i\beta$

$\lambda_2 = \alpha - i\beta$ $\quad \beta \neq 0$

$M_2(\lambda_1) = M_2(\lambda_2) = 0$

In this case the general solution to Eq. (1) is given by

$$\boxed{y_g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \qquad (4)$$

with $c_1, c_2 \in \mathbb{C}$ indicating arbitrary constant.

(B).. continuations

Alternatively ~~Eq.(4)~~ the general solution can be written as

$$\boxed{y_g(x) = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)} \qquad (5)$$

where $A, B \in \mathbb{R}$ are arbitrary constants.

Let us show that (4) is equivalent to (5)

Starting from Eq. (4)

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Inserting $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$

$$y(x) = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

Now using Euler formulas

$$e^{i\beta x} = \cos\beta x + i\sin\beta x$$
$$e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

We obtain

$$y(x) = e^{\alpha x}\left(c_1\cos\beta x + i c_1\sin\beta x + c_2\cos\beta x - i c_2\sin\beta x\right)$$

$$= e^{\alpha x}\left[\overbrace{(c_1+c_2)}^{=A}\cos\beta x + i\overbrace{(c_1-c_2)}^{B}\sin\beta x\right] =$$

$$\boxed{\begin{array}{l} A = c_1 + c_2 \\ B = i(c_1 - c_2) \end{array}}$$

$$= y\, e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right) \qquad \square$$

(B) continuing..

Alternatively the general solution can be written as

$$\boxed{y_g(x) = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)} \qquad (5)$$

where $A, B \in \mathbb{R}$ are arbitrary constants.

(C) Case of a single solution λ_1 to the characteristic equation with multiplicity 2

$\lambda_1 \in \mathbb{R}$ $\qquad M_2(\lambda_1) = 0$

In this case the general solution to (1) is

$$\boxed{y_g(x) = e^{\lambda_1 x}\left(c_1 x + c_2\right)}$$

where $c_1, c_2 \in \mathbb{R}$ arbitrary constant

Let us show that Eq (4) is equivalent to Eq. (5)

Starting from Eq. (4)

$$y_g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Inserting $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

$$y_g(x) = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

Now, using Euler formulas

$$e^{i\beta x} = \cos\beta x + i \sin\beta x$$

$$e^{-i\beta x} = \cos\beta x - i \sin\beta x$$

We obtain

$$y_g(x) = e^{\alpha x}\Big(\underline{c_1 \cos\beta x} + i\, c_1 \sin\beta x + \underline{c_2 \cos\beta x} - i\, c_2 \sin\beta x \Big)$$

$$= e^{\alpha x}\Big[(c_1 + c_2) \cos\beta x + i\,(c_1 - c_2) \sin\beta x \Big]$$

$$A = c_1 + c_2 \qquad B = i(c_1 - c_2)$$

$$y_g(x) = e^{\alpha x}\big(A \cos\beta x + B \sin\beta x \big) \qquad \text{Eq. (5)}$$

$\square$