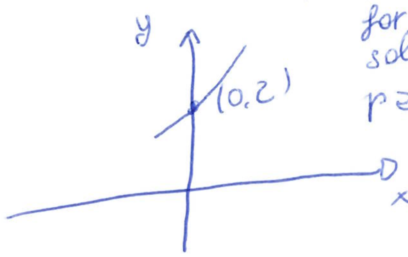
Mock Quiz # 3

$y' = \frac{xy}{x-1}$ $\qquad y(0)=2$ $\quad$ I.C.

$x_0 = 0 \quad y_0 = 2$



We look for solutions passing through the point (0,2).

Solve the I.V.P.

$$\frac{dy}{y} = \frac{x}{x-1}\,dx$$

$$\int \frac{dy}{y} = \int \frac{x}{x-1}\,dx + C$$

LHS: $\quad H(y) = \int \frac{1}{y}\,dy = \ln|y|$

RHS: $\quad F(x) = \int \frac{x}{x-1}\,dx =$

$$= \int 1 + \frac{1}{x-1}\,dx$$

$$= x + \ln|x-1|$$

$$H(y) = F(x) + C$$

$$\ln|y| = x + \ln|x-1| + C'$$

$$|y| = e^{C} e^{x} |x-1|$$

$$y = \underbrace{\boxed{\pm e^{C}}}_{D}\, e^{x}(x-1) = D e^{x}(x-1)$$

$y(0)=2 \qquad 2 = -D$

$$\boxed{y(x) = 2e^{x}(1-x)}$$

$$|y| = e^{x + \ln|x-1| + C'} = e^{C'} e^{x} |x-1|$$

$$y = \boxed{\pm e^{C'}} e^{x} (x-1) = D e^{x} (x-1)$$

$$\boxed{y(x) = D e^{x} (x-1)}$$ Explicit general solution

Impose the I.C.

$$2 = y(0) = D e^{0} (0-1) = -D \qquad D = -2$$

$$\boxed{y(x) = 2e^{x}(1-x)}$$ ✓

Consider D: $|x - x_0| \leq A$, $|y - y_0| \leq B$

Under which conditions on A & B the Picard-Lindelöf theorem guarantees that the solution exist and is unique?

D: $|x| \leq A$ $\quad |y-2| \leq B$

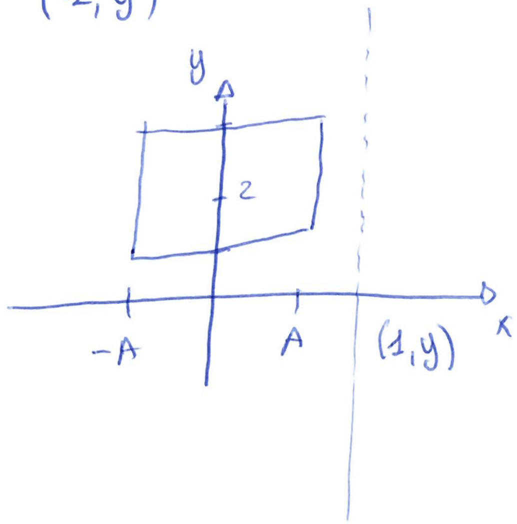
$$y' = f(x,y) = \frac{x}{x-1} \cdot y$$

1. $f(x,y)$ ~~are~~ is continuous in D
2. $\frac{\partial f}{\partial y}(x,y)$ is bounded in D
3. $A \leq \frac{B}{M}$ where $M = \max\limits_{(x,y) \in D} |f(x,y)|$

① $f(x,y)$ is continuous in D

$$f(x,y) = \frac{x}{x-1} \cdot y$$

continous for all (x,y) except $\cancel{(\bar{x},\bar{y})}$ $(1, y)$



$|x| \le A \qquad -A \le x \le A < 1$

$|y-2| \le B \qquad 2-B \le y \le 2+B$

$$\boxed{0 < A < 1}$$

② $\frac{\partial}{\partial y} f(x,y)$ should be bounded in D

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left(\frac{x}{x-1} \cdot y\right) = \frac{x}{x-1}$$

is continous for $x \ne 1$

$\Downarrow$

is continous in D provided $0 < A < 1$

$\Downarrow$

it is bounded in D ✓

③ $A \le \frac{B}{M}$ where $M = \max\limits_{(x,y) \in D} |f(x,y)|$

$$M = \max_{(x,y)\in D} \left| \frac{x}{x-1} \cdot y \right| = \max_{(x,y)\in D} \left| \frac{x}{x-1} \right| \cdot \max_{(x,y)\in D} |y|$$

$-A \leq x \leq A$

$2-B \leq y \leq 2+B$

$$\max_{(x,y)\in D} |y| = 2+B$$

$$\max_{(x,y)\in D} \left| \frac{x}{x-1} \right| = \max_{(x,y)\in D} \left| 1 + \frac{1}{x-1} \right| = \frac{A}{1-A}$$

$$A \leq \frac{B}{M} = \frac{B}{(2+B)A} (1-A) \quad \checkmark \qquad M = (2+B) \cdot \frac{A}{1-A}$$

There exist one and only one solution in the rectangular region D

$|x| \leq A$

$|y-2| \leq B$

with $\quad 0 < A < 1 \qquad A \leq \frac{B}{2+B} \frac{1-A}{A}$

$B > 0$

$$\frac{A^2}{1-A} \leq \frac{B}{2+B}$$

A.1)

$$\begin{cases} 1 - y \sin x + (\cos x)\, y' = 0 \\ P(x,y) + Q(x,y)\, y' = 0 \end{cases}$$

$P(x,y) = \dfrac{\partial F}{\partial x}$ $\quad$ $Q(x,y) = \dfrac{\partial F}{\partial y}$

$$\begin{cases} P(x,y) = 1 - y \sin x \\ Q(x,y) = \cos x \end{cases}$$

$F(x,y) = 0$ Implicit solution

The ODE is exact if and only if

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} F\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} F\right)$$

$$\frac{\partial}{\partial x} Q(x,y) = \frac{\partial}{\partial y} P(x,y)$$

$$\frac{\partial}{\partial x} \cos x \overset{?}{=} \frac{\partial}{\partial y} (1 - y \sin x)$$

LHS: $\dfrac{\partial}{\partial x} \cos x = -\sin x$

RHS: $\dfrac{\partial}{\partial y} (1 - y \sin x) = -\sin x$

✓ Yes! The ODE is exact.

Integrate

$$F(x,y) = \int P(x,y)\,dx + g(y) = \int (1 - y\sin x)\,dx + g(y)$$

$$F(x,y) = x - y\int \sin x\,dx + g(y) = x + y\cos x + g(y)$$

$$\frac{\partial F}{\partial y} = Q(x,y)$$

$$\frac{\partial F}{\partial y} = \cos x + g'(y) = \cos x \quad \Rightarrow \quad g'(y) = 0$$

$$g(y) = C'$$

$$\boxed{F(x,y) = x + y\cos x + C' = 0}$$ Implicit form.

$$\boxed{y = -\frac{(x + C')}{\cos x}}$$ Explicit form.