Motivation for the Picard - Lindelöf theorem (existence and uniqueness of solution)

Initial value problem I.V.P. for 1^{st}-order ODEs

We consider the I.V.P.

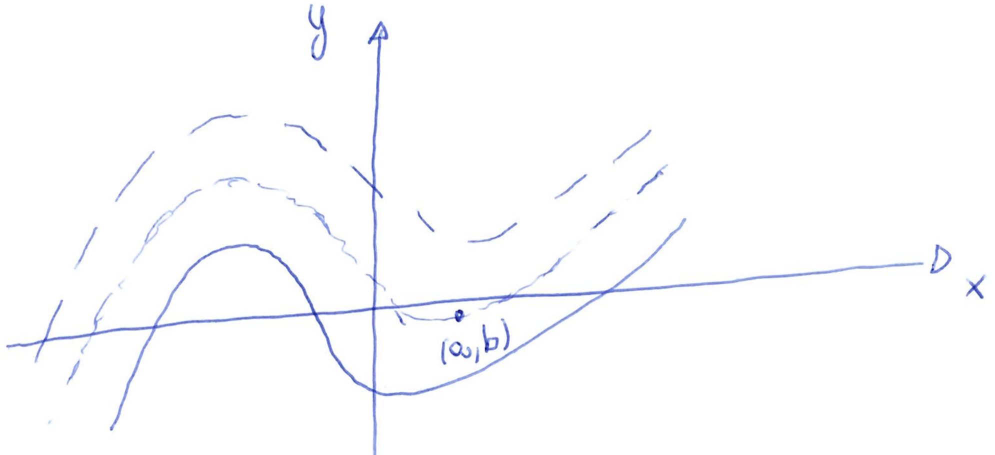
1) ODE: $\qquad y' = f(x,y)$

2) I.C. $\qquad y(a) = b$

The ODE has a general solution that depends on an arbitrary constant C.

$$F(x, y(x)) = C \quad \text{implicit general solution.}$$

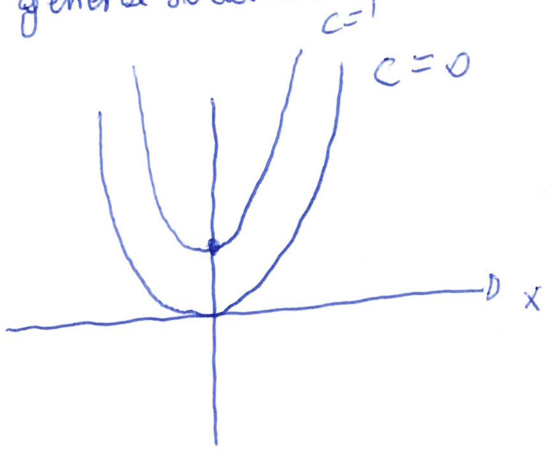
The I.V.P. impose to find the function / functions $y(x)$ that solve $y' = f(x,y)$ and satisfy $y(a) = b$



Example

$y' = x$ & $y(0) = 1$

$y(x) = \int x\,dx + C = \frac{1}{2}x^2 + C$ general solution



$1 = y(0) = C \Rightarrow \boxed{C = 1}$

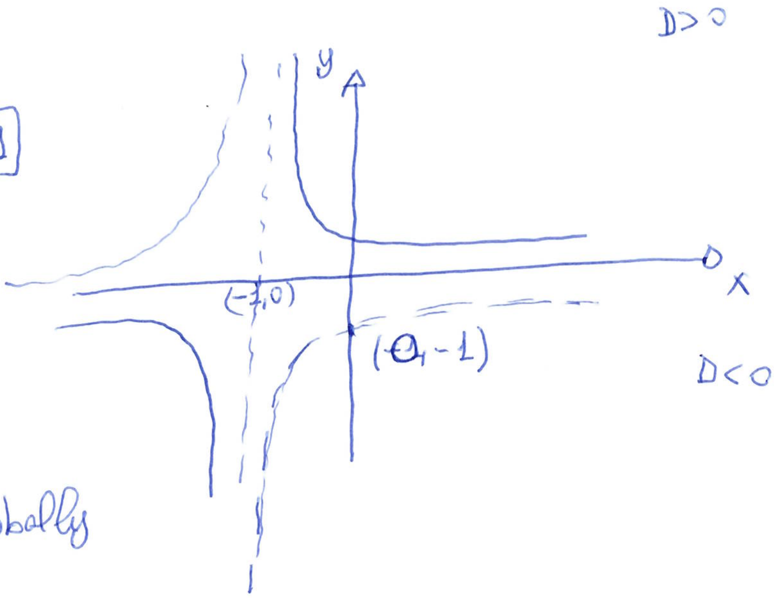
$$\boxed{y' = \frac{1}{2}x^2 + 1}$$

Example

$y' = -\frac{y}{x+1}$ & $y(0) = -1$

separable

General solution $y(x) = \frac{D}{x+1}$ $x \neq -1$



$-1 = y(0) = \frac{D}{1} = D$ $\boxed{D = -1}$

$$y(x) = -\frac{1}{x+1}$$

There are other solution, globally

$$y(x) = \begin{cases} -\frac{1}{x+1} & x > -1 \\ \frac{D}{x+1} & x < -1 \end{cases}$$

Locally there is only one solution

Definition

An initial value problem formed by an ODE + I.C.

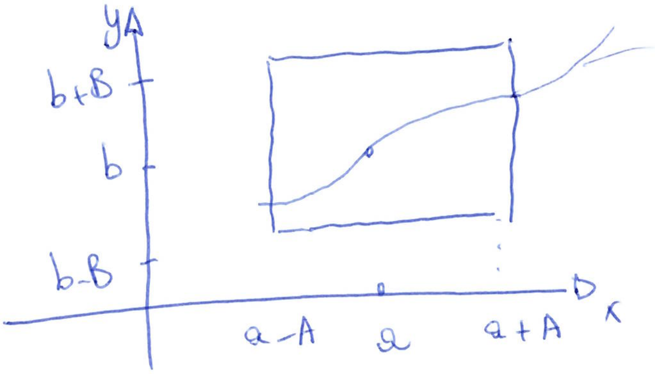
$y(a)=b$ has a

UNIQUE SOLUTION if

for any two solutions $y_1(x)$, $y_2(x)$ satisfying the I.V.P.

there exist $A>0$, $B>0$ such that

$$y_1(x) = y_2(x) \qquad \forall x \in (a-A, a+A)$$
$$\forall y \in (b-B, b+B)$$



However there are cases in which the solution to the IVP. is NOT UNIQUE in any D $|x-a| \le A$
$|y-b| \le B$.

Example $\quad y' = \frac{1}{2y} \quad \& \quad y(0) = b$

This ODE is separable

$$\frac{dy}{dx} = \frac{1}{2y} \Rightarrow \int 2y\, dy = \int dx + C$$

LHS: $\quad H(y) = \int 2y\, dy = y^2$

RHS: $\quad F(x) = \int dx = x$

Implicit solution $\quad H(y) = F(x) + C$

$$y^2 = x + C$$

Explicit solution $\quad y(x) = \pm \sqrt{x + C}$

We impose the I.C. $\quad y(0) = b$

- If $b > 0$ $\quad b = y(0) = + \sqrt{C} \Rightarrow C = b^2$

  $y(x) = + \sqrt{x + b^2}$ UNIQUE SOLUTION

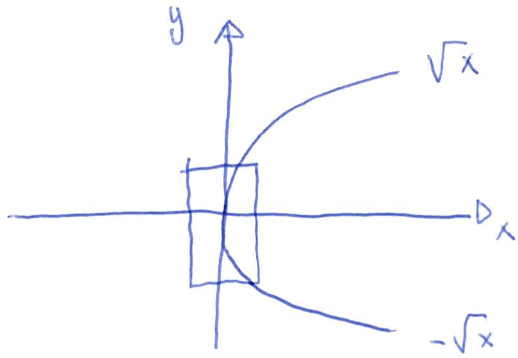
- If $b < 0$ $\quad b = y(0) = - \sqrt{C} \Rightarrow C = b^2$

  $y(x) = - \sqrt{x + b^2}$ UNIQUE SOLUTION

If $b=0$ $\quad 0=b=y(0)=\pm\sqrt{C}$ $\quad C=0$

$$y(x)=\sqrt{x}$$

$$y(x)=-\sqrt{x}$$



Example $\quad y'=3y^{2/3}$ $\quad$ & $\quad y(0)=0$

ODE $\quad y'=g(y)$ $\quad g(y)=3y^{2/3}$

I.C. $\quad y(a)=b$ $\quad a=0 \quad b=0$

$$\frac{dy}{dx}=3y^{2/3} \Rightarrow \frac{dy}{3y^{2/3}}=dx$$

Special solution

$g(y_0)=0$ $\quad y=y_0$

$g(0)=0$ $\quad \boxed{y=0}$ $\quad$ Constant solution (1)

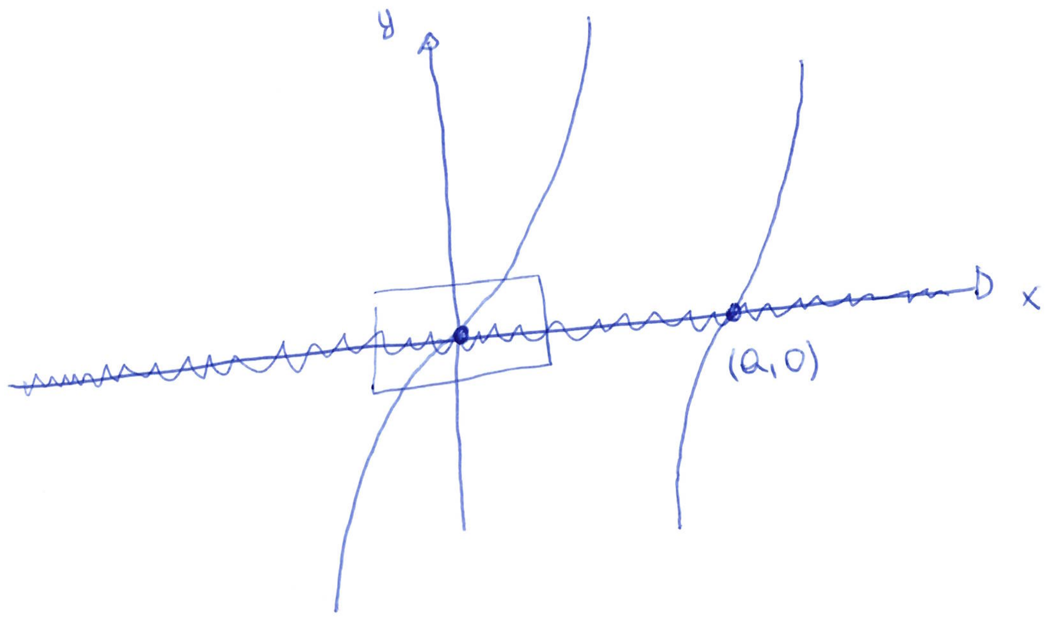
Solving the ODE by separation of variable

$$y(x)=(x+C)^3 \qquad (2)$$

Let us impose $y(0)=0$

(1) $y(x)=0$ satisfies the I.C.

(2) $0=y(0)=C^3$ $\quad C=0$ $\quad y(x)=x^3$



(3) $$y(x)=\begin{cases} 0 & \text{if } x<0 \\ x^3 & \text{if } x\geq 0 \end{cases}$$

Let us impose I.C. $y(a)=0$

(1) $y(x)=0$ is a solution

(2) $0=y(a)=(a+C)^3$ $\quad C=-a$

$y(x)=(x-a)^3$