III. A $\quad \frac{dy}{dx} + \frac{1}{2} y = \frac{1}{2} e^{x/3}$

$y(x), \quad y(0) = 1$

1. Solve homogeneous eq.: $\quad \frac{dy}{dx} + \frac{1}{2} y = 0$

$$\frac{dy}{dx} = -\frac{1}{2} y$$

$$\int \frac{dy}{y} = \int dx \left(-\frac{1}{2}\right) \qquad D \cdot \left(-\tfrac{1}{2}\right) e^{-\frac{x}{2}} = -\frac{1}{2} D \cdot e^{-\frac{x}{2}} \quad \checkmark$$

$$\ln|y| = -\frac{1}{2} x + C \quad \Rightarrow \quad |y| = e^{-\frac{x}{2} + C}$$

$$y = \pm e^{-\frac{x}{2} + C}$$

$$= D \cdot e^{-\frac{x}{2}}$$

$$D = \pm e^{C}$$

2. Variation of the constant: $\quad y(x) = D \cdot e^{-\frac{x}{2}}$

$$y(x) = D(x) \cdot e^{-\frac{x}{2}}$$

substitute: $\quad \frac{d}{dx} y(x) = \frac{d}{dx} \left(D(x) \cdot e^{-\frac{x}{2}} \right)$

Substitute: $\frac{d}{dx} y(x) = \frac{d}{dx}\left(D(x) \cdot e^{-\frac{x}{2}}\right)$

ODE $\Rightarrow$ $\frac{d}{dx}\left(D(x) \cdot e^{-\frac{x}{2}}\right) + \frac{1}{2} y = \frac{1}{2} e^{\frac{x}{3}}$

$$D'(x) \cdot e^{-\frac{x}{2}} + D(x) \cdot \left(-\frac{1}{2} e^{-\frac{x}{2}}\right) + \frac{1}{2} y = \frac{1}{2} e^{\frac{x}{3}}$$

$$\uparrow$$

$$D(x) \cdot e^{-\frac{x}{2}}$$

$$\Rightarrow \quad D'(x) \cdot e^{-\frac{x}{2}} = \frac{1}{2} e^{\frac{x}{3}}$$

$\int e^{ax}\, dx$ $\quad D'(x) = \frac{1}{2} e^{\frac{x}{3} + \frac{x}{2}} = \frac{1}{2} e^{\frac{5x}{6}}$

$= \frac{1}{a} e^{ax}$ $\quad \Rightarrow \quad D(x) = \frac{1}{2} \cdot \frac{6}{5} \cdot e^{\frac{5x}{6}} + C$

$$y(x) = D(x)\, e^{-\frac{x}{2}} = \frac{3}{5} \cdot e^{\frac{5x}{6} - \frac{x}{2}} + C e^{-\frac{x}{2}}$$

$y(0) = 1$ $\quad y(0) = 1 = \frac{3}{5} + C \quad \Rightarrow \quad \frac{2}{5} = C$

$$\Rightarrow \quad y(x) = \frac{3}{5} e^{\frac{x}{3}} + \frac{2}{5} e^{-\frac{x}{2}}$$

Integrating factor: ODE $\frac{dy}{dx} + \frac{1}{2} y = \frac{1}{2} e^{\frac{x}{3}}$

$\mu(x)$

$$\mu(x) \cdot \frac{dy}{dx} + \mu(x) \cdot \frac{1}{2} y = \mu(x) \frac{1}{2} e^{\frac{x}{3}}$$

$$y(x) = D(x)\, e^{-\frac{x}{2}} = \frac{3}{5} \cdot e^{\frac{5x}{6} - \frac{x}{2}} + C e^{-\frac{x}{2}}$$

$$y(0) = 1 \qquad y(0) = 1 = \frac{3}{5} + C \Rightarrow \frac{2}{5} = C$$

$$\Rightarrow \quad y(x) = \frac{3}{5} e^{\frac{x}{3}} + \frac{2}{5} e^{-\frac{x}{2}}$$

Integrating factor: ODE $\quad \frac{dy}{dx} + \frac{1}{2} y = \frac{1}{2} e^{\frac{x}{3}}$

$\mu(x)$

$$\mu(x) \cdot \frac{dy}{dx} + \mu(x) \cdot \frac{1}{2} y = \mu(x) \frac{1}{2} e^{\frac{x}{3}}$$

$$\frac{d}{dx}\left(\mu(x) \cdot y(x)\right) = y(x) \cdot \frac{d\mu}{dx} + \mu(x) \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d\mu}{dx} = \frac{1}{2} \mu(x) \quad \Rightarrow \mu(x) = e^{\frac{x}{2}}$$

$$\frac{d}{dx}\left(\mu(x) \cdot y(x)\right) = \mu(x) \frac{1}{2} e^{\frac{x}{3}}$$

$$\frac{d}{dx}\left(e^{\frac{x}{2}} y(x)\right) = \frac{1}{2} e^{\frac{x}{3} + \frac{x}{2}} = \frac{1}{2} e^{\frac{5x}{6}}$$

integrate

$$e^{\frac{x}{2}} y(x) = \frac{1}{2} \frac{6}{5} e^{\frac{5x}{6}} + C$$

$$\Rightarrow \quad y(x) = \frac{3}{5} e^{\frac{x}{3}} + C \cdot e^{-\frac{x}{2}} \quad \checkmark$$