Wolfram Just

Recap:

- $\frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)\,dx + C$ "Calculus I"

- $\frac{dy}{dx} = f(x)\,g(y)$ "separable"

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

- Scale invariant ODE ("homogeneous")

Assume that the right hand side depends on the ratio $\frac{y}{x}$ only.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \text{"Scale invariant"}$$

Example:

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{x^2}{y^2} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^{-2} \quad ✓$$

$$\frac{dy}{dx} = \sin(y+x) = \sin y \cos x + \cos y \sin x$$

✗

$$x \frac{dy}{dx} = y - x e^{\frac{y}{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - e^{\frac{y}{x}}$$

✓

How to solve scale invariant ODEs

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad ?$$

Idea: Introduce a "new" variable

$$z(x) = \frac{y(x)}{x}, \quad y(x) = x \cdot z(x)$$

Consider $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{\frac{y}{x}}$

Use $z = \dfrac{y}{x}$, $\underline{y = x\,z}$

Recall $y(x)$, $z(x)$

$$\underline{\frac{dy}{dx}} = z - e^{z}$$

How to express $\dfrac{dy}{dx}$ in terms of "z":

$$\frac{dy}{dx} = \frac{d(x.z)}{dx} = z + x \cdot \frac{dz}{dx}$$

$$\cancel{z} + x\frac{dz}{dx} = \cancel{z} - e^{z}$$

The result is a separable ODE!!

$$\Rightarrow \quad \int e^{-z}\,dz = -\int \frac{dx}{x}$$

$$+e^{-z} + \cancel{C_1} = +\ln|x| + \underbrace{C_2 - C_1}_{-C}$$

$$e^{-z} = \ln|x| - C$$

$$\Rightarrow \quad -z = \ln(\ln|x| - C)$$

$$\Rightarrow \quad z = -\ln(\ln|x| - C)$$

$$\Rightarrow \quad y = x \cdot z = -x\ln(\ln|x| - C)$$

- Linear ODEs

The order of an ODE is the order of the highest derivative

Examples:

- $e^x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - (\ln x)^2 \cdot y = \tanh x$

  2nd order Linear.

- $x \frac{d^3y}{dx^3} + e^y = 0$ 3rd order nonlinear

- $y \frac{dy}{dx} + x = 0$ 1st order. non linear.

An ODE is said to be a linear ODE if the equation is a linear expression in terms of y and its derivatives.

In formal terms:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_0(x) \cdot y = f(x)$$

a linear ODE of order n.

- 1st order linear ODEs

A first order linear ODE can be written as

$$\frac{dy}{dx} = a(x)\, y + b(x)$$

If $b(x)$ is not identically zero then the ODE is said to be <u>inhomogeneous</u>

If $b(x) \equiv 0$ is identically to zero then

$$\frac{dy}{dx} = a(x)\, y$$

is said to be a homogeneous ODE (!?!)

How to solve:

Step I: Solve the homogenous ODE

Step II: (Miracle) Use variation of parameters.

Consider

$$\frac{dy}{dx} = -2xy + x$$

Step I: consider the corresponding homogeneous ODE

$$\frac{dy}{dx} = -2xy$$

$$\Rightarrow \int \frac{dy}{y} = -\int 2x \, dx$$

$$\ln|y| = -x^2 + C_1$$

$$|y| = e^{-x^2 + C} = e^{C_1} \cdot e^{-x^2}$$

$$y = \underbrace{\pm e^{C_1}}_{D} \cdot e^{-x^2}$$

$$y = D e^{-x^2}$$ (Solution to the homogenous eq.)

Step II: Pulled from the air.

Variation of parameters. Take the solution of the homogenous ODE and replace the constant D by an unknown function $D(x)$,

$$y(x) = D(x)\, e^{-x^2}$$

Claim: This solves the inhomogeneous ODE <u>and</u> we are able to compute $D(x)$!?!

$D(x)e^{-x^2}$ $\quad \dfrac{dy}{dx} = -2xy + x$

$D(x)e^{-x^2}$

$$\frac{d\,D(x)e^{-x^2}}{dx} = -2x\,D(x)\,e^{-x^2} + x$$

!!!

$$\frac{dD}{dx}e^{-x^2} + \cancel{D(x)\cdot\left(-2x e^{-x^2}\right)} = \cancel{-2x D(x) e^{-x^2}} + x$$

$$\frac{dD}{dx}e^{-x^2} = x$$

$$\int \frac{dD}{dx}\,dx = \int x e^{x^2}\,dx$$

$$D(x) = \frac{1}{2}e^{x^2} + C$$

$\int x e^{x^2} dx$

sub: $u = x^2$

$du = 2x\,dx$

$= \int \frac{1}{2} e^{u} du$

$= \frac{1}{2} e^{u} = \frac{1}{2} e^{x^2} (+C)$

Hence:

$$Y(x) = D(x) \cdot e^{-x^2}$$

$$= \left(\frac{1}{2}e^{x^2} + C\right) e^{-x^2} = \frac{1}{2} + Ce^{-x^2}$$

So far: 1^{st} order Linear ODEs

Now: 1^{st} order general ODEs

- Exact 1^{st} order ODEs

Basic idea:

$$x^2 + y^2 = C \qquad (C > 0)$$

gives a curve in the x-y plane.

Take the derivative with respect to x
(recall: $y = y(x)$)

$$2x + \underbrace{\frac{dy^2}{dx}}_{2y \cdot \frac{dy}{dx}} = 0$$

$2x + 2y \frac{dy}{dx} = 0$ is a first order ODE

Aim: Given a first order ODE, say

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

is it possible to find a solution which can be written as

$$F(x,y) = C$$

Task I: Figure out conditions on P and Q so that F exists.

Task II: Compute F.

Task I: Assume a solution is given by

$$F(x, y(x)) = C_1$$

Compute the derivative with respect to x

$$\underbrace{\frac{\partial F(x,y)}{\partial x}}_{P(x,y)} + \underbrace{\frac{\partial F(x,y)}{\partial y}}_{Q(x,y)} \cdot \frac{dy}{dx} = 0$$

1st order ODE.

F exists if

$$P(x,y) = \frac{\partial F(x,y)}{\partial x} \qquad Q(x,y) = \frac{\partial F(x,y)}{\partial y}$$

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial^2 F(x,y)}{\partial x \partial y} \qquad \frac{\partial Q(x,y)}{\partial x} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

$$\Rightarrow \quad \frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

A fistorde ODE $\quad P(x,y) + Q(x,y)\dfrac{dy}{dx} = 0$
is said to be exact if $\quad \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Example:

$$\underbrace{3x^2 + y}_{P} \underbrace{- (3y^2 - x)}_{Q} \frac{dy}{dx} = 0$$

$$P(x,y) = 3x^2 + y \qquad Q(x,y) = -(3y^2 - x)$$

$$\frac{\partial P}{\partial y} = 0 + 1 = \frac{\partial Q}{\partial x} = 0 + 1$$

The ODE is exact! The solution can be written as $F(x,y) = C_1$

Task II: Compute F.

Recall:

$$P(x,y) = \frac{\partial F}{\partial x}, \quad Q(x,y) = \frac{\partial F}{\partial y}$$

(1) $3x^2 + y = \frac{\partial F}{\partial x}$, $\quad -(3y^2 - x) = \frac{\partial F}{\partial y}$ (2)

(1) Integration with respect to x

$$F(x,y) = \int \underset{\substack{\uparrow \\ \text{treat } y \text{ as} \\ \text{a constant.}}}{3x^2 + y}\, dx + C_1(y)$$

$$F(x,y) = x^3 + y \cdot x + G(y)$$

(2) $$-(3y^2 - x) = \frac{\partial F}{\partial y} = 0 + x + \begin{cases} \cancel{\frac{\partial G}{\partial y}} \\ \frac{dC}{dy} \end{cases}$$

$$-3y^2 + \cancel{x} = \cancel{x} + \frac{dC}{dy}$$

$$\frac{dC}{dy} = -3y^2 \Rightarrow C(y) = -y^3 + K$$

$$F(x,y) = x^3 + yx - y^3 + K$$

General solution of the exact ODE

$F(x,y) = C_1$

$$x^3 + yx - y^3 = C_1$$ (an implicit solution)