Solution of 1^{st} order separable ODE
(More formally)

We consider the separable ODE

$$\boxed{\frac{dy}{dx} = f(x)\,g(y)}$$

and an interval $y \in (A, B)$ in the domain $g(y)$ such that $g(y) \neq 0$

We want to show that the ODE has implicit solution

$$\boxed{H(y(x)) = F(x) + C}$$

where $H(y)$ is the anti-derivative of $\frac{1}{g(y)}$

$F(x)$ is the anti-derivative of $f(x)$

<u>Proof</u>

$$H(y) = \int \frac{1}{g(y)}\, dy$$

We have $$\frac{d}{dy} H(y) = \frac{1}{g(y)}$$

We want to show that $H(y(x))$ is the anti-derivative $f(x)$

$$\frac{d\, H(y(x))}{dx} = f(x) \qquad *$$

because this implies

$$H(y(x)) = F(x) + C$$

Let us consider

$$\frac{d}{dx} H(y(x)) = \overset{= \frac{1}{g(y)}}{\frac{dH}{dy}}\, \frac{dy}{dx} = \frac{1}{g(\cancel{y})}\, \overbrace{f(x)\, g(\cancel{y})}^{\frac{dy}{dx} \,=}$$

$$\frac{d}{dx} H(y(x)) = f(x)$$

$$\boxed{H(y(x)) = F(x) + C}$$

If $H(y) = u$ admits an explicit expression for its inverse function/functions when we need to consider all inverse functions

$$y = H^{-1}(u)$$

Explicit solution

$$\boxed{y = H^{-1}(F(x) + C)}$$

Reducible to separable 1st order ODE

We consider ODE of the type

$$\frac{dy}{dx} = f(ax + by + c)$$

where $a, b, c \in \mathbb{R}$ are constant parameters

These equations are NOT separable but they are REDUCIBLE TO SEPARABLE

Solve of 1st order reducible to separable ODE.

Step 1 Introduce the variable z

$$z = ax + by + c \quad *$$

$$f(ax + by + c) = f(z)$$

$$\frac{dy}{dx} = f(z)$$

$$\Rightarrow \quad y = \frac{z - ax - c}{b}$$

<u>Step 2</u> Observe that the ODE for $z(x)$ is SEPARABLE

$$\frac{dz}{dx} = \frac{d}{dx}(ax + by + c) = a + b\frac{dy}{dx}$$

But $\frac{dy}{dx} = f(z)$

$$\boxed{\frac{dz}{dx} = a + b f(z)}$$

SEPARABLE
**

<u>Step 3</u> Solve ** by separation variables

<u>Step 4</u> Set $y(x) = \frac{z(x) - ax - c}{b}$

Example $\quad y' = e^{-(3x+y)} - 3$.

$y' = f(ax+by+c)$

$a = +3$
$b = +1$
$c = 0$

Step 1 $\quad z = 3x+y \implies y = z - 3x$

$y' = e^{-z} - 3$

Step 2

$$\frac{dz}{dx} = 3 + \frac{dy}{dx} = \cancel{3} + e^{-z} - \cancel{3}$$

$$\frac{dz}{dx} = e^{-z} \cdot 1 \quad ***$$

Step 3 Solve ***

$$\int \frac{dz}{e^{-z}} = \int dx + C_1$$

LHS: $$H(z) = \int \frac{1}{e^{-z}}\,dz = \int e^{z}\,dz = e^{z}$$

RHS: $$F(x) = \int dx = x$$

$$H(z) = F(x) + C_1$$

$$e^{z} = x + C_1$$

$$\boxed{z = \ln(x + C_1)} \quad \text{for } x + C_1 > 0$$

<u>Step 4</u>

$$y = z - 3x$$

$$\boxed{y(x) = \ln(x + C_1) - 3x} \quad \text{for } x + C_1 > 0$$

Summary for solving separable ODEs

Given $\frac{dy}{dx} = f(x)\,g(y)$

① Identify $f(x)$, $g(y)$

② Separate the variables

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$$

for $g(y) \neq 0$

LHS: $H(y) = \int \frac{dy}{g(y)}$

RHS: $F(x) = \int f(x)\,dx$

$$\boxed{H(y) = F(x) + C}$$

Implicit solution

④ Express $y(x) = H^{-1}(F(x)+C)$

if you can for all inverse functions

otherwise just give the implicit solution

A) $\frac{dy}{dx} = 3y^{2/3}$

① $f(x) = 1$

$g(y) = 3y^{2/3}$

$y=0$ is a root

$y(x)=0$ is a solution

② $H(y) = \int \frac{1}{g(y)}\,dy = \int \frac{1}{3y^{2/3}}\,dy$

$= \int \frac{1}{3} y^{-2/3}\,dy = y^{1/3}$

$$F(x) = \int 1\,dx = x$$

$$H(y) = F(x) + C$$

$$\boxed{y^{1/3} = x + C}$$

$$\boxed{y = (x + C)^3}$$

$y = 0$

(B)

$$\frac{dy}{dx} = \frac{x}{y}$$

$f(x) = x$

$g(y) = \frac{1}{y}$

$$H(y) = \int y\,dy = \frac{y^2}{2}$$

$$F(x) = \int x\,dx = \frac{x^2}{2}$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$H(y) = u \qquad \frac{y^2}{2} = u$$

$$y = H^{-1}(u) \qquad y = \begin{cases} \sqrt{2u} \\ -\sqrt{2u} \end{cases}$$

$$y(x) = \begin{cases} +\sqrt{2(F(x)+C)} = \sqrt{x^2+2C} \\ -\sqrt{2(F(x)+C)} = -\sqrt{x^2+2C} \end{cases}$$