The simplest ODE

$$\boxed{\frac{dy}{dx} = f(x)} \qquad (1)$$

Eq. (1) has solution

$$y(x) = \int f(x)\,dx + C$$ General solution

<u>Example</u> $\frac{dy}{dx} = x$ & I.C. $y(1)=0$

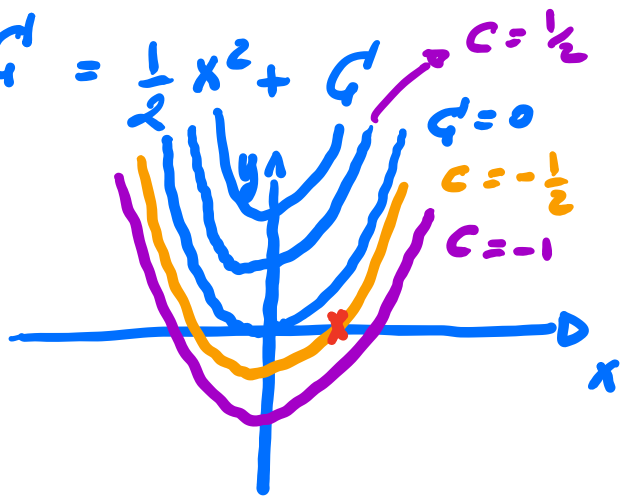
General solution

$$y(x) = \int x\,dx + C = \frac{1}{2}x^2 + C$$

I.C. $y(1) = \underline{0}$

$y(x_0) = \underline{y_0}$

$y_0 = 0 \quad x_0 = 1$



$y(x) = \frac{1}{2} x^2 + C_1$. general solution

Impose the I.C $\Rightarrow$ find C_1

$0 = y(1) = \frac{1}{2} \cdot 1^2 + C$

$\frac{1}{2} + C_1 = 0$

$\boxed{C = -\frac{1}{2}}$

$$\boxed{y(x) = \frac{x^2}{2} - \frac{1}{2}}$$

Solution to the ODE + I. C.

The general solution is given in terms of an arbitrary constant
The constant can be fixed by imposing the I.C.

1^{st} - order separable ODE

A 1^{st} order ODE in normal form

$$y' = \tilde{f}(x,y)$$

is separable if $\tilde{f}(x,y) = f(x)\,g(y)$

$$\boxed{\begin{array}{c} y' = f(x)\,g(y) \\ \text{or equivalently} \\ \dfrac{dy}{dx} = f(x)\,g(y) \end{array}}$$

Separable 1^{st} order ODE

(2)

Examples

$$y' = e^x \cdot y$$

$f(x) = e^x$

$g(y) = y$

Yes!

$$y' = \frac{e^x y}{y+1} = e^x \frac{y}{1+y}$$

$f(x) = e^x$

$g(y) = \frac{y}{1+y}$

Yes!

$$y' = e^x = e^x \cdot 1$$

$f(x) = e^x$

$g(y) = 1$

Yes!

$$y' = e^{x+3y} = e^x \cdot e^{3y}$$

$f(x) = e^x$

$g(y) = e^{3y}$

Yes!

$$y' = (x + 3y)^{2/3}$$

No!

$$y' = (x + 3y)$$

No!

Constant solutions

Given a 1^{st}-order separable ODE

$$\frac{dy}{dx} = f(x)\,g(y) \qquad (2)$$

Let $y_0, y_1, \dots y_k$ be the roots of $g(y)$

therefore $g(y_r) = 0 \quad \forall r \in \{1,2..k\}$

Then the constant function $\bar{y}(x) = y_r$ is a solution of Eq. (2)

Example $g(y) = (y-1)(y-2)$, $f(x) = 1$

$$\frac{dy}{dx} = (y-1)(y-2) \qquad *$$

Roots $g(y)$ $\quad y_1 = 1$, $y_2 = 2$

$\Rightarrow \quad \bar{y}(x) = 1 \quad$ and $\quad \bar{y}(x) = 2$

are solutions of *

Proof Given that $\bar{y}(x) = y_r$ is a constant function.

$$\frac{d\bar{y}}{dx} = 0 = \underbrace{g(\bar{y})}_{=0} f(x) = 0$$

Eq (2) is an identity for $\bar{y}(x) = y_r$

$\Rightarrow \bar{y}(x)$ is a solution to the ODE

$$\frac{dy}{dx} = f(x)\, g(y)$$

Solving 1^{st}- order separable ODEs

Proceed formally

$$\boxed{\frac{dy}{dx} = f(x)g(y)}$$

Step 1 Identify $f(x)$ $g(y)$

Separate the variable

$$\frac{dy}{g(y)} = f(x)\,dx \qquad g(y) \neq 0$$

Step 2 Integrate both sides

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$$

$H(y)$ the antiderivative of $\frac{1}{g(y)}$

$F(x)$ the antiderivative of $f(x)$

$$\boxed{H(y) = F(x) + C}$$ Implicit solution

Step 3 If the inverse of $H(y)=u$ has an explicit expression/s $y = H^{-1}(u)$

$$\boxed{y(x) = H^{-1}\left(F(x) + C \right)}$$ Explicit solution (3)

Notice that there might be more than one inverse function $H^{-1}(u)$

You need to list all solutions of type (3)

Not always $H(y)$ can be inverted in that case you can only provide the implicit solution

Example $y' = x y^2$

Step 1 $f(x) = x \quad , \quad g(y) = y^2$

$$\frac{dy}{dx} = x y^2 \Rightarrow \boxed{\frac{dy}{y^2} = x\, dx}$$

Step 2 $$\int \frac{dy}{y^2} = \int x\, dx + C$$

LHS: $H(y) = \int \frac{dy}{y^2} = \int y^{-2}\, dy = -y^{-1} = -\frac{1}{y}$

RHS: $F(x) = \int x\, dx = \frac{x^2}{2}$

$$\boxed{H(y) = F(x) + C}$$ Implicit solution

$$\boxed{-\frac{1}{y} = \frac{x^2}{2} + C}$$

Step 3

$$H(y) = -\frac{1}{y}$$

$$H(y) = u \qquad y = H^{-1}(u)$$

$$H(y) = -\frac{1}{y} = u$$

$$y = -\frac{1}{u} = H^{-1}(u)$$

$$\boxed{y = H^{-1}(F(x) + C)}$$

$$\boxed{y = -\frac{1}{\frac{x^2}{2} + C_1}}$$

Explicit solution