MTH5123 Differential Equations

Revision calculus and algebra questions – Solutions G. Bianconi

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Problem 1

Compute the derivative $f'(x)$ of the following functions

a) $f(x) = (x-1)(x^2+1), \quad \Rightarrow \quad f'(x) = (x^2+1) + 2x(x-1) = 3x^2 - 2x + 1$

b) $f(x) = 1/(1-x^2), \quad \Rightarrow \quad f'(x) = -1/(1-x^2)^2\,(-2x) = \frac{2x}{(1-x^2)^2}$

c) $f(x) = x/(1+x^2), \quad \Rightarrow \quad f'(x) = \frac{1}{1+x^2} - x\frac{2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

d) $f(x) = (1-x^4)/(1+x^2), \quad (1-x^4) = (1-x^2)(1+x^2) \Rightarrow \quad f(x) = 1-x^2, \quad f'(x) = -2x$

e) $f(x) = xe^{-x}, \quad \Rightarrow \quad f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

f) $f(x) = x\sin x, \quad \Rightarrow \quad f'(x) = \sin x + x\cos x$

g) $f(x) = x\cos(x^2), \quad \Rightarrow \quad f'(x) = \cos(x^2) - x\cdot 2x\cdot\sin(x^2)$

h) $f(x) = x\ln|x|, \quad \Rightarrow \quad f'(x) = \ln|x| + x\cdot\frac{1}{x} = \ln|x| + 1$

i) $f(x) = 1/(x\ln|x|), \quad \Rightarrow \quad f'(x) = -\frac{1}{(x\ln|x|)^2}(\ln|x|+1)$

j) $f(x) = \sqrt{1+\sin^2 x}, \quad \Rightarrow \quad f'(x) = \frac{\sin x\,\cos x}{\sqrt{1+\sin^2 x}}$

k) $f(x) = \arctan\frac{2x}{1-x^2}, \quad \Rightarrow \quad f'(x) = \frac{1}{1+(2x/(1-x^2))^2}\frac{2(1-x^2)-2x(-2x)}{(1-x^2)^2} = \cdots = \frac{2}{(1+x^2)}$

l) $f(x) = \ln\tan\frac{x}{2}, \quad \Rightarrow \quad f'(x) = \frac{1}{\tan\frac{x}{2}}\frac{1}{2}\sec^2\frac{x}{2} = \cdots = \csc x$

m) $f(x) = \sinh x\,\cosh x, \quad \Rightarrow \quad f'(x) = \cosh^2 x + \sinh^2 x$

n) $f(x) = (\cos x + \sin x\tan x)\cos x = \cos^2 x + \sin x\frac{\sin x}{\cos x}\cos x = 1, \quad \Rightarrow \quad f'(x) = 0$

Problem 2

Compute the indefinite integral $\int f(x)\,dx$ of the following functions:

a) $f(x) = (x-1)(x^2+1), \quad \Rightarrow \quad \int f(x)dx = \int(x^3-x^2+x-1)\,dx = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + C$

b) $f(x) = 1/(1-x^2) = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) \quad \Rightarrow \quad \int f(x)dx = \frac{1}{2}(-\ln|1-x| + \ln|1+x|) + C$

c) $f(x) = x/(1+x^2), \quad \Rightarrow \quad \int f(x)dx = \frac{1}{2}\int\frac{d(1+x^2)}{1+x^2} = \frac{1}{2}\ln(1+x^2) + C$

d) $f(x) = (1-x^4)/(1+x^2), \quad \Rightarrow \quad \int f(x)dx = \int(1-x^2)\,dx = x - \frac{x^3}{3} + C$

e) $f(x) = xe^{-x}, \quad \Rightarrow \quad \int f(x)dx = -\int xd(e^{-x}) = -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} + C$

f) $f(x) = x\sin x, \Rightarrow \int xd(-\cos x) = x(-\cos x) - \int 1(-\cos x)\, dx = -x\cos x + \sin x + C$

g) $f(x) = x\cos(x^2), \quad \Rightarrow \quad \int f(x)dx = \frac{1}{2}\int d\left(\sin\left(x^2\right)\right) = \frac{1}{2}\sin\left(x^2\right) + C$

h) $f(x) = x\ln|x|, \quad \Rightarrow \quad \int f(x)dx = \frac{1}{2}\int \ln|x|\, d(x^2)$

$$= \frac{1}{2}x^2\ln|x| - \frac{1}{2}\int x^2\cdot\frac{1}{x}\, dx = \frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$$

i) $f(x) = 1/(x\ln|x|), \quad \Rightarrow \quad \int f(x)dx = \int \frac{d(\ln|x|)}{\ln|x|} = \ln|\ln|x|| + C$

m) $f(x) = \sinh x\, \cosh x, \quad \Rightarrow \quad \int f(x)dx = \int \sinh x\; d\left(\sinh x\right) = \frac{1}{2}\sinh^2 x + C$

n) $f(x) = (\cos(x) + \sin(x)\tan(x))\cos(x) = 1, \quad \Rightarrow \quad \int f(x)dx = \int dx = x + C$

Problem 3

Compute the zeros, maxima/minima, and limit when $x \to \infty$ of the following functions.

a) $f(x) = 2e^{-2x} - e^{-x} = e^{-2x}\left(2 - e^{x}\right)$. Zeroes are at $e^x = 2$, that is $x = \ln 2$. Derivative: $f'(x) = -4e^{-2x} + e^{-x} = e^{-2x}\left(-4 + e^x\right)$. Max or min are at $f'(x) = 0$, which gives $-4 + e^x = 1$, hence $e^x = 4$, $x = \ln 4 = 2\ln 2$. It is easy to see that $f'(x) < 0$ for $x < 2\ln 2$, and becomes positive for $x > 2\ln 2$, hence the function has the minimum at $x = 2\ln 2$. Finally, $f(x) \to 0$ for $x \to \infty$.

b) $f(x) = e^{-x}(\cos(x) + \sin(x))$. Zeroes are at $\cos(x) = -\sin(x)$, that is $x = -\frac{\pi}{4} + n\pi$. The derivative is $f'(x) = -e^{-x}(\cos(x) + \sin(x)) + e^{-x}(-\sin(x) + \cos(x)) = -2e^{-x}\sin x$. Minima/maxima are at $x = n\pi = 0, \pm\pi, \pm 2\pi, \ldots$. Finally, $f(x) \to 0$ for $x \to \infty$ as $e^{-x} \to 0$ and $\sin x, \cos x$ are *bounded*: $|\sin x| \leq 1$, $|\cos x| \leq 1$.

Problem 4

Describe and/or sketch the curves representing solutions to the algebraic equations:

$$\textbf{a)}\; x^2 + y^2 = 4, \qquad \textbf{b)}\; (x-1)y = 2, \qquad \textbf{c)}\; y^2 + 3x = 0$$

a) circle of radius 2 centered at the origin

b) $y = 2/(x-1)$ is a hyperbolic curve, with y tending to $-\infty$ when $x \to 1^-$ (from the left), and y tending to ∞ when $x \to 1^+$ (from the right).

c) $y = \pm\sqrt{-3x}$ for $x < 0$ is a parabola with its branches looking to the left and symmetric with respect to the negative x semi-axis.

Problem 5

Rewrite the system of equations in matrix form and solve by applying matrix inversion

$$\textbf{a)}\, 2x + 3y = 5, \quad 3x + 2y = 5 \qquad \textbf{b)}\, c_1 + \frac{1}{2}c_2 = 1, \quad 2c_1 + 3c_2 = 2$$

a) The system in the matrix form is $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ which is solved as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{(4-9)}\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

hence $x = 1, y = 1$.

b) The system in the matrix form is $\begin{pmatrix} 1 & 1/2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which is solved as

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{(3-1)}\begin{pmatrix} 3 & -1/2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

hence $c_1 = 1$, $c_2 = 0$.

Problem 6

The follow exercise requires understanding complex numbers.

a) Find all roots of the polynomial $x^8 - 1$.

b) Describe the nature (real vs. complex, distinct vs. coincident) of the solutions of the equation $x^2(1-x) = x + \lambda x^2$, as a function of the real parameter λ. *Hint: The values $\lambda = -1, 3$ are the interesting ones.*

(a) Assume x is a complex number, written in polar form as $x = re^{i\theta}$. Then $x^8 - 1 = 0$ can be rewritten as $r^8 e^{8i\theta} = 1 = e^{2k\pi i}$, for $k \in \mathbb{Z}$, which is solved by $r = 1$ and $\theta = \frac{k\pi}{4}$, for $k = 0, 1, \ldots, 7$ *(Why?)*. We also list these 8 points in Cartesian form

$$e^0 = 1, \quad e^{1\pi i/4} = \frac{1}{\sqrt{2}}(1+i), \quad e^{\pi i/2} = i, \quad e^{3\pi i/4} = \frac{1}{\sqrt{2}}(-1+i),$$

$$e^{\pi i} = -1, \quad e^{5\pi i/4} == \frac{1}{\sqrt{2}}(-1-i), \quad e^{3\pi i/2} = -i, \quad e^{7\pi i/4} == \frac{1}{\sqrt{2}}(1-i).$$

(b) Rearranging the equation and factoring, we obtain

$$x\left[x^2 - (\lambda-1)x + 1\right] = 0 \Longleftrightarrow x = 0, \frac{1}{2}\left[(\lambda-1) \pm \sqrt{(\lambda-1)^2 - 4}\right].$$

We notice, then, that for all choices of $\lambda \in \mathbb{R}$, this cubic equation always has the real root $x = 0$, however, the other two roots are determined by the value of the discriminant, $\Delta := (\lambda - 1)^2 - 4$. *Case 1:* If $\Delta = 0$, then there is one real, repeated (coincident) root at $\frac{1}{2}(1 + \lambda)$. But this occurs precisely when $\lambda = -1$ or $\lambda = 3$, in which case, the roots are $x = 0, -1, -1$ or $x = 0, 1, 1$, respectively. *Case 2:* If $\Delta > 0$, then there are two distinct real roots to the quadratic equation and the roots of the polynomial are given by $x = 0, \frac{1}{2}\left[(\lambda - 1) + \sqrt{(\lambda-1)^2 - 4}\right], \frac{1}{2}\left[(\lambda - 1) - \sqrt{(\lambda-1)^2 - 4}\right]$. *Case 3:* If $\Delta < 0$, then there are two distinct complex roots for the quadratic equation and the roots of the polynomial are again, given by the formula as in Case 2, however the complex roots are conjugates of each other.

Problem 7

Study the function $|\csc(x) + \cot(x)|$, and its logarithm.

Notes: First, start by examining $f(x) = \csc(x) + \cot(x) = \frac{1+\cos x}{\sin x}$, in particular, notice that the domain is given by all real x such that $x \neq k\pi$, for $k \in \mathbb{Z}$. On the other hand, we also see that when $x = k\pi$, for k odd, the numerator of $f(x)$ vanishes, so we need to examine the (left and right) limits approaching these values carefully. *You may need to use L'Hopital's rule).* Once we have established the behaviour of $f(x)$, the original function $\log |f(x)|$ can be studied.

Problem 8

Consider the function $f(x) = \ln \frac{x^2}{x-1}$. Determine the domain of f. Show that f is not invertible, and find the two branches of f^{-1}.

Observe that the domain of $\ln x$ is $x > 0$, Thus, we require $\frac{x^2}{x-1} > 0$, which means $x > 1$, since the numerator is nonnegative for all x. Next, note that a formal calculation to solve for a possible inverse function shows

$$y = \ln \frac{x^2}{x-1} \Rightarrow x^2 - e^y x + e^y = 0,$$

which shows that there are two branches (when we solve for x using the quadratic formula). Hence the function is not invertible until we specify a branch. *(Fill in the details here!)*