MTH5123 Differential Equations

Lecture Notes

Week 5

School of Mathematical Sciences
Queen Mary University of London

Autumn 2021

2.2.2.2 Euler type equation

Note:
Certain second-order linear ODEs with *nonconstant* coefficients can be reduced to corresponding ODEs with constant coefficients by special substitutions. Consider, for example, the *Euler type* equation

$$ax^2y'' + bxy' + cy = 0\,,\ x > 0,\ a, b, c = \text{const.}\,. \tag{2.28}$$

This equation can be reduced to one with constant coefficients by introducing the new variable $x = e^t$ so that $y(x) = y[x(t)] = z(t)$. By the chain rule we have

$$\dot{z} = \frac{dz}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t \ \Rightarrow\ y' = e^{-t}\dot{z}\,. \tag{2.29}$$

Differentiating z another time yields

$$\ddot{z} = \frac{d}{dt}\dot{z} = \frac{d}{dt}\left[\frac{dy}{dx}e^t\right] = \left(\frac{d}{dt}\frac{dy}{dx}\right)e^t + \frac{dy}{dx}e^t = \frac{d^2y}{dx^2}\frac{dx}{dt}e^t + \frac{dy}{dx}e^t = \frac{d^2y}{dx^2}e^{2t} + \frac{dy}{dx}e^t\,.$$

Solving this equation for $y'' = \frac{d^2y}{dx^2}$ gives

$$y'' = e^{-2t}\left(\ddot{z} - \dot{z}\right)\,. \tag{2.30}$$

Substituting (2.29)and (2.30) into (2.28) the latter is reduced to the equation with constant coefficients

$$a\ddot{z} + (b - a)\dot{z} + cz = 0\,, \tag{2.31}$$

which can be solved for $z(t)$ by the standard method given above. One then recovers the original solution $y(x)$ by $y(x) = z(t)|_{t=\ln x}$.

2.2.2.3 Linear inhomogeneous 2nd-order ODEs with constant coefficients and $f(x) \neq 0$

The general solution of the inhomogeneous equation (2.17) with $f(x) \neq 0$, viz. (2.14), can be recovered from the general solution of the corresponding homogeneous equation with $f(x) = 0$, viz. (2.13), by extending the *variation of parameter* method that we already applied successfully to solving first-order linear inhomogeneous equations.
As an example, we consider **second-order** ODE's of the form (2.17)

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = f(x)\,. \tag{2.32}$$

First: find $y_h(x)$ to the corresponding homogenous ODE

The characteristic equation corresponding to (2.32) is given by $M_2(\lambda) = a_2\lambda^2 + a_1\lambda + a_0 = 0$. It has the two roots

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2}\,,\ \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_0}}{2a_2}\,, \tag{2.33}$$

which are real and distinct as long as $a_1^2 - 4a_2a_0 > 0$. Considering for simplicity only this case, we have learned that we can write the general solution to the *homogeneous* equation as

$$y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, \tag{2.34}$$

where c_1 and c_2 are two real constants.

Second: Find $y_p(x)$ based on The variation of parameter method

According to the variation of paramter method we will look for a solution of the full *inhomogeneous* equation (2.32) in the form of

$$y(x) = c_1(x) e^{\lambda_1 x} + c_2(x) e^{\lambda_2 x} . \tag{2.35}$$

We have to show that this ansatz works, and if so, whether it will yield a *particular* or possibly even the *general* solution. Differentiating (2.35) yields

$$\frac{dy}{dx} = c_1(x)\lambda_1 e^{\lambda_1 x} + c_2(x)\lambda_2 e^{\lambda_2 x} + c_1'(x) e^{\lambda_1 x} + c_2'(x) e^{\lambda_2 x} \tag{2.36}$$

with $c_{1,2}' \equiv \frac{dc_{1,2}}{dx}$. To simplify this expression before we proceed further, we impose the additional condition

$$c_1'(x) e^{\lambda_1 x} + c_2'(x) e^{\lambda_2 x} = 0 \tag{2.37}$$

on the two functions $c_{1,2}(x)$. This implies that

$$\frac{dy}{dx} = c_1(x)\lambda_1 e^{\lambda_1 x} + c_2(x)\lambda_2 e^{\lambda_2 x} , \tag{2.38}$$

which facilitates the following second differentiation

$$\frac{d^2y}{dx^2} = c_1(x)\lambda_1^2 e^{\lambda_1 x} + c_2(x)\lambda_2^2 e^{\lambda_2 x} + c_1'(x)\lambda_1 e^{\lambda_1 x} + c_2'(x)\lambda_2 e^{\lambda_2 x} . \tag{2.39}$$

Now we substitute (2.35), (2.38) and (2.39) into the left-hand side of (2.32) giving

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = c_1(x) e^{\lambda_1 x} \left(a_2\lambda_1^2 + a_1\lambda_1 + a_0\right) \tag{2.40}$$

$$+ c_2(x) e^{\lambda_2 x} \left(a_2\lambda_2^2 + a_1\lambda_2 + a_0\right) + a_2 \left(c_1'(x)\lambda_1 e^{\lambda_1 x} + c_2'(x)\lambda_2 e^{\lambda_2 x}\right) .$$

Remembering that both λ_1 and λ_2 are roots of the characteristic equation, that is $a_2\lambda_1^2 + a_1\lambda_1 + a_0 = 0$ and $a_2\lambda_2^2 + a_1\lambda_2 + a_0 = 0$ we see that (2.32) and (2.40) together imply the relation

$$c_1'(x)\lambda_1 e^{\lambda_1 x} + c_2'(x)\lambda_2 e^{\lambda_2 x} = f(x)/a_2 . \tag{2.41}$$

Now we compare (2.37) and (2.41). Multiplying (2.37) with the factor $-\lambda_2$ and adding to (2.41) gives

$$c_1'(x) e^{\lambda_1 x}(\lambda_1 - \lambda_2) = f(x)/a_2 , \tag{2.42}$$

which allows us to find $c_1(x)$ by straightforward integration,

$$c_1(x) = \frac{1}{(\lambda_1 - \lambda_2)a_2} \left(\int f(x) e^{-\lambda_1 x} dx + C_1 \right) , \tag{2.43}$$

where C_1 is a real constant. Similarly, multiplying (2.37) with the factor λ_1 and subtracting from (2.41) gives

$$c_2'(x)e^{\lambda_2 x}(\lambda_2 - \lambda_1) = f(x)/a_2 \,. \tag{2.44}$$

Hence

$$c_2(x) = -\frac{1}{(\lambda_1 - \lambda_2)a_2}\left(\int f(x)e^{-\lambda_2 x}dx + C_2\right) \,, \tag{2.45}$$

where C_2 is a real constant. Collecting everything together we find that *a solution* to the *inhomogeneous* equation (2.32) is given by

$$y(x) = \frac{1}{(\lambda_1 - \lambda_2)a_2}\left\{e^{\lambda_1 x}\left(\int f(x)e^{-\lambda_1 x}dx + C_1\right) - e^{\lambda_2 x}\left(\int f(x)e^{-\lambda_2 x}dx + C_2\right)\right\} \,. \tag{2.46}$$

Putting $f(x) = 0$ in (2.46) and introducing the notation

$$\frac{C_1}{\lambda_1 - \lambda_2} = c_1 \,, \; -\frac{C_2}{\lambda_1 - \lambda_2} = c_2$$

we see that the solution (2.46) reduces to

$$y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \,, \tag{2.47}$$

which is the *general* solution of the corresponding *homogeneous* equation. Accordingly, our corresponding solution $y(x)$ of the *inhomogeneous* equation can be written as $y_g(x) = y_h(x) + y_p(x)$, where

$$y_p(x) = \frac{1}{(\lambda_1 - \lambda_2)a_2}\left\{e^{\lambda_1 x}\int f(x)e^{-\lambda_1 x}dx - e^{\lambda_2 x}\int f(x)e^{-\lambda_2 x}dx\right\} \tag{2.48}$$

is a *particular* solution of the *inhomogeneous* equation. Hence, the variation of parameter method did indeed yield the general solution $y_g(x) = y(x)$ of (2.32).

Note:

1. Although the solution (2.46) (or equivalently the pair (2.47), (2.48)) was formally derived for *real* $\lambda_1 \neq \lambda_2$ it retains its validity for *complex conjugate* roots $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\beta \neq 0$ as long as one uses complex coefficients c_1, c_2. To bring the solution onto a real form one uses Euler's formula (2.27). One can even use (2.46) in the limit $\lambda_1 \to \lambda_2$ (that is, $\beta \to 0$) by using L'Hopital's rule, as will be demonstrated later on by an example.

2. Although the variation of parameter method is of general validity for arbitrary $f(x)$, its implementation relies on our ability to perform the integrals $\int f(x)e^{-\lambda_2 x}dx$ explicitly. In practical terms, for finding explicit forms of the solution it is sometimes easier to guess a *particular* solution $y_p(x)$ of the inhomogeneous equation and then to combine it with the *general solution* $y_h(x)$ of the corresponding homogeneous equation into the general solution $y_g(x) = y_h(x) + y_p(x)$ of the inhomogeneous one according to our theory.

2.2.2.4 Educated guess method for linear inhomogeneous 2nd-order ODEs with constant coefficients and $f(x) = P(x)e^{ax}$

If the right-hand side has the form $f(x) = P(x)e^{ax}$, where $P(x)$ is a polynomial of degree k, and $a \neq \lambda_1, a \neq \lambda_2$ (which means that e^{ax} is *not* a solution of the *homogeneous* equation), then a particular solution can *always* be found in the form $y_p(x) = Q(x)e^{ax}$ with some polynomial $Q(x) = d_k x^k + \ldots + d_1 x + d_0$ of the *same* degree. We may refer to such a method of finding particular solutions as the **educated guess** method.

Example:
Find a particular solution of the ODE

$$y'' + 2y' - 3y = xe^{2x} .$$

Solution: Here $a = 2$ and $P(x) = x$ is of first degree. First we need to check that e^{2x} is not a solution of $y'' + 2y' - 3y = 0$, which is indeed the case. Then we look for a solution in the form $y_p(x) = (d_1 x + d_0)e^{2x}$. Differentiating gives

$$y_p' = e^{2x}(2d_0 + d_1 + 2d_1 x) \, , \; y_p'' = e^{2x}(4d_0 + 4d_1 + 4d_1 x) \, ,$$

which by substitution into the left-hand side of the inhomogeneous equation and collecting similar terms yields

$$y_p'' + 2y_p' - 3y_p = e^{2x}(5d_0 + 6d_1 + 5d_1 x) \, .$$

Matching the coefficients to the right-hand side xe^{2x} we find $d_1 = 1/5$ and $d_0 = -\frac{6d_1}{5} = -6/25$. Thus a particular solution to the given ODE is

$$y_p(x) = \left(\frac{1}{5}x - \frac{6}{25} \right) e^{2x} \, .$$

Another version of the *educated guess* method exits in the case of two complex conjugate roots $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$. Here, if the right-hand side has the form $f(x) = P(x)\cos(ax)$ or $f(x) = P(x)\sin(ax)$, where $P(x)$ is a polynomial of degree k, and $ia \neq \lambda_1, ia \neq \lambda_2$ (which means that $e^{iax} = \cos(ax) + i\sin(ax)$ is not a solution of the homogeneous equation), then such a particular solution can always be found in the form

$$y_p(x) = Q(x)(A\cos(ax) + B\sin(ax)) \tag{2.49}$$

with some coefficients A, B and some polynomial $Q(x) = d_k x^k + \ldots + d_1 x + 1$ (note that the last coefficient of the polynomial can be chosen to be equal to one).