Q1 Write in terms of the density functions, what it means for distributions of $B(1/2)$ and $(B(1/2), B(3/2))$ to be consistent.

Literally this means

$$\frac{1}{\sqrt{2\pi\cdot\sqrt{\frac{1}{2}}}}\exp\left(\frac{-x^2}{2\cdot\frac{1}{2}}\right)=\frac{1}{\sqrt{2\pi\cdot\sqrt{\frac{1}{2}}}}\exp\left(\frac{-x^2}{2\cdot\frac{1}{2}}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sqrt{\frac{3}{2}-\frac{1}{2}}}}\exp\left(\frac{-(y-x)^2}{2\cdot\left(\frac{3}{2}-\frac{1}{2}\right)}\right)\mathrm{d}y,$$

which can be simplified.

Q2 Show that the process $(X^k(t)-t,\ t\geq 0)$ is a martingale for (a) $k=1$ and X the Poisson process with unit rate, (b) $k=2$ and X the BM.

For $t>s\geq 0$ write $X(t)-t=(X(s)-s)+[(X(t)-X(s)-(t-s)]$. Take the conditional expectation given $X(s)-s$; for the Poisson process the right-hand side becomes $X(s)-s$. The conditioning on $\mathcal{F}_s$ (natural filtration) gives the same, since the process is Markovian.

In the Brownian case use the decomposition $B(t)=B(s)+[B(t)-B(s)]$, where the terms are independent.

Q4 For constants $\mu,\sigma>0$ the process $B_{\mu,\sigma}(t):=\mu t+\sigma B(t)$ is 'a Brownian motion with drift μ and diffusion/volatility σ'. Show that $B_{\mu,\sigma}$ is Gaussian, find its mean and covariance functions. Also find the quadratic variation of the process on $[0,t]$.

Let $X(t)=B_{\mu,\sigma}(t)$ for shorthand. It is enough to show that

$$\sum_{j=1}^{k}\alpha_i X(t_i)$$

has one-dimensional normal distribution for any choice of times and coefficients. But this follows from the analogous property of the BM. We have from the properties of the BM

$$\mathbb{E}X(t)=\mu t,\quad \mathrm{Var}X(t)=\sigma^2\min(s,t)$$

and $\langle X\rangle(t)=\sigma^2 t$.

Q5 The process $B^\circ(t):=B(t)-tB(1),\ \ t\in[0,1]$, is known as the Brownian bridge. Find the covariance function of B°. Is this process Gaussian? Markov? Martingale? Explain your answers.

The covariance function is, for $s<t$,

$$\mathrm{Cov}(B^\circ(s),B^\circ(t))=s(1-t),$$

as follows from the Brownian covariance function. The process is Gaussian, as in Q4.

The Markov property can be derived from the joint density formula for $B^\circ(t_i), i=1,\cdots,k-1$:

$$\frac{1}{p(1,1)}\prod_{i=1}^{k}p(t_i-t_{i-1},x_i-x_{i-1}),$$

where $0=t_0<t_1<\cdots<t_{k-1}<t_k=1, x_0=0, x_k=1$ and $p(t,x)$ is the $\mathcal{N}(0,t)$ density.

Alternatively, note that solution to the the stochastic differential equation

$$dX(t) = \frac{X(t)}{1-t}dt + dB(t), \quad X(0) = 0,$$

is a Gaussian process with mean zero and the covariance as above. Hence it is a Brownian bridge. Hence this process is Markovian, and is not a martingale.

Q6 Prove that $B(t)$ has no limit as $t \to \infty$ almost surely. Hint: it is enough to show that $B(n) - B(n-1)$ is not a Cauchy sequence, $n \in \mathbb{N}$.

$p := \mathbb{P}(|B(n) - B(n-1)| > 1) > 0$ implies that the independent events $|B(n) - B(n-1)| > 1$ occur infinitely often. Hence the sequence $B(n)$ does not converge,

Q7 Show that $\limsup_{t\to\infty} B(t)/\sqrt{t} = \infty$. [Hint: use Kolmogorov's 0-1 law.]

Let (t_n) be any positive increasing sequence with $t_n \to \infty$. The random variables $X_n = B(t_n) - B(t_{n-1})$ (where $t_0 = 0$) are independent, and $B(t_n) = X_1 + \cdots + X_n$. Write

$$\frac{B(t_n)}{\sqrt{t_n}} = \frac{X_1 + \cdots + X_m}{\sqrt{t_n}} + \frac{X_{m+1} + \cdots + X_n}{\sqrt{t_n}}.$$

As $n \to \infty$ the first term vanishes, hence $Y := \limsup_{n\to\infty} B(t_n)/\sqrt{t_n}$ is measurable with respect to the tail σ-algebra $\mathcal{T}$ of the sequence $X_1, X_2, \cdots$. This σ-algebra $\mathcal{T}$ has only events of probability 0 and 1 by Kolmogorov's 0-1 law., therefore $\mathbb{P}(Y = \infty)$ should be either 0 or 1. Now, $\{Y \geq M\} \in \mathcal{T}$ for any integer $M > 0$,

$$\{Y \geq M\} \supset \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{B(t_m) \geq M\sqrt{t_m}\},$$

and thus, using that $B(t)$ has $\mathcal{N}(0, t)$ distribution,

$$\mathbb{P}(Y \geq M) \geq \mathbb{P}\left(\bigcup_{m=n}^{\infty} \{B(t_m) \geq M\sqrt{t_m}\}\right) \geq \mathbb{P}(B(t_n) \geq M\sqrt{t_n}\}) = 1 - \Phi(M) > 0.$$

Since this probability is not 0, we conclude that $\mathbb{P}(Y \geq M) = 1$. Letting $M \to \infty$ gives

$$\mathbb{P}(Y = \infty) = \mathbb{P}\left(\bigcap_{M=1}^{\infty} \{Y \geq M\}\right) = 1.$$