Q1 Suppose $X_n \stackrel{d}{\to} c$ for constant c. Show that $X_n \stackrel{\mathbb{P}}{\to} c$. Hint: use functions

$$f_\epsilon(x) = (1 - |x - c|\epsilon^{-1})_+$$

to estimate $\mathbb{P}(|X_n - c| \le \epsilon)$ from below.

The convergence in distribution $|X_n - c| \stackrel{d}{\to} 0$ implies $\mathbb{E}(1-|X_n-c|\epsilon^{-1})_+ \to f_\epsilon(c) = 1$ because f_ϵ is a continuous function. But since $f_\epsilon(x) > 0$ only for $|x - c| < \epsilon$ we get

$$\mathbb{E}(1 - |X_n - c|\epsilon^{-1})_+ \ge \mathbb{P}(|X_n - c| \le \epsilon),$$

hence this probability approaches 1 as $n \to \infty$.

Q2 Suppose $\sum_{n=1}^{\infty} \mathbb{E}\,|X_n| < \infty$. Using Chebyshev inequality and Borel-Cantelli lemma show that $X_n \stackrel{\text{a.s.}}{\to} 0$.

Recall that for numerical sequence, $x_n \to x$ means that $|x_n - x| > \epsilon$ does not hold for infinitely many n, for every ϵ. It is enough to ensure that this holds for some sequence of ϵ_k converging to 0. For random variables, $X_n \stackrel{\text{a.s.}}{\to} X$ means that $\mathbb{P}(|X_n - X| > \epsilon_k \text{ i.o.}) = 0$ for $\epsilon_k \to 0$.

We have by Chebyshev's inequality and the assumption on convergence of the series

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \ge \epsilon) \le \epsilon^{-1} \sum_{n=1}^{\infty} \mathbb{E}X_n < \infty$$

By the Borel-Cantelli lemma we get

$$\mathbb{P}(|X_n| > \epsilon \;\text{ i.o.}) = \mathbb{P}(\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} |X_n| > \epsilon) = 0.$$

Letting $\epsilon = 1/k$ we get

$$\mathbb{P}(\cup_{k=1}^{\infty}\{|X_n| > 1/k \;\text{ i.o.}\}) = 0,$$

which is the same as the convergence $X_n \stackrel{\text{a.s.}}{\to} 0$.

Q3 Suppose that $X_n \stackrel{\mathbb{P}}{\to} X$. Show that there exists subsequence (n_k) such that $X_{n_k} \stackrel{\text{a.s.}}{\to} X$. By the convergence in probability, it is possible to choose, using induction, $n_1 < n_2 < \cdots$ so large that

$$\mathbb{P}(|X_{n_k} - X| > k^{-2}) < k^{-2}, \quad k = 1, 2, \cdots$$

Since

$$\sum_{k=1}^{\infty} \mathbb{P}(|X_{n_k} - X| > k^{-2}) < \infty,$$

the Borel-Cantelli lemma ensures that $\mathbb{P}(|X_{n_k} - X| > k^{-2} \text{ i.o.}) = 0$. Then also

$$\mathbb{P}(|X_{n_k} - X| > \epsilon \text{ i.o.}) = 0$$

and so $X_{n_k} \stackrel{\text{a.s.}}{\to} X$.

Q4 Suppose $X_n \stackrel{\mathbb{P}}{\to} Y$ and $X_n \stackrel{\mathbb{P}}{\to} Z$. Prove that $\mathbb{P}(Y \ne Z) = 0$. Using the result of Q3, choose n_k's to achieve $X_{n_k} \stackrel{\text{a.s.}}{\to} Y$, $X_{n_k} \stackrel{\text{a.s.}}{\to} Z$. Then, of course, $Y = Z$ a.s.

Q5 Show that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ holds if and only if $\mathbb{E}[X_{n+1} \cdot 1_A] = \mathbb{E}[X_n \cdot 1_A]$ for every $A \in \mathcal{F}_n$. By definition of the conditional expectation, $\mathbb{E}[Y|\mathcal{F}]$ is a $\mathcal{F}$-measurable random variable such that

$$\mathbb{E}(\mathbb{E}[Y|\mathcal{F}]1_A) = \mathbb{E}[Y1_A]$$

for all $A \in \mathcal{F}$. Thus

$$\mathbb{E}(\mathbb{E}[X_{n+1}|\mathcal{F}_n]1_A) = \mathbb{E}[X_{n+1}1_A] = \mathbb{E}[X_n 1_A].$$

Now use that $Y = Z$ a.s. for $\mathcal{F}$-measurable Y, Z if and only if $\mathbb{E}[Y1_A] = \mathbb{E}[Z1_A]$ for $A \in \mathcal{F}$.

Q6 For martingale (X_n), show that $\mathbb{E}[X_n|\mathcal{F}_m] = X_m$ for $n \geq m$. What are the analogues for sub- and supermartingales?

Use the tower property and induction. The analogues are abvious.

Q7 Let $\xi_1, \xi_2, \ldots$ be r.v.'s with $\mathbb{E}|\xi_j| < \infty$ and $\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = 0$. Show that $X_n = \sum_{k=1}^n \xi_k$ is a martingale (with $X_0 = 0$).

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_n|\mathcal{F}_n] + \mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = X_n$$

by assumptions and because X_n is $\mathcal{F}_n$-measurable.

Q9 Two dice are rolled until a sum of 7 is thrown. Find the expectation of the sum of scores over all rolls.

The probability to roll $X_n = 7$ is $1/6$. For τ the number of rolls, then τ is a stopping time with $\mathbb{E}\,\tau = 6$. The mean score in one roll is $\mathbb{E}\,X_n = 3.5$, hence the expected sum of scores if $6 \cdot 3.5 = 21$ by Wald's identity.

Q10 Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $\mathbb{E}\,\xi_j = 0$, $X_n = \xi_1 + \cdots + \xi_n$, $\tau = \min\{n : X_n \geq 0\}$. Prove that $\mathbb{E}\,\tau = \infty$.

We need to exclude the trivial case $\xi_n = 0$ a.s. (when the claim is false).

Consider $\sigma = \min\{n > 1 : \xi_2 + \cdots + \xi_n \geq x\}$, for some fixed $x > 0$. If $\mathbb{E}\,\sigma < \infty$, Wald's identity gives $\mathbb{E}(\xi_2 + \cdots + \xi_\sigma) = 0$, which is not true as this should be at least x; hence $\mathbb{E}\,\sigma = \infty$.

Now, given $X_1 = -x$, τ coincides with such σ and has conditional expectation ∞. But then $\mathbb{E}\,\tau = \infty$ follows also unconditionally, because $\mathbb{P}(X_1 < 0) > 0$.

Q12 Show that for submartingales $(X_n), (Y_n)$ also $(X_n \wedge Y_n)$ is a submartingale.

Using $\mathbb{E}[X \wedge Y|\mathcal{F}] \leq \mathbb{E}[X|\mathcal{F}]$ and $\mathbb{E}[X \wedge Y|\mathcal{F}] \leq \mathbb{E}[Y|\mathcal{F}]$ yields

$$\mathbb{E}[X_{n+1} \wedge Y_{n+1}|\mathcal{F}_n] \leq \mathbb{E}[X_{n+1}|\mathcal{F}_n] \wedge \mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n \wedge Y_n.$$