LTCC course: Harmonic Analysis

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1 Lecture 1

1.1 Motivating examples

1.1.1 Heat equation

Circular chain of length N. Denote $u_t(j)$ the temperature of j-th piece at time $t \in \mathbb{Z}_+$.

$$\underbrace{u_{t+1}(j) - u_t(j)}_{\text{temperature increment at } j} = \frac{\kappa}{2}\left[(u_t(j+1) - u_t(j)) - (u_t(j-1) - u_t(j)))\right]$$

where $0 < x < 1$.

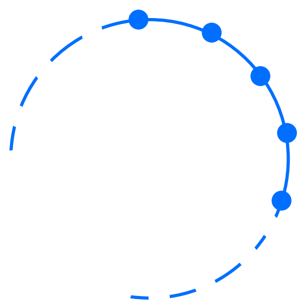

Figure 1: Circular chain of length N.

What happens as t grows; particularly, as $t \to \infty$?

* $\overline{u}_t = \frac{1}{N}\sum_{j=0}^{N-1} u_t(j)$ is preserved, i.e. equal to $\overline{u}_0$

* $u_t \to \overline{u}_0$

*Corrections are most welcome

More quantitative question: set $u_0 = N\delta_0$ and $N \gg 1$, after how much time $u_t \approx 1$? Is it N, $\sqrt{N}$, N^2? The idea: fast and slow fluctuations. Fast fluctuations get smoothed out fast. How to separate u_t into scales, which could be analysed separately?

Main trick is in using $e_p(j) = \exp\left(\frac{2\pi ijp}{N}\right)$.

* "wavelength $\frac{N}{p}$" but with arithmetic nuances

* character property: $e_p(j+k) = e_p(j)e_p(k)$

Exercise 1. Check that there are no other characters.

Claim. *e_p form an orthogonal basis: $\frac{1}{N}\sum_{j=}^{N-1} e_p(j)\overline{e_q(j)} = \delta_{pq}$.*

Expand

$$\begin{aligned} u_t &= \sum \hat{u}_t(p)e_p \\ \hat{u}_t(p) &= \frac{1}{N}\sum u_t(j)\overline{e_p(j)} \\ u_t(j \pm 1) &= \sum \hat{u}_t(p)e_p(\pm 1)e_p(j) \end{aligned}$$

hence,

$$\hat{u}_{t+1}(p) - \hat{u}_t(p) = \frac{x}{2}\left(e_p(1) + e_p(-1) - 2)\hat{u}_t(p)\right).$$

All p are uncoupled, i.e. one equation for each p!

$$\begin{aligned} \hat{u}_{t+1}(p) &= (1 - x(1 - \cos\frac{2\pi p}{N}))\hat{u}_t(p) \qquad (1) \\ \implies \hat{u}_t(p) &= (1 - x(1 - \cos\frac{2\pi p}{N}))^t\hat{u}_0(p). \end{aligned}$$

Analysis:

- constants are proportional to e_0, $\hat{u}_t(0) = \bar{u}_t$, and, indeed, it does not change

- the greater $|p|$ is, the faster $\hat{u}_t(p) \to 0$. The slowest one is $\hat{u}_t(1)$ and $\hat{u}_t(-1)$:

$$\begin{aligned} |\hat{u}_t(1)| &= (1 - x(1 - \cos\frac{2\pi}{N}))^t\hat{u}_0(1) \\ &\sim 1 - \frac{2x\pi^2}{N^2} \\ &\sim \exp\left(-\frac{2x\pi^2}{N^2}\right). \end{aligned}$$

It takes $\lesssim N^2$ steps to converge to $\bar{u}_0$

Exercise 2. Let $u_0 = N\delta_0$, then $\max_j |u_t(j) - 1| = \begin{cases} \leq C \exp\{\frac{-ct}{N^2}\}, & t \geq CN^2 \\ \geq \frac{1}{2}, & t \leq \frac{1}{C}N^2 \end{cases}$

A more realistic version $t \in \mathbb{R}_+$, $x \in \mathbb{T}$ ($\mathbb{R}$? $\mathbb{R}^2$?)

(a) $t \in \mathbb{R}_+$, still on $\mathbb{Z}/n\mathbb{Z}$.

$$\frac{\partial}{\partial t} u_t(j) = \frac{\kappa}{2} \left[(u_t(j+1) - u_t(j)) + u_t(j-1) - u_t(j) \right]$$

Exercise 3. Develop this theory.

(b) x - continual

$$\frac{\partial}{\partial t} u_t(x) = \frac{\kappa}{2} \frac{\partial^2 u_t(x)}{\partial x^2} \quad \text{or} \quad \dot{u}_t = \frac{\kappa}{2} u_t''.$$

$\mathbb{T}$: periodic because $u_t(1) = u_t(0)$ - "circular rod".

$$e_p(x) = \exp(2\pi i p x) \qquad p \in \mathbb{Z}$$
$$\mathrm{L}_2(\mathbb{T}) = \{ f : \mathbb{T} \to \mathbb{C} : \int_0^1 |f(x)|^2 \, \mathrm{d}x < \infty \}$$

Claim. *e_p form an orthonormal basis of* $\mathrm{L}_2(\mathbb{T})$.

Proof. $\langle e_p, \overline{e_p} \rangle = \int_0^1 e_p(x) \overline{e_p(x)} \, \mathrm{d}x = \delta_{pq}$. Completeness: Weierstrass theorem. $\square$

Want: $u_t(x) \stackrel{?}{=} \sum \hat{u}_t(p) e_p(x)$, where $\hat{u}_t(p) = \langle u_t, e_p \rangle$. Ignoring convergence et al.:

$$\begin{aligned} e_p''(x) &= -4\pi^2 p^2 e_p(x) \\ &\implies \dot{\hat{u}}_t(p) = -2\pi^2 p^2 \hat{u}_t(p) \\ &\implies \hat{u}_t(p) = \exp\big(-2\pi^2 p^2 t\big) \hat{u}_t(0) \end{aligned}$$

$$\begin{aligned} u_t(x) &= \sum \exp\big(-2\pi^2 p^2 t\big) e_p(x) \int u_0(x) \overline{e_p(y)} \, \mathrm{d}y \\ &= \int u_0(y) \underbrace{\sum \exp\big(-2\pi^2 t\big) e_p(x-y)}_{P_t(x-y)} \, \mathrm{d}y \\ &= (P_t * u_0)(x) \end{aligned}$$

- For $t > 0$, $\dot{u}_t = \frac{1}{2} u_t''$

- $u_t \underset{t\to\infty}{\rightrightarrows} \bar{u}_0$, $(P_t(y) \overset{?}{\to} 1)$ $\qquad t \to \infty$

- $u_t \underset{t\to+0}{\rightrightarrows} u_0$, $(P_t(y) \overset{??}{\to} \delta(y))$ $\qquad t \to 0$ (initial condition). We prove this item in section 4 (example 17).

Difficulty: $f = \sum \hat{f}(p)e_p$ in L_2, i.e. $\int \left| f(x) - \sum_{|p|\le k} \hat{f}(p)e_p(x) \right|^2 \mathrm{d}x \underset{u\to\infty}{\to} 0$, but not pointwise (or uniformly). We shall see an example:

Exercise 4. Solve the equation $\dot{u_t} = \frac{1}{2}u''$, $x \in [0, \frac{1}{2}]$ with $u_t(0) = u_t(\frac{1}{2}) = 0$. Hint: extend $u_t(-x) = -u_t(x)$, $0 \le x \le \frac{1}{2}$.

Exercise 5. $P_t(x) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{2}\frac{(x-n)^2}{t}\right)$. Hint: Gaussian integral, $\int_{-\infty}^{\infty} \left(-A\xi^2 + iB\xi\right) \mathrm{d}\xi = \sqrt{\frac{\pi}{4}} \exp\left(\frac{-B^2}{4A}\right)$, $A > 0$.

1.1.2 Equidistribution mod 1

$\alpha \in \mathbb{R}/\mathbb{Q}$, $\{k\alpha\}$, $\#\left\{k \in \{1, \dots, K\} : \{k\alpha\} \in I\right\} \overset{?}{=} |I|K + o(K)$?, where I is an arc. (Not hard to prove. What about $\{k^2\alpha\}$?)

Theorem 1 (Weyl)**.** If $P(x)$ is a polynomial of degree ≥ 1 with at least one irrational coefficient, then $\{P(k)\}$ is equidistributed mod 1.

Let denote μ as probability measure on $\mathbb{T}$ (δ_0 for example), μ_t is a shift by t, s.t. $\int f(x)\,\mathrm{d}\mu_t = \int f(x+t)\,\mathrm{d}\mu$.

$$T_{K\mu} = \frac{1}{K}\left(\mu + \mu_\alpha + \mu_{2\alpha} + \cdots + \mu_{(K-1)\alpha}\right),$$

becomes uniform (similarly to the solution of the heat equation), but in a weaker sense.

Reminder (Weak convergence). Set $\hat{\nu}(p) = \int e_{-p}\,\mathrm{d}\nu$ (ν is not in $L_2(\mathbb{T})$, but we should not be dogmatic). $\nu_k \to \nu \iff \forall p \ge 1 \hat{\nu}_k(p) \to \hat{\nu}(p)$.

For the uniform distribution: $\widehat{mes}(p) = 0$ for $p \neq 0$.

$$\widehat{T_{K\mu}}(p) = \frac{1}{K}\sum_{k=0}^{K-1} \exp(-2\pi i k p \alpha) = \frac{1 - \exp(-2\pi i k p \alpha)}{K(1 - \exp(-2\pi i k p \alpha))} \underset{k\to\infty}{\to} 0.$$

This is quantative: depends on the approximability of α by rationals. But how to bound $|T_{K\mu}(I) - |I||$?

Theorem 2 (Erdős–Turán)**.** Let μ be a probability measure on $\mathbb{T}$. Then for any $N \geq 1$.

$$\sup_I |\mu(I) - |I|| \leq C\left\{\frac{1}{N} + \sum_{p=1}^{N} \frac{|\hat{\mu}(p)|}{p}\right\}$$

We prove Erdős–Turán theorem in section 5.

For $\mu_k = T_{K\mu}$: assume $\forall p \in \mathbb{Z}\diagdown\{0\}$, $\|p\alpha\| \stackrel{def}{=} dist(p\alpha, \mathbb{Z}) \geq \frac{\alpha}{|P|^\tau}$, for some $a, \tau > 0$ (for $\alpha = \sqrt{2}$ we can take $\tau = 1$). Naive:

$$\begin{aligned}\sum_{p=1}^{N} \frac{|\hat{\mu}_k(p)|}{p} &\leq \sum_{p=1}^{N} \frac{2}{pK} \frac{1}{2 - 2\cos(2\pi p\alpha)} \\ &\leq \sum_{p=1}^{N} \frac{2}{apK} \frac{\sqrt{2\pi}}{\|p\alpha\|} \leq \frac{2\pi}{aK} \sum_{p=1}^{N} \frac{1}{p^{1-\tau}} \\ &\leq \frac{C_\tau}{aK} N^\tau\end{aligned}$$

Equate, $\frac{1}{N} = \frac{C_\tau}{aK} N^\tau \implies N^{\tau+1} = \frac{aK}{C_\tau} \implies N = \left(\frac{aK}{C_\tau}\right)^{\frac{1}{\tau+1}}$. Thus:

$$\sup_I |\mu_k(I) - |I|| \leq C'_\tau \frac{1}{(aK)^{\tau+1}}.$$

The bound is not very sharp! Ideal is $\frac{1}{K}$, which is inachievable, but we expect $\frac{\log K}{K}$ for $\tau = 1$. Better idea would be to use dyadic chunks. W.l.o.g. we assume that $N = 2^M - 1$.

$$\min_{2^{m-1} \leq p \leq 2^m - 1} \|p\alpha\| \geq \frac{a}{2^{m\tau}}$$

Although, this cannot be achieved for all p! If $\|p\alpha\|$, $\|p'\alpha\| \leq \frac{2^{l-1}a}{2^{m\tau}}$, for some $l \leq m\tau$, $p \neq p'$. Then:

$$\begin{aligned}&\left\|(p - p')\alpha \leq \frac{2^l a}{2^{m\tau}}\right\| \\ &\implies \frac{a}{|p - p'|^\tau} \leq \frac{2^l a}{2^{m\tau}} \\ &\implies |p - p'| \geq 2^{m - l/\tau},\end{aligned}$$

i.e. there are $\le 2^{l/\tau}+1$ such p-s. We get:

$$\begin{aligned}
&C\sum_{m=1}^{M}\left[\sum_{l=1}^{m\tau}2^{l/\tau}\left(\frac{2^l a}{2^{m\tau}}\right)^{-1}+2^m a^{-1}\right]\frac{1}{2^m}\\
&=Ca^{-1}\sum_{m=1}^{M}\left[2^{m(\tau-1)}\sum_{l=1}^{m\tau}2^{l\left(\frac{1}{\tau}-1\right)}\right]\\
&\le C'a^{-1}\sum_{m=1}^{M}\left[1+2^{m(\tau-1)}\begin{cases}m, & \tau=1\\ 1, & \tau>1\end{cases}\right]\\
&\le C''a^{-1}\begin{cases}M^2, & \tau=1\\ 2^{M(\tau-1)}, & \tau>1\end{cases}\\
&\le C'''a^{-1}\begin{cases}\log^2 N\\ N^{\tau-1}\end{cases}
\end{aligned}$$

$$\min_N\left[\frac{a^{-1}}{K}\begin{cases}\log^2 N\\ N^{\tau-1}\end{cases}+\frac{1}{N}\right]\sim\begin{cases}\frac{\log^2 K}{K}, & \tau=1\\ \frac{1}{K^{1/\tau}}, & \tau>1\end{cases}$$ can be improved to the $\frac{\log K}{K}$ sharp!

Conclusion:

$$\sup_I\left||I|-\frac{1}{K}\#\{1\le k\le K:\{k\alpha\}\in I\}\right|\lesssim\begin{cases}\frac{\log^2 K}{K}, & \tau=1\\ K^{-1/\tau}, & \tau>1\end{cases}$$

Exercise 6. For $\epsilon_1,\dots,\epsilon_k$ i.i.d. ±1, show that

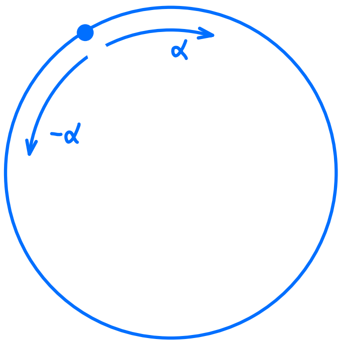

Figure 2: Random steps of length α on the rod.

$$\sup_I\left|P\left\{\underbrace{\left\{\sum_{k=1}^{K}\epsilon_k\alpha\right\}}_{\text{fraction part}}-|I|\right\}\right|\le\frac{Ca}{K^{\tau/2}}$$

(under the same assumption on α)

2 Lecture 2

2.1 Construction

Recall that $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $e_p(x) = e^{2\pi i p x}$

$$f \in \mathrm{L}_1(\mathbb{T}) \qquad \longmapsto \qquad \hat{f}(p) = \int f(x)\overline{e_p(x)}\,\mathrm{d}x\,, \quad \text{(Fourier coefficients of } f\text{)}$$

Set $\{e_p\}$ forms an orthonormal basis of $f \in \mathrm{L}_2(\mathbb{T})$, therefore:

$$\int |f|^2\,\mathrm{d}x = \sum \left|\hat{f}(p)\right|^2 \tag{2}$$

$$\int f\overline{g}\,\mathrm{d}x = \sum \hat{f}(p)\overline{\hat{g}(p)} \tag{3}$$

Example 1. $\hat{e}_q(p) = \delta_{pq}$.

Example 2. $f(x) = x, \quad 0 \le x < 1$

$$\begin{aligned}\hat{f}(p) &= \int_0^1 x\exp(-2\pi i p x)\,\mathrm{d}x \\ &= \begin{cases} x\frac{\exp(-2\pi i p x)}{-2\pi i p x}\Big|_0^1 + \frac{1}{2\pi i p}\int_0^1 \exp(-2\pi i p x)\,\mathrm{d}x = \frac{i}{2\pi p}, & \qquad p \neq 0 \\ \frac{1}{2}, \quad p = 0 \end{cases}\end{aligned}$$

$$\Longrightarrow \sum_{p=1} \frac{1}{p^2} = \frac{\pi^2}{6}$$

Exercise 7. Compute $\hat{f}$ for $f(x) = \begin{cases} 1, & |x| \le \frac{a}{2} \\ 0, & \frac{a}{2} < x \le \frac{1}{2} \end{cases}$.

The definition of Fourier coefficients is not always general enough. Sometimes we want to consider more general functions, such as the Dirac δ-function $\delta(x)$, s.t. $\delta(x) \ge 0$, $\delta(x) = 0$, for $x \neq 0$, s.t. $\int \delta(x)\,\mathrm{d}x = 1$. Mathematically $\int \delta(x) f(x)\,\mathrm{d}x = f(0)$ functional on $\mathcal{C}(\mathbb{T})$.

A function g in $\mathrm{L}_1(\mathbb{T})$ defines a functional on $\mathcal{C}(\mathbb{T})$ by $\phi_g(f) = \int fg\,\mathrm{d}x$. We now shall define the Fourier coefficients of measures and even more general objects.

Let $\mathcal{B} \subset \mathrm{L}_1(\mathbb{T})$ be a nice space of functions, e.g. $\mathrm{L}_2(\mathbb{T})$, $\mathrm{L}_p(\mathbb{T})$ for $p < \infty$, $\mathcal{C}(\mathbb{T})$, $\mathcal{C}^k(\mathbb{T})$, or $\mathcal{C}^\infty(\mathbb{T})$.

Definition 1. $\mathcal{B}$ is called a homogeneous space if:

- Banach (or Fréchet); contains $e_p(x) = \exp(2\pi i p x)$

- $f \in \mathcal{B} \implies f_y \in \mathcal{B}$, where $f_y(x) = f(x+y)$
- $f_y \underset{y\to 0}{\longrightarrow} f$ in the topology of $\mathcal{B}$.

Remark. The third property holds in $\mathrm{L}_1(\mathbb{T})$.

Let's denote $\mathcal{B}'$ as a dual space, the space of continuous functionals $\phi : \mathcal{B} \to \mathbb{C}$. Examples of dual spaces:

Example 3. $\mathcal{C}(\mathbb{T})'$ – complex measures, for example δ, $\delta_0 - i\delta_{\frac{1}{2}} : f \mapsto f(0) - if(\frac{1}{2})$.

Example 4. $\mathcal{C}^1(\mathbb{T})'$ – also contains derivatives of measures, e.g. $\delta'(f) = -f'(0)$, the reason for this notation $\int \delta'(x) f(x)\,\mathrm{d}x = \delta(x)f(x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int \delta(x) f'(x)\,\mathrm{d}x$.

Example 5. $\mathcal{C}^\infty(\mathbb{T})'$ – contains derivatives of δ-functions of arbitrary finite order.

Let's define $\hat{\phi}(p) = \phi(e_{-p})$ for $\phi \in \mathcal{B}'$. *Sanity check:* if $\mathcal{B} \subset \mathrm{L}_\infty(\mathbb{T})$, then $\mathcal{B}' \supset \mathrm{L}_1(\mathbb{T}) : \phi_f(g) = \int fg\,\mathrm{d}x$ and in this case definitions coincide.

Exercise 8. Compute Fourier coefficients: $\hat{\delta}$, and $\widehat{\delta'}$

2.2 Algebraic properties

All algebraic properties are corollary of $e_p(x+y) = e_p(x)e_p(y)$.

Property 1. $\hat{f}_y(p) = e_p(y)\hat{f}(p)$

Property 2. $\hat{\phi}_y(p) = e_p(y)\hat{\phi}(p)$

Property 3. $\widehat{(f * g)}(p) = \hat{f}(p)\hat{g}(p)$, for $f, g \in \mathrm{L}_1(\mathbb{T}) \longrightarrow (f*g)(x) = \int f(y)g(x-y)\,\mathrm{d}y$.

Remark. $f * g \in \mathrm{L}_1(\mathbb{T})$ since:

$$\begin{aligned}
\|f*g\|_1 &= \int \left| \int f(y)g(x-y)\,\mathrm{d}y \right| \mathrm{d}x \\
&\leq \int\int \mathrm{d}x\,\mathrm{d}y\,|f(y)||g(x-y)| \\
&\overset{u=x-y}{=} \int\int \mathrm{d}x\,\mathrm{d}y\,|f(y)||g(u)| \\
&= \|f\|_1\|g\|_1
\end{aligned}$$

Property 4. $\widehat{(\mu * \nu)}(p) = \hat{\mu}(p)\hat{\nu}(p)$ for $\mu, \nu \in \mathcal{C}(\mathbb{T})'$ where $(\mu * \nu)(f) = \int f(x+y)\mu(x)\nu(y)$

Let's define an linear operator $T_y : f \mapsto f_y$ (shift), and $\phi \mapsto \phi_y$, where $f_y = f(x+y)$, and $\phi_y(f) = \phi(f_{-y})$.

$T_y e_p = e_p(y) e_p$, s.t. e_p is an eigenvalue of shift operators T_y and thus also an eigenvalue of any operator that commutes with shifts, i.e. for any such T, $T e_p = \lambda_p e_p$, and consequently $\widehat{Tf}(p) = \lambda_p \hat{f}(p)$.

Example 6. $\hat{f}'(p) = +2\pi i p \hat{f}(p)$, and more generally differential operators with constant coefficients.

Example 7. We look for a given function $f(x)$ another real function $g(x)$, s.t. $\widehat{(f+ig)}(-p) = 0, p = 0, 1, \ldots$

$$f(x) = \hat{f}(0) + \sum_{p=1}^{\infty} \left[\hat{f}(p) e_p(x) + \overline{\hat{f}(p)} e_{-p}(x) \right]$$

$$g(x) = \sum_{p=1}^{\infty} \left[-i\hat{f}(p) e_p(x) + i\overline{\hat{f}(p)} e_{-p}(x) \right]$$

Thus $\hat{g}(p) = \begin{cases} -i\hat{f}(p), & p > 0 \\ 0 & \\ i\hat{f}(p), & p < 0 \end{cases}$, g is called the conjugate function, $g = \tilde{f}$.

Reason. $(f+ig)(x) = u(e^{2\pi i x})$ is the boundary value of an analytic function $u(z) = \sum_{p=0}^{\infty} \widehat{(f+ig)}(p) z^p$ in the unit disk.

Exercise 9. Compute eigenvalues for Laplacian $\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2}$ on torus.

Example 8. Solution to heat equation: $f_t = P_t * f_0$

2.3 Decay rate of Fourier coefficients

Motto: "The nicer is f, the faster $\hat{f}$ decays". This works cleanly for L_2 and spaces defined using $\sum \left|\hat{f}(p)\right|^2 = \int |f|^2 \,\mathrm{d}x$, i.e. $f \in \mathrm{L}_2(\mathbb{T})$ iff Fourier coefficients are square summable:

Property 1. $f \in \mathrm{L}_2(\mathbb{T}) \Leftrightarrow \hat{f} \in l_2$

Property 2. Using property 1 and example 6, $f^{(k)} \in \mathrm{L}_2(\mathbb{T})$, i.e. f is the k-fold integral of $g \in \mathrm{L}_2(\mathbb{T})$, $\iff \sum \left|\hat{f}(p)\right|^2 |p|^{2k} < \infty$.

Property 3. $f \in \mathcal{C}^\infty(\mathbb{T}) \Leftrightarrow \left|\hat{f}(p)\right|$ decays faster than any power of p. (Exercise. Hint: $\|g\|_\infty \le \|g\|_1 + \|g'\|_1 \le \|g\|_2 + \|g'\|_2$).

Theorem 3. Let $f \in \mathrm{L}_1(\mathbb{T})$, $a > 0$. TFAE ($f \in \mathcal{C}^\infty(\mathbb{T})$):

(i) f admits an analytic extension to $|\mathrm{Im}(z)| < a$

(ii) $f \in \mathcal{C}^\infty(\mathbb{T})$ and $\forall 0 < \tilde{a} < a, \quad \exists C(\tilde{a}) : \left\| f^{(k)} \right\|_\infty \leq C(\tilde{a})\tilde{a}^{-k}k!$

(iii) $f \in \mathcal{C}^\infty(\mathbb{T})$ and $\forall 0 < \tilde{a} < a, \quad \exists C(\tilde{a}) : \left\| f^{(k)} \right\|_2 \leq C(\tilde{a})\tilde{a}^{-k}k!$

(iv) $\forall 0 < \tilde{a} < a, \quad \exists C(\tilde{a}) : \left| \hat{f}(p) \right| \leq C(\tilde{a}) \exp(-2\pi |p| \tilde{a})$. (Exponential decay of Fourier coefficients)

Proof. theorem 3.(i) $\implies$ theorem 3.(ii) using Cauchy formula: $f^{(k)}(x) = \frac{k!}{2\pi i} \oint \frac{f(z)}{(z-x)^{k+1}} \,\mathrm{d}z$. □

3 Lecture 3

Proof. Theorem 3.(ii) $\implies$ theorem 3.(iii) is obvious, since $\left\|f^{(k)}\right\|_\infty \leq \left\|f^{(k)}\right\|_2$. □

Proof. Theorem 3.(iii) $\implies$ theorem 3.(iv), and theorem 3.(iv) $\implies$ theorem 3.(i). Remind that $(\frac{k}{e})^k \leq k! \leq C_\delta(1+\delta)^k(\frac{k}{e})^k$. Let $\tilde{a} < a_1 < a$, apply (3) with a_1 in place of $\tilde{a}$:

$$\begin{aligned}
\int \left|f^{(k)}\right|^2 \mathrm{d}x &\leq C(a_1)^2 a_1^{-2k}(k!)^2 \\
&\leq C(a_1)^2 C_\delta^2 \left[\frac{k(1+\delta)}{ea_1}\right]^{2k} \\
&\text{, where } 1+\delta = \frac{a_1}{\tilde{a}} \\
&\leq \tilde{C}(a_1)^2 \left(\frac{k}{\tilde{a}e}\right)^{2k}
\end{aligned}$$

The LHS is $\sum \left|\hat{f}(p)\right|^2 (2\pi|p|)^{2k}$, hence for any k and $p \neq 0$:

$$\begin{aligned}
\left|\hat{f}(p)\right| &\leq \tilde{C}(a_1)\left(\frac{k}{\tilde{a}e2\pi|p|}\right)^k \\
&\text{, take } k = 2\pi\tilde{a}|p| \quad \text{then} \\
&\leq \tilde{\tilde{C}}(a_1)\exp\{-2\pi\tilde{a}|p|\}.
\end{aligned}$$

□

Proof. Theorem 3.(iv) $\implies$ theorem 3.(i): Let $F(z) = \sum \hat{f}(p)e^{2\pi i p z}$

- Converges uniformly and therefore analytic for $|z| < a$
- On $\mathbb{T}$ coincides with f in L_2, hence also almost everywhere.

□

Example 9. $P_t * f$ is an entire function (i.e. analytic in $\mathbb{C}$) for $t > 0$, whenever $f \in \mathcal{C}(\mathbb{T})$ or even $f \in L_1(\mathbb{T})$.

Exercise 10. If f is analytic in $|\mathrm{Im}(z)| < A$ with a single simple pole at z_0 and $|\mathrm{Im}(z_0)| = a \in (0, A)$, then Fourier coefficients for f, $\hat{f}(p) = \exp(2\pi i p z_0) + O(\exp(-(A-\delta)2\pi|p|)$.

For other function spaces the connection is less tight.

Example 10. $f \in \mathcal{C}^k(\mathbb{T}) \implies \sum \left|\hat{f}(p)\right|^2 |p|^{2k} < \infty$, but $\sum \left|\hat{f}(p)\right|^2 |p|^{2k} < \infty \implies f \in \mathcal{C}^{k-1}(\mathbb{T})$. Both implications are "unimprovable".

Example 11. What can be said about $\hat{f}(p)$ for $f \in \mathrm{L}_1(\mathbb{T})$? Clearly, $\left|\hat{f}(p)\right| \leq \|f\|_1$, but $|\hat{\mu}|$ is bounded even for measures!

We now discuss an improvement of Riemann-Lebesgue: if $f \in \mathrm{L}_1(\mathbb{T})$ then $\hat{f}(p) \to 0$. We prove a quantitative version as follows: recall that $\omega(\delta; f) = \sup_{|y|<\delta} \|f - f_y\|_\infty$ and f is uniformly continuous $\iff \omega(\delta; f) \underset{\delta\to+0}{\to} 0$. Let's denote $\omega_{\mathcal{B}}(\delta; f) = \sup_{|y|<\delta} \|f - f_y\|_{\mathcal{B}}$, e.g. $f \in \mathrm{L}_1(\mathbb{T}) \implies \omega_{\mathrm{L}_1(\mathbb{T})}(\delta; f) \underset{\delta\to+0}{\to} 0$.

Theorem 4 (Riemann-Lebesgue)**.**

$$\left|\hat{f}(p)\right| \leq \frac{1}{2}\omega_{\mathrm{L}_1(\mathbb{T})}(\frac{1}{2|p|}; f)$$

And in particular, $\hat{f}(p) \to 0$, $p \to \pm 0$.

Proof.

$$\begin{aligned}\hat{f}(p) &= \int f(x)\overline{e_p(x)}\,\mathrm{d}x = \int f(x + \frac{1}{2p})\overline{e_p(x + \frac{1}{2p})}\,\mathrm{d}x \\ &= \underbrace{\int f(x)\overline{e_p\left(x + \frac{1}{2p}\right)}\,\mathrm{d}x}_{\text{term 1}} + \underbrace{\int \left(f\left(x + \frac{1}{2p}\right) - f(x)\right)\overline{e_p\left(x + \frac{1}{2p}\right)}\,\mathrm{d}x}_{\text{term 2}}\end{aligned}$$

* term 1 $= -\int f(x)\overline{e_p(x)}\,\mathrm{d}x = -\hat{f}(p)$
* $|\text{term 2}| \leq \int \left|f\left(x + \frac{1}{2p}\right) - f(x)\right| \mathrm{d}x \leq \omega_{\mathrm{L}_1(\mathbb{T})\left(\frac{1}{2|p|}; f\right)}$

□

Example 12. When the function f is α-Lipschitz, i.e. modulus of the continuity is bounded by $|f(x) - f(y)| \leq C|x - y|^\alpha$, $\alpha \in (0, 1]$, therefore $f \in Lip_\alpha \implies \left|\hat{f}(p)\right| \leq p^{-\alpha}$.

Exercise 11. Prove: for $0 < \alpha < 1$, $f_\alpha(x) = \sum_{p=1} 3^{-p\alpha}\cos(2\pi 3^p x)$ lies in $Lip_\alpha \cap \mathrm{L}_1$, but $\left|\hat{f}(p)\right| \geq |p|^{-\alpha}$ on a sub-sequence. I.e. example 12 is sharp.

Corollary 1 (Riemann-Lebesgue)**.** $f^{(k)} \in \mathrm{L}_1(\mathbb{T}) \implies \left|\hat{f}(p)\right| = o\left(|p|^{-k}\right)$.

We can use corollary 1 to compute asymptotics of $\hat{f}$.

Example 13. If $f \in \mathcal{C}^1(\mathbb{T}\backslash\{0\})$ is k-th the integrable piecewise function and has a jump discontinuity at 0, then p-th Fourier coefficient $\hat{f}(p) = \frac{1}{2\pi i p}\left(f(+0) - f(-0)\right) + o\left(\frac{1}{|p|}\right)$.

Proof. Let $g(x) = f(+0) + \left(f(-0) - f(+0)\right)x, \quad 0 \le x < 1$. $\hat{g}(p)$ has this asymptotics, $(f - g)' \in \mathrm{L}_1(\mathbb{T})$. $\square$

Exercise 12. $f \in \mathcal{C}^1\left(\mathbb{T}\backslash\{0\}\right)$, $f(\pm x) \sim A_\pm x^{-\alpha}$ for $\alpha \in [0,1)$ and $x \to +0$, $\implies A_+\Gamma(\alpha+1)(2\pi i p)^{\alpha-1} + A_-\Gamma(\alpha-1)(-2\pi i p)^{\alpha-1} + o\left(\frac{1}{|p|}\right)$.

3.1 Convergence of Fourier series

Convergence of the partial sum $S_n(f;x) = \sum_{-n}^{n} \hat{f}(p)e_p(x)$. Clearly, $\hat{S}_n = \hat{f}\mathbb{1}_{[-n,n]}$ therefore by item property 1, we expect that S_n is a convolution of f with something. This is indeed so: $S_n = D_n * f$, where $D_n = \sum_{-n}^{n} e_p(x)$ (Dirichlet kernel), since $S_n(f;x) = \sum_{-n}^{n} \int f(y)\overline{e_p(y)}\,\mathrm{d}y\, e_p(x) = \int f(y) \sum_{-n}^{n} e_p(x - y)\,\mathrm{d}y$.

$$
\begin{aligned}
D_n &= \sum_{-n}^{n} e_p(x) = e_{-n}(p) \sum_{0}^{2n} e_p(x) \\
&= e_{-n}(p) \frac{1 - e_{2n+1}(x)}{1 - e_p(x)} = \frac{\sin((2n+1)\pi x)}{\sin(\pi x)}
\end{aligned}
$$

And, clearly, $\hat{D}_n(0) = 1$. When is is true that $D_n * f \to f$? In $\mathrm{L}_2(\mathbb{T})$, S_n is the best approximation of f by a trigonometric polynomial of degree $\le n$, and hence $S_n \longrightarrow f$ in $\mathrm{L}_2(\mathbb{T})$ if $f \in \mathrm{L}_2(\mathbb{T})$. However, this is not the case for $\mathcal{C}(\mathbb{T})$, $\mathrm{L}_1(\mathbb{T})$ and etc.

Example 14 (de la Vallée-Poussin)**.** $f \in \mathcal{C}(\mathbb{T})$, but $(D_n * f)(0)$ diverges. As well as, $f \in \mathrm{L}_1(\mathbb{T})$, but $(D_n * f) \underset{\mathrm{L}_1(\mathbb{T})}{\not\to} f$.

Example 15 (Kolmogorov)**.** $f \in \mathrm{L}_1(\mathbb{T})$, but $(D_n * f)$ diverges everywhere.

Example 16 (Carleson-Hunt)**.** $f \in \mathrm{L}_p(\mathbb{T})$, $1 < p < \infty$, and $(D_n * f)$ converges almost everywhere.

Theorem 5 (Dini)**.** If $f \in \mathrm{L}_1(\mathbb{T})$ and $\int \frac{|f(y) - f(x)|}{|y - x|}\,\mathrm{d}y < \infty$ for some $x \in \mathbb{T}$, then $S_n(f;x) \to f(x)$.

Proof. W.l.o.g. assume $x = 0$, $f(0) = 0$

$$S_n(f;0) = \int S_n(y)f(y)\,\mathrm{d}y = \int \frac{\sin(2\pi ny)\cos(\pi y) + \cos(2\pi ny)\sin(\pi y)}{\sin(\pi y)} f(y)\,\mathrm{d}y$$
$$= \int \sin(2\pi ny) \underbrace{\left[\frac{\cos(\pi y)}{\sin(\pi y)} f(y)\right]}_{\text{in } \mathrm{L}_1(\mathbb{T})} \mathrm{d}y + \int \cos(2\pi ny) \underbrace{f(y)}_{\text{in } \mathrm{L}_1(\mathbb{T})} \mathrm{d}y$$

By theorem 4 it is implied that both last terms tend to 0. □

Corollary 2 (Localisation principle)**.** *If $f = g$ in $(x-\epsilon, x+\epsilon)$ and $S_n(f;x) \to A$, then $S_n(g;x) \to A$.*

3.1.1 Absolute convergence

When is it true that $\sum \left|\hat{f}(p)\right| < \infty$? Note that in this case $\sum \hat{f}(p)e_p \to f$ absolutely and uniformly $\implies$ $f \in \mathcal{C}(\mathbb{T})$ (as a uniform limit of continuous functions).

Theorem 6 (Bernstein)**.** If $\int_0^1 \omega_{\mathrm{L}_2(\mathbb{T})}(h;f)\frac{\mathrm{d}h}{h^{3/2}} < \infty$, then $\sum \left|\hat{f}(p)\right| < \infty$.

Remark. In particular, the convergence of the integral implies that $f \in \mathcal{C}(\mathbb{T})$. Is there a direct proof?

Proof.

$$\begin{aligned}
\omega_{\mathrm{L}_2(\mathbb{T})}(h;f)^2 &\geq \int |f(x+h) - f(x)|^2 \,\mathrm{d}x \\
&= \sum_p \left|\hat{f}(p)\right|^2 |1 - e_p(h)|^2 \\
&\geq 2 \sum_{\frac{1}{4h} \leq |p| \leq \frac{1}{2h}} \left|\hat{f}(p)\right|^2
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{2^m \leq |p| \leq 2^{m+1}} \left|\hat{f}(p)\right|^2 &\lesssim \frac{1}{2}\omega_{\mathrm{L}_2(\mathbb{T})}(2^{-m};f)^2 \\
\sum_{2^m \leq |p| \leq 2^{m+1}} \left|\hat{f}(p)\right| &\lesssim 2^{\frac{m}{2}}\omega_{\mathrm{L}_2(\mathbb{T})}(2^{-m};f)
\end{aligned}$$

On the other hand

$$\int_{2^{-m}}^{2-m+1} \omega_{\mathrm{L}_2(\mathbb{T})}(2;f)\frac{\mathrm{d}h}{h^{\frac{3}{2}}} \geq \omega_{\mathrm{L}_2(\mathbb{T})}(2^{-m};f)2^{\frac{m}{2}}$$

□

Definition 2. $A(\mathbb{T}) = \Big\{ f \in \mathrm{L}_1(\mathbb{T}) : \hat{f} \in l_1 \Big\}$.

Remark. If $a \in l_1, \quad \sum a_p e_p \in \mathcal{C}(\mathbb{T})$.

$A(\mathbb{T})$ is an algebra (subalgebra of $\mathcal{C}(\mathbb{T})$: $f, g \in A(\mathbb{T}) \implies fg \in A(\mathbb{T})$. *Reason:* $(l_1, *)$ is an algebra.

4 Lecture 4

Reminder. Last lecture motto, "The nicer is f, the faster $\hat{f}(p)$ decays", works nice for some function spaces:

- $f \in \mathrm{L}_2(\mathbb{T}) \iff \hat{f} \in l_2$
- $f \in \mathcal{C}^\infty(\mathbb{T}) \iff \left|\hat{f}(p)\right| = O(|p|^{-\infty})$
- f is analytic $\iff$ $\hat{f}(p)$ decays exponentially

For other spaces there are no simple necessary and sufficient conditions.

- $f \in \mathrm{L}_1(\mathbb{T}) \implies \hat{f}(p) \to 0$, or moreover $\left|\hat{f}(p)\right| \leq \omega_{\mathrm{L}_1}(h; f) = \sup_{0 \leq y \leq h} \|f - f_y\|_1$
- $f \in Lip_\alpha \implies \left|\hat{f}(p)\right| \lesssim |p|^{-\alpha}$, or more generally $\left|\hat{f}(p)\right| \lesssim \omega_{\mathrm{L}_1}\left(\frac{1}{|p|}; f\right)$

The decay is dominated by the most singular singularity. We also discussed convergence:

- $f \in \mathrm{L}_2(\mathbb{T}) \implies \sum \hat{f}(p) e_p \to f$ in $\mathrm{L}_2(\mathbb{T})$

  Remark. Carleson showed that this is also holds a.e.
- $f \in \mathrm{L}_1(\mathbb{T})$, $\int \frac{f(y)-f(x)}{y-x}\,\mathrm{d}x < \infty \implies \sum_{-N}^{N} \hat{f}(p) e_p(x) \to f(x)$ (for this x).
  In general, $S_N = \sum_{-N}^{N} \hat{f}(p) e_p(x)$ may not converge to f in any sense.

  Reason. $S_N = D_N * f$, $D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$

Absolute uniform convergence: $A(\mathbb{T}) = \left\{f \in \mathrm{L}_1(\mathbb{T}), \sum \left|\hat{f}(n)\right| < \infty\right\} \subset \mathcal{C}(\mathbb{T})$. Check out theorem 6, and in particular, $f \in Lip_\alpha$ for some $\alpha > \frac{1}{2} \implies f \in A(\mathbb{T})$. But not for $\alpha = \frac{1}{2}$ and vice versa.

What to do with general $f \in \mathrm{L}_1(\mathbb{T})$ and $f \in \mathcal{C}(\mathbb{T})$.

Paradox. Although $f \in \mathcal{C}(\mathbb{T})$ can be approximated uniformly by trigonometric polynomials $S_N = D_N * f$, the orthogonal projection of f onto the space of trigonometric polynomials of degree $\leq N$ may be very far from f. The *solution* is $D_n * f$ minimises $\|P - f\|_2$, but not $\|P - f\|_1$, and not $\|P - f\|_\infty$.

New motto: "The nicer f is, the better it can be approximated by trigonometric polynomials". But not by $D_N * f$, how then?

Theorem 7 (Fejér)**.** If $f \in \mathcal{C}(\mathbb{T})$, then $\frac{S_0(f)+\cdots+S_{N-1}(f)}{N} \rightrightarrows f$, which implies Weierstrass theorem without circular reasoning. More generally: for any homogeneous space $\mathcal{B}$ and any $f \in \mathcal{B}$, $\frac{S_0(f)+\cdots+S_{N-1}(f)}{N} \rightrightarrows f \in \mathcal{B}$.

Reminder. $\quad * \; a_n \to A \implies \frac{a_0+\cdots+a_{N-1}}{N} \to A$

* Not vice versa, e.g. $a_n = (-1)^n$

  What is $\frac{S_0+\cdots+S_{N-1}}{N}$? $S_n = D_n * f$, hence $\frac{S_0+\cdots+S_{N-1}}{N} = (\frac{D_0+\cdots+D_{N-1}}{N}) * f$.

Exercise 13. If $|a_n - a_{n-1}| \le \frac{C}{n}$ and $(a_0 + \cdots + a_{N-1})/N \to A$, show that $a_n \to A$.

Exercise 14. Show that $K_N(x) = \frac{D_0(x)+\cdots+D_{N-1}(x)}{N} = \frac{\sin^2(\pi N x)}{N\sin^2(\pi x)}$.

Property 1. $\int K_N(x)\,\mathrm{d}x = 1$. Since, it holds for D_N
Property 2. $\int |K_N(x)|\,\mathrm{d}x \le C$. Since $K_N \ge 0$
Property 3. $\forall 0 < \delta < \frac{1}{2}$, $\int_{|x|>\delta} |K_N(x)|\,\mathrm{d}x \underset{N\to\infty}{\to} 0$. Since, $|K_N(x)| \lesssim \frac{1}{Nx^2}$

Definition 3. A sequence of functions k_N satisfying property 1, property 2, property 3 called a summability kernel or approximate δ-function.

Remark. Sometimes instead of a large parameter $N \to \infty$, we shall encounter a small parameter $t \to 0$.

Theorem 8. If $\mathcal{B}$ is a homogeneous space, $f \in \mathcal{B}$ then $k_N * f \to f \in \mathcal{B}$ for any summability kernel k_N.

Corollary 3. *Uniqueness in* $\mathrm{L}_1(\mathbb{T})$: $\hat{f} = \hat{g} \implies f = g$ *(and also explicit reconstruction).*

Proof. We assume that $\mathcal{B}$ is a Banach space. $(k_n * f)(x) = \int k_n(y)f(x-y)\,\mathrm{d}y$, and briefly $k_N * f = \int k_N(y) f_y\,\mathrm{d}y$.

$$\begin{aligned}\|k_N * f - f\|_{\mathcal{B}} &\underset{\text{by property 1}}{=} \left\|\int k_N(-y)(f_y - f)\,\mathrm{d}y\right\|_{\mathcal{B}} \\ &\le \int |k_N(-y)|\|f_{-y} - f\|_{\mathcal{B}}\,\mathrm{d}y \\ &\le \underbrace{\int_{|y|\le\delta} |k_N(y)|\,\mathrm{d}y\,\omega_{\mathcal{B}}(f;\delta)}_{\text{term 1}} + \underbrace{\int_{\delta<|y|\le\frac{1}{2}} |k_N(y)|\,\mathrm{d}y\,2\|f\|_{\mathcal{B}}}_{\text{term 2}}\end{aligned}$$

* by property 2, term $1 \le C\omega_{\mathcal{B}}(f;\delta)$

* by property 3, term $2 \to 0$, as $N \to \infty$

So we let $N \to \infty$ and $\delta \to +0$. □

4.1 Additional applications

Example 17. The heat kernel $P_t(t \to +0)$ is a summability kernel, and hence $(P_t * f) \underset{t\to 0}{\to} f \in \mathcal{B}$ for any $f \in \mathcal{B}$, e.g. $\mathcal{C}(\mathbb{T})$, i.e. $u_t(x) = \begin{cases} f(x), & t = 0 \\ (P_t * f)(x), & t > 0 \end{cases}$ is continuous at $t = 0$ and thus indeed is a solution to the heat equation. Hint: use exercise 4.

Example 18. Given a continuous $g : \{|z| = 1\} \to \mathbb{R}$, we want $G : \{|z| \leq 1\} \to \mathbb{R}$ which is continuous and harmonic (in the interior). $G(r\exp(2\pi ix)) = (P_r^o * f)(x)$, and $f(x) = g(\exp(2\pi ix))$ where $P_r^o(x) = \sum_p r^{|p|}e_p(x)$ is the Poisson kernel.

Exercise 15. - P_r^o $(r \to 1-0)$ is a summability kernel ($\implies$ G is continuous at the boundary)

- $\Delta G = 0$ is $\{|z| < 1\}$

Example 19. Another useful kernel is Jackson kernel: $\mathcal{J}_N(x) = \frac{K_N(x)^2}{\int K_N(y)^2 \mathrm{d}y}$

Exercise 16. Show that $\mathcal{J}_N$ is a summability kernel and $\mathcal{J}_N(x) \lesssim N^{-3}|x|^{-4}$ for $|x| \leq \frac{1}{2}$

4.2 Amplifications of the Weierstrass theorem

Let denote trigonometric polynomials of degree $\leq N$ as $\mathcal{T}_N = \{P = \sum c_p e_p : |p| \leq N\}$, and the rate of approximation by trigonometric polynomials $E_N(f;\mathcal{B}) = \inf_{P\in\mathcal{T}_N} \|f - P\|_{\mathcal{B}}$.

Theorem 9 (Jackson$^+$)**.** if $f \in \mathcal{B}$, then $E_N(f;\mathcal{B}) \lesssim C\omega_{\frac{1}{N}}(f;\mathcal{B})$ and, moreover, if $f^{(k)} \in \mathcal{B}$, then $E_N(f;\mathcal{B}) \lesssim C^k N^{-k}\omega_{\frac{1}{N}}(f^{(k)};\mathcal{B})$.

Exercise 17. Prove second part of the theorem 9.

First part. Assume w.l.o.g. $k = 0$, and $\omega_{\mathcal{B}}(2\delta; f) \leq 2\omega_{\mathcal{B}}(\delta; f)$.

$$\begin{aligned} E_{2N}(f;\mathcal{B}) &\leq \|\mathcal{J}_N * f - f\|_{\mathcal{B}} \\ &\leq 2\int_0^{\frac{1}{2}} \mathcal{J}_N(y)\|f_y - f\|_{\mathcal{B}}\,\mathrm{d}y \\ &\leq 2\int_0^{\frac{1}{2}} \mathcal{J}_N(y)\omega_{\mathcal{B}}(y; f)\,\mathrm{d}y \end{aligned}$$

$$\int_{\frac{(K-1)}{2N}}^{\frac{K}{2N}} \lesssim \overbrace{\frac{1}{N}}^{\text{term 1}} \overbrace{\frac{1}{N^3(\frac{K}{N})^4}}^{\text{term 2}} \overbrace{K\omega_{\mathcal{B}}(\frac{1}{2N}; f)}^{\text{term 3}} = \frac{1}{K^3}\omega_{\mathcal{B}}(\frac{1}{2N}; f)$$

Term 1 is the length of the interval. Term 2 is the bound on $\mathcal{J}_N$ using exercise 16. Term 3 is built using $\omega_{\mathcal{B}}(\frac{k}{2N}; f) \leq k\omega_{\mathcal{B}}(\frac{1}{2N}; f)$. The final step is the summing over k. □

Remarkably, Jackson's theorem is sharp in a much stronger sense than the results from the previous section.

Theorem 10 (Bernstein)**.** If $E_N(f;\mathcal{B}) \leq AN^{-(k+r)}$ for some $k \in \mathbb{Z}_+$, $\quad 0 < r < 1$ then $f^{(k)} \in \mathcal{B}$ and $\omega_\delta(f^{(k)}; \mathcal{B}) \leq C_{r,k} A\delta^r$.

Remark. $r = 1$ requires some more care.

Proof. For $k = 0$, let $P_n \in \mathcal{T}_{2n}$ be such that $\|P_n - f\|_{\mathcal{B}} = E_{2n}(f;\mathcal{B})$, then $f = \lim^{\mathcal{B}} P_n = \sum_{n=0}^{\infty} (P_n - P_{n+1})$, where

$$\|Q_n\|_{\mathcal{B}} \leq 2E_{2^n}(f;\mathcal{B}) \lesssim 2A2^{-nr} \tag{4}$$

Theorem 11 (Berstein)**.** If $P \in \mathcal{T}_N$, $\|P'\|_{\mathcal{B}} \leq KN\|P\|_{\mathcal{B}}$ where $K = 2\pi$ (we do not need the sharp value of k).

Continuing the proof of theorem 10:

$$\begin{aligned}
\|f_y - f\|_{\mathcal{B}} &\leq \sum \|Q_{n,y} - Q_n\|_{\mathcal{B}} \\
&\lesssim \sum \min(2A2^{-nr}, 2^{n(1-r)}AK|y|) \\
&\lesssim A(|y| \sum_{2^n \leq \frac{1}{y}} 2^{2(1-r)} + \sum_{2^n > \frac{1}{y}} 2^{-nr}) \\
&\lesssim A(|y||y|^{r-1} + |y|^r) \\
&\lesssim A|y|^r.
\end{aligned}$$

□

Exercise 18. Prove $k \geq 1$ case.

Exercise 19. Prove theorem 11. Plan for the proof:

$$\begin{aligned}
P &= V_N * P \qquad \text{for} \qquad V_N = 2K_{2N} - K_N \\
P' &= V_N' * P = \int V_N'(y) P_y \, \mathrm{d}y \\
\implies \|P'\|_{\mathcal{B}} &\leq \int |V_N'(y)| \|P_y\|_{\mathcal{B}} \, \mathrm{d}y
\end{aligned}$$

One can check that $\int |V_N'(y)| \, \mathrm{d}y \leq KN$.

5 Lecture 5

5.1 Fourier coefficients of linear functionals

Reminder. $\mathcal{B}$ - homogeneous space, $\phi \in \mathcal{B}'$ - linear functional, $\hat{\phi}(p) = \phi(e_{-p})$, $p \in \mathbb{Z}$.

5.1.1 Reconstruction of ϕ from $\hat{\phi}$

We proved this for $\mathcal{B}$ (extended theorem 7)

Theorem 12. Let $\forall f \in \mathcal{B}$, $\phi(f) = \frac{1}{N} \lim_{N\to\infty} \sum_{n=0}^{N-1} \int S_n(\phi) f \,\mathrm{d}x$, where $S_n(\phi) = \sum_{-N}^{N} \hat{\phi}(x) e_p$. (Briefly: $\phi = \lim \frac{1}{N} \sum_{n=0}^{N-1} S_n(\phi)$ in $\mathcal{B}'$)

Corollary 4. $\hat{\phi} = \hat{\psi} \implies \phi = \psi$.

Proof of theorem 12.

$$\frac{1}{N} \sum_{n=0}^{N-1} S_n(\phi) = \sum_{-N}^{N} \left(1 - \frac{|p|}{N}\right) \hat{\phi}(p) e_p,$$

hence,

$$\begin{aligned}
\int \frac{1}{N} \sum_{n=0}^{N-1} S_n(\phi) f \,\mathrm{d}x &= \int \sum_{-N}^{N} \left(1 - \frac{|p|}{N}\right) \hat{\phi}(p) e_p f \,\mathrm{d}x \\
&= \sum_{-N}^{N} \left(1 - \frac{|p|}{N}\right) \hat{\phi}(p) \hat{f}(-p) \\
&= \phi\left(\sum_{-N}^{N} \left(1 - \frac{|p|}{N}\right) \hat{f}(-p) e_{-p}\right) \\
&= \phi\left(\sum_{-N}^{N} \left(1 - \frac{|p|}{N}\right) \hat{f}(p) e_p\right) \\
&= \phi\left(\underbrace{\frac{1}{N} \sum_{n=0}^{N-1} S_n(f)}_{\text{Converges to } f \in \mathcal{B} \text{ by theorem 7}}\right) \to \phi(f)
\end{aligned}$$

□

5.1.2 Focus on $\mathcal{B} = \mathcal{C}(\mathbb{T})$

$\mathcal{B}' = \mu(\mathbb{T})$ - measures. $\mu^+(\mathbb{T}) \subset \mu(\mathbb{T})$ - positive measures. ($\mu(\mathbb{T}) = \mu^+(\mathbb{T}) + i\mu^+(\mathbb{T}) - \mu^+(\mathbb{T}) - i\mu^+(\mathbb{T})$, any measure can be decomposed in this way and follows from Hahn theorem).

Inequality 1 (Erdős–Turán)**.** if μ is a probability measure on $\mathbb{T}$, then $\forall N \geq 1$:

$$\sup_{I-\text{arc}} |\mu(I) - mes(I)| \geq C \left[\frac{1}{N} + \sum_{p=1}^{N} \frac{\hat{\mu}(p)}{p} \right]$$

Lemma 1 (Ganelius)**.** Let $f \in \mathrm{L}_1(\mathbb{T})$ real-valued, and let $\omega^+(\delta; f) = \sup_{x \leq y \leq x+\delta} (f(y) - f(x))$. Then

$$\sup |f| \leq C \left[\sum_{p=0}^{N-1} \left| \hat{f}(p) \right| + \omega^+(\frac{1}{N}; f) \right]$$

Proof of inequality 1. Let $f(t) = x - \mu[0,t] - A$, where $A = \int_0^1 (x - \mu[0,x])\,\mathrm{d}x$. Then

(1) $\omega^+(\delta; f) \leq \delta$

(2) $\hat{f}(0) = 0$

(3) $\hat{f}(p) = \int \overline{e_p(x)}\,(x - \mu[0,x] - A)\,\mathrm{d}x = \frac{1}{2\pi i p}\hat{\mu}(p)$, $p \geq 1$

Hence,

$$|\mu[a,b] - (b-a)| \geq 2C \left(\frac{1}{N} + \sum_{p=1}^{N} \frac{|\hat{\mu}(p)|}{2\pi p} \right)$$

□

Proof of lemma 1. W.l.o.g. $M = \sup |f| = |f(0)|$

Case 1. $M > 0$, then $f(x) \geq M - \omega^+(\frac{2k}{N}; f)$ for $0 \geq x \geq -\frac{2k}{N}$.

$$(K_N * f)(-\frac{k}{N}) = \underbrace{\int_{-\frac{k}{N}}^{\frac{k}{N}} K_N(y) f(-\frac{k}{N} - y)\,\mathrm{d}y}_{term1} + \underbrace{\int_{\frac{k}{N} \geq |y| \geq \frac{1}{2}} K_N(y) f(-\frac{k}{N} - y)\,\mathrm{d}y}_{term2}$$

For sufficiently large $k \in \mathbb{N}$, $\int_{-\frac{k}{N}}^{\frac{k}{N}} K_N(y)\,\mathrm{d}y \geq \frac{9}{10}$, whence *term 1* $\geq \frac{9}{10}(M - \omega^+\left(\frac{2k}{N}; f\right))$, *term 2* $\geq -\frac{1}{10}M$, and therefore

$$\begin{aligned}(K_N * f)(-\frac{k}{N}) &\geq \frac{9}{5}M - \frac{9}{10}\omega^+\left(\frac{2k}{N}; f\right) \\ &\geq \frac{4}{5}M - 2k\omega^+\left(\frac{1}{N}; f\right)\end{aligned}$$

On other hand:

$$\begin{aligned}\|K_N * f\|_\infty &\leq \sum_{|p|\leq N} \left|\hat{f}(p)\right| \\ &= |f(0)| + 2\sum_{p=1}^{N} \left|\hat{f}(p)\right|\end{aligned}$$

Whence

$$M \lesssim \left[\omega^+\left(\frac{1}{N}; p) + \sum_{p=0}^{N} \left|\hat{f}(p)\right|\right)\right]$$

Case 2. $M < 0$. Similar argument.

□

Now start with $a = (a_p)_{p\in\mathbb{Z}}$

- when does there exist $\mu \in \mu^+(\mathbb{T})$ such that $\hat{\mu}(p) = a_p$?
- same question for $\mu \in \mu(\mathbb{T})$

Definition 4. a is called positive-semidefinite ($a \succcurlyeq 0$) if $\forall k \geq 1, \forall z_1, \ldots, z_n \in \mathbb{C}$

$$\sum_{p,q=-k}^{k} a_{p-q}\overline{z_p}z_q \geq 0$$

Theorem 13 (Herglotz)**.** $a \succcurlyeq 0 \iff \exists \mu \in \mu^+(\mathbb{T})$, such that $\hat{\mu} = a$.

Proof. If $\mu \in \mu^+(\mathbb{T})$

$$\begin{aligned}\sum \hat{\mu}(p-q)\overline{z_p}z_p &= \sum \int e_{q-p}\,\mathrm{d}\mu\, z_q\overline{z_p} \\ &= \int \left|\sum z_q e_q\right|^2 \mathrm{d}\mu \geq 0\end{aligned}$$

Assume $a \succcurlyeq 0$. Denote: $\mu\left(\sum_{|p|\leq N} c_p e_p\right) = \sum c_p a_{-p}$. We claim that if $P \in \mathcal{T}_N$ is ≥ 0 on $\mathbb{T}$, then $\mu(P) \geq 0$. This follows from:

Lemma 2. If $P \in \mathcal{T}_N \geq 0$, then $P = |Q|^2$ for some trigonometric polynomials Q.

Explicitly: $P = \sum c'_p \overline{c'_q} e_{p-q}$, whence $\mu(P) = \sum c'_p \overline{c'_q} a_{q-p} \geq 0$. Thus $P \geq A \implies \mu(P) \geq Aa_0$, for some $A \in \mathbb{R}$, and $|P| \leq A \implies |\mu(p)| \leq Aa_0$. Hence μ is a bounded functional on $\bigcup \mathcal{T}_N$ and can be extended to $\mathcal{C}(\mathbb{T})$, i.e. defines a measure $\mu \in \mu(\mathbb{T})$. To prove that $\mu \in \mu^+(\mathbb{T})$, take $f \geq 0$ in $\mathcal{C}(\mathbb{T})$; then,

$$\begin{aligned} \exists P_n \rightrightarrows f, \quad P_n \in \mathcal{T}_n, \quad \text{and } P_n \geq 0, \quad \text{e.g. } P_n = K_n * f \\ \implies \mu(f) = \lim \mu(P_n) \geq 0 \end{aligned}$$

□

Proof of lemma 2.

$$P(x) = \sum_{-N}^{N} c_p e_p \geq 0 \implies c_{-p} = \overline{c_p}$$

Let $P^{\#}(z) = \sum_{-N}^{N} c_p z^p = C_N z^{-N} \prod (z - \xi_j)^{m(\xi_j)}$

- if $|\xi_j| = 1$, $m(\xi_j)$ is even,
- if $|\xi_j| \neq 1$,

$$\begin{aligned} P^{\#}\left(\frac{1}{\overline{\xi_j}}\right) &= \sum c_p \overline{\xi_j^{-p}} = \overline{\sum c_{-p} \xi_j^{-p}} = P^{\#}(\xi_j) = 0 \\ \implies m\left(\frac{1}{\overline{\xi_j}}\right) &= m(\xi_j) \end{aligned}$$

Let $Q(x) = \sqrt{|C_N|} \prod_{|\xi|>1} (e_1(x) - \xi_j)^{m(\xi_j)/2} \prod_{|\xi|=1} (e_1(x) - \xi_j)^{m(\xi_j)/2}$, then

$$|Q(x)|^2 = |P(x)| = P(x)$$

□

Exercise 20. Complete the details of the proof when $m(\xi_j) > 1$.

5.2 Applications

5.2.1 Stationary Gaussian process

(X_n) - stationary Gaussian process ($\mathbb{C}$-valued), $\mathbb{E}X_n = 0$.

Claim. *$\exists \rho \in \mu^+(\mathbb{T})$ (semi-positive measure) such that $\mathbb{E}\overline{X_p}X_q = \hat{\rho}(p-q)$.*

Proof.

$$\sum \mathbb{E}\overline{X_p}X_q\overline{z_p}z_q$$
$$= \mathbb{E}\Big|\sum z_p X_p\Big|^2 \geq 0$$

□

Exercise 21. Show that if $\frac{X_1+\cdots+X_N}{N} \to 0\ (= \mathbb{E}X_1) \iff \rho$ has no atom at 0.

5.2.2 Predictability

Can one predict $X_0, X1, \ldots$ if we know $X_{-1}, X_{-2}, \ldots$?

Reminder. (*Szegő-Krein*) ρ' - density of the absolutely continuous (ac) part of f, as in Lebesgue decomposition theorem; $\mathfrak{g}(\rho') = exp\left(\int \log \rho'(x)\,\mathrm{d}x\right)$. (*Jensen*) $\int \log \rho'(x)\,\mathrm{d}x \leq \log \int \rho'(x)\,\mathrm{d}x \leq \log \rho_{ac}(\mathbb{T})$, hence $\int \log \rho'(x)\,\mathrm{d}x \in [-\infty, 0]$, $\mathfrak{g}(\rho') \in [0,1]$.

Theorem 14 (Szegő-Krein)**.** TFAE:

(i) $\mathfrak{g}(\rho') = 0$, i.e. $\int \log_- \rho'\,\mathrm{d}x = +\infty$

(ii) $\forall k,\ \epsilon > 0,\ \exists c_{1,k}, \ldots, c_{N,k} : \mathbb{E}\Big|X_k - \sum_{j=0}^{N} c_{j,k}X_{-j}\Big|^2 \leq \epsilon$

Exercise 22. Show that theorem 14.(ii) $\iff$ $span(e_p)_{p\geq 1}$ is dense in $\mathrm{L}_2(\rho)$.

5.2.3 Spectral theorem for unitary operators

U - unitary operator on $\mathcal{H}$, $f \in \mathcal{H}$, and

$$a_n = \langle U^n f, f\rangle, \quad a \succcurlyeq 0$$

The measure ρ such that $\hat{\rho} = a$ is called the spectral measure of U at f.

Exercise 23. Construct an isometry $V : \overline{span}\{U^n f\} \leftrightarrow \mathrm{L}_2(\rho)$, such that $(V^{-1}UV)g = e_1 g$. (spectral theorem for unitary operators).

6 Overview

Functions on $\mathbb{T} \leftrightarrow$ sequence.

- $f \in L_1(\mathbb{T}) \to \hat{f}(p) = \int f(x)\overline{e_p(x)}\,\mathrm{d}x$, where $e_p(x) = \exp(2\pi i p x)$
- more generally: $\mu \in \mu(\mathbb{T}) \to \hat{\mu}(p) = \int \overline{e_p(x)}\,\mathrm{d}\mu(x)$.
- even more generally: $\phi \in \mathcal{B}'$

Idea: sometimes $\hat{f}$ is more accessible than f, especially problems invariant under shifts (heat equation, equidistribution mod 1, etc.).

Basic questions:

(a) Given f, how does $\hat{f}$ behave?

(b) Given a sequence a, does there exist f such that $\hat{f} = a$, is it unique, and how to reconstruct f from a?

Brief answers:

(a) "The nicer f is, the faster $\hat{f}$ decays" (works nicely in some spaces, less cleanly in others)

(b) Uniqueness / reconstruction: Fejer method - very general. Existence: only special results when answer for the previous item is very precise. More intuitive method for reconstruction: "$f = \sum \hat{f}(p) e_p$":

  - works in L_2
  - in $L_1 \to \mathcal{C}(\mathbb{T})$ - requires additional assumptions

Generalisations: often locally compact abelian groups (LCA), particularly $\mathbb{R}$.