Random $G(N,p)$ ensemble

It is a canonical network ensemble satisfying the constraint

$$\sum_{\underline{a}} \left(\sum_{i<j} a_{ij} \right) P(\underline{a}) = \bar{L}$$

This ensemble has probability

$$\boxed{P(\underline{a}) = \frac{1}{\boxed{Z}} e^{-\lambda \sum_{i<j} a_{ij}}}$$

We want to show

$$P(\underline{a}) = p^{L} (1-p)^{N(N-1)/2 - L}$$

where $\quad p = \dfrac{2\bar{L}}{N(N-1)}$

We start by calculating Z.

$$Z = \sum_{\underline{a}} \underbrace{e^{-\lambda \sum_{i<j} a_{ij}}} =$$

$$Z = \underbrace{\sum_{a_{12}=0,1} \sum_{a_{13}=0,1} \cdots \sum_{a_{N-1,N}=0,1}}_{\sum_{\underline{a}}} \underbrace{\prod_{i<j} e^{-\lambda a_{ij}}}_{e^{-\lambda \sum_{i<j} a_{ij}}} =$$

$$= \left(\sum_{a_{12}=0,1} e^{-\lambda a_{12}} \right) \left(\sum_{a_{13}=0,1} e^{-\lambda a_{13}} \cdots \right) \left(\sum_{a_{N-1,N}=0,1} e^{-\lambda a_{N-1,N}} \right)$$

$$= \prod_{i<j} \left(\sum_{a_{ij}=0,1} e^{-\lambda a_{ij}} \right) = \prod_{i<j} \left(1 + e^{-\lambda} \right)$$

$$\boxed{= \left(1 + e^{-\lambda}\right)^{\frac{N(N-1)}{2}} = Z}$$

I want to calculate $\underline{P}_{ij}$

$$\underline{P}_{ij} = \sum_{\underline{a}} a_{ij} \, P(\underline{a}) = \sum_{\underline{a}} a_{ij} \, \frac{e^{-\lambda \sum_{r<s} a_{rs}}}{Z}$$

$$p_{ij} = \sum_{a_{12}=0,1} \sum_{a_{13}=0,1} \cdots \sum_{a_{N-1,N}=0,1} a_{ij} \frac{1}{Z} \prod_{r<s} \underbrace{e^{-\lambda a_{rs}}}$$

$$= \frac{1}{Z} \left(\sum_{a_{12}=0,1} e^{-\lambda a_{12}} \right) \left(\sum_{a_{13}=0,1} e^{-\lambda a_{13}} \right) \cdots \underbrace{\left(\sum_{a_{ij}=0,1} a_{ij} e^{-\lambda a_{ij}} \right)}$$

$$\cdots \left(\sum_{a_{N,N-1}=0,1} e^{-\lambda a_{N,N-1}} \right)$$

$$p_{ij} = \frac{1}{Z} \left[\prod_{\substack{r<s \\ (r,s) \neq (i,j)}} \left(\sum_{a_{rs}} e^{-\lambda a_{rs}} \right) \right] \sum_{a_{ij}=0,1} \underbrace{a_{ij} \; e^{-\lambda a_{ij}}}$$

$$= \frac{1}{Z} \left(1 + e^{-\lambda}\right)^{\frac{N(N-1)}{2} - 1} e^{-\lambda}$$

$$Z = \left(1 + e^{-\lambda}\right)^{\frac{N(N-1)}{2}}$$

$$= \frac{\left(1 + e^{-\lambda}\right)^{\frac{N(N-1)}{2} - 1}}{\left(1 + e^{-\lambda}\right)^{\frac{N(N-1)}{2}}} e^{-\lambda} = \boxed{\frac{e^{-\lambda}}{1 + e^{-\lambda}} = p_{ij}}$$

p_{ij} is independent of (i,j), it is the same for every links.

$$p = p_{ij} = \frac{e^{-\lambda}}{1+e^{-\lambda}}$$

Our original constraint was

$$\sum_{\underline{a}} \left(\sum_{i<j} a_{ij} \right) P(\underline{a}) = \bar{L}$$

$$\sum_{i<j} \underbrace{\left(\sum_{\underline{a}} a_{ij} \, P(\underline{a}) \right)}_{p_{ij}} = \bar{L}$$

$$\sum_{i<j} p_{ij} = \sum_{i<j} p = p \, \frac{N(N-1)}{2} = \bar{L}$$

$$\boxed{p = \frac{2\bar{L}}{N(N-1)}}$$

You can also see that

$$P(\underline{g}) = \frac{1}{Z} e^{-\lambda \sum_{i<j} a_{ij}} = \prod_{ij} p_{ij}^{a_{ij}} (1-p_{ij})^{1-a_{ij}}$$

Left as an exercise

$$p_{ij} = \frac{e^{-\lambda}}{1+e^{-\lambda}}$$

Canonical ensemble with given expected degree sequence.

$$H(G) = \sum_{i=1}^{N} \lambda_i \sum_{j=1}^{N} a_{ij} = \sum_{i<j} (\lambda_i + \lambda_j) a_{ij}$$

$$H(G) = \sum_{i=1}^{N} \lambda_i \sum_{j=1}^{N} a_{ij} = \underbrace{\sum_{ij} \lambda_i a_{ij}}_{H} =$$

$$= \frac{1}{2} \left(\underbrace{\sum_{ij} a_{ij} \lambda_i}_{H} + \underbrace{\sum_{ij} a_{ji} \lambda_j}_{H} \right) =$$

$$= \frac{1}{2} \sum_{i,j} \Big(a_{ij} \lambda_i + \underbrace{a_{ji}}_{a_{ji} = a_{ij}} \lambda_j \Big) \xrightarrow{\text{symmetry}} = \frac{1}{2} \sum_{ij} a_{ij} (\lambda_i + \lambda_j)$$

Symmetry

$$\boxed{H(G) = \sum_{i<j} a_{ij} (\lambda_i + \lambda_j)}$$

We want to show that in the canonical ensemble

$$P(\underline{a}) = \frac{1}{Z} e^{-H(G)} = \frac{1}{Z} e^{-\sum_{i<j} a_{ij} (\lambda_i + \lambda_j)}$$

we have

$$\boxed{Z = \prod_{i<j} \left(1 + e^{-\lambda_i - \lambda_j} \right)}$$

$$\boxed{p_{ij} = \sum_{\underline{a}} a_{ij} P(\underline{a}) = \frac{e^{-\lambda_i - \lambda_j}}{1 + e^{-\lambda_i - \lambda_j}}}$$

where

$$\bar{k}_i = \sum_{j=1}^{N} p_{ij} = \sum_j \frac{e^{-\lambda_i - \lambda_j}}{1 + e^{-\lambda_i - \lambda_j}}$$

$$Z = \sum_{\underline{a}} e^{-\sum_{r<s} a_{rs}(\lambda_r + \lambda_s)} =$$

$$= \sum_{a_{12}=0,1} \sum_{a_{13}=0,1} \cdots \sum_{a_{N-1\,N}=0,1} \prod_{r<s} e^{-a_{rs}(\lambda_r + \lambda_s)}$$

$$= \left(\sum_{a_{12}=0,1} e^{-a_{12}(\lambda_1 + \lambda_2)} \right) \cdots \left(\sum_{a_{N-1,N}=0,1} e^{-a_{N-1,N}(\lambda_{N-1} + \lambda_N)} \right)$$

$$= \prod_{r<s} \left(\sum_{a_{rs}=0,1} \underbrace{e^{-a_{rs}(\lambda_r + \lambda_s)}} \right) =$$

$$= \prod_{r<s} \left(1 + e^{-(\lambda_r + \lambda_s)} \right) = Z$$

$$p_{ij} = \sum_{\underline{a}} a_{ij}\, P(\underline{a}) = \sum_{\underline{a}} a_{ij} \frac{e^{-\sum_{r<s} a_{rs}(\theta_r+\theta_s)}}{Z}$$

$$= \frac{1}{Z}\left[\prod_{\substack{r<s \\ (r,s)\neq(i,j)}} \left(1+e^{-\theta_r-\theta_s}\right)\right] e^{-\theta_i-\theta_j}$$

$$= \frac{e^{-(\theta_i+\theta_j)}}{1+e^{-(\theta_i+\theta_j)}} = p_{ij}$$

is now dependent on i and j

Constraint

$$\bar{k}_i = \sum_{\underline{a}} \left(\sum_{j=1}^{N} a_{ij}\right) P(\underline{a})$$

$$= \sum_{j=1}^{N}\left(\sum_{\underline{a}} a_{ij}\, P(\underline{a})\right) = \sum_{j=1}^{N} p_{ij}$$

$$\boxed{\bar{k}_i = \sum_{j=1}^{N} \frac{e^{-\lambda_i - \lambda_j}}{1 + e^{-\lambda_i - \lambda_j}}} \quad \forall i$$