Q1. Let $Y_1, Y_2, \dots$ be independent, exponentially distributed random variables with $\mathbb{E}\,Y_i = 1$. Show that $\mathbb{P}(Y_n > \log n \text{ i.o.}) = 1$.

Let $A_n := \{Y_n > \log n\}$. We have $\mathbb{P}(A_n) = \exp(-\log n) = 1/n$, and the events are independent because the Y_n's are independent. Since $\sum_n \mathbb{P}(A_n) = \infty$ the assertion follows from the Borel-Cantelli Lemma (part (b)).

Q2. Show that condition (i) in the three series theorem is necessary for convergence of the series.

If the condition fails then, as in Q1, by the Borel-Cantelli Lemma $|X_n| > c$ for infinitely many n, with probability one. But then $X_n \not\to 0$, and so $\sum_n X_n$ cannot converge.

Q3. Suppose rv's $X_1, \cdots, X_n$ independent, rv's $Y_1, \cdots, Y_m$ independent, and random vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_m)$ are independent. Show that the $(n+m)$ random variables $X_1, \cdots, X_n, Y_1, \cdots, Y_m$ are independent.

It is sufficient to show that the probability

$$\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n, Y_1 \le y_1, \cdots, Y_m \le y_n)$$

factors, because events $\{Z \le z\}$ generate $\sigma(Z)$ for real r.v. Z. We have the above probability equal to

$$\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n)\,\mathbb{P}(Y_1 \le y_1, \cdots, Y_m \le y_m)$$

by the independence of vectors, and

$$\mathbb{P}(X_1 \le x_1, \cdots, X_n \le x_n) = \prod_{j=1}^{n} \mathbb{P}(X_j \le x_j)$$

by the independence of X_j's. Similarly, $\mathbb{P}(Y_1 \le y_1, \cdots, Y_m \le y_m)$ factors.

Q4. Let $X_1, X_2, \dots$ be arbitrary random variables. Prove that if $\sum_{j=1}^{\infty} \mathbb{E}|X_j| < \infty$ then the series $\sum_{j=1}^{\infty} X_j$ converges absolutely with probability one.

Any number can be written as the difference of its positive and negative parts, $x = x_+ - x_-$, where $x_+ = \max(x, 0), x_- = \max(-x, 0)$. Note that $|x| = x_+ + x_-$. Representing this way each X_j, the claim is reduced to the case $X_j \ge 0$. Assuming X_j's nonnegative, we have nondecreasing sequence $S_n = X_1 + \cdots + X_n$ of partial sums, which has $\mathbb{E}S_n = \sum_{j=1}^{n} \mathbb{E}X_j < M$ for some large enough but finite M (by the assumption). By the Monotone Convergence Theorem $\mathbb{E}\sum_{j=1}^{\infty} X_j = \lim_{n\to\infty} \mathbb{E}S_n < M$. But this implies that $\sum_{j=1}^{\infty} X_j < \infty$ a.s., since otherwise the expectation of the series would be infinite.

Q5. Suppose $\mathbb{E}X$ exists. Argue that for every ϵ there exists δ such that $\mathbb{P}(A) < \delta$ implies

$$\mathbb{E}(|X| \cdot 1_A) < \epsilon$$

(where 1_A indicator of event A).

As in Q4, it is enough to consider $X \ge 0$. Let $X_n = X \cdot 1(X \le n)$, then $X_n \uparrow X$ and by the Theorem of Monotone Convergence also $\mathbb{E}X_n \uparrow \mathbb{E}X$. Thus for large enough n we have $\mathbb{E}(X \cdot 1(X > n)) < \varepsilon/2$. Set $\delta = \varepsilon/(2n)$, then $\mathbb{P}(B) < \delta$ implies

$$\mathbb{E}(X \cdot 1_B) = \mathbb{E}(X \cdot 1_B \cdot 1(X > n)) + \mathbb{E}(X \cdot 1_B \cdot 1(X \le n)) < \varepsilon/2 + n\varepsilon/(2n) = \varepsilon.$$

Q6. Show that $\mathbb{E}[XY] = \mathbb{E}X\,\mathbb{E}Y$ if the rv's are independent.

Start with $X = 1_A, Y = 1_B$, when the claim follows from independence of A and B. By linearity the identity is extended to *simple* r.v.'s of the form $X = \sum_{j=1}^n a_j 1_{A_j},\ Y = \sum_{i=1}^m b_j 1_{B_j}$. To the general case the identity is extended using definition of the Lebesque integral.

Q7. For three measures suppose $\mu \gg \nu \gg \rho$ and that μ, ν, ρ are σ-finite. Prove the chain rule for the Radon-Nikodým derivative:

$$\frac{d\rho}{d\mu} = \frac{d\nu}{d\mu}\frac{d\rho}{d\nu}.$$

Let

$$f = \frac{d\rho}{d\nu}, \quad g = \frac{d\nu}{d\mu}.$$

Choose pointwise increasing sequence of simple functions f_n to have $f_n \to f$. By the Monotone Convergence Theorem

$$\int_E f_n d\nu \to \int_E f d\nu = \rho(E), \quad \int_E f_n g d\mu \to \int_E f g d\mu$$

for any measurable set E. Now for measurable set A

$$\int_E 1_A d\nu = \nu(E \cap A) = \int_{E\cap A} g d\mu = \int_E 1_A g d\mu.$$

Since we can write $f_n = \sum a_j 1_{A_j}$ by linearity we have

$$\int_E f_n d\nu = \int_E f_n g d\mu.$$

Letting $n \to \infty$, we get $\rho(E)$ in the left-hand side, so passing to the limit

$$\rho(E) = \int_E f g d\mu.$$

Since E arbitrary, this completes the proof.

Q8. Let μ be a normal distribution $\mathcal{N}(m, \sigma^2)$, and ν the exponential distribution with parameter β. Argue that $\mu \gg \nu$ and find the Radon-Nikodym derivative $d\nu/d\mu$.

If $B \in \mathcal{B}(\mathbb{R})$ is a nullset under the normal distribution then (by the absolute continuity) also under the Lebesque measure, hence also under the exponential distribution. By the chain rule from Q7

$$\frac{d\nu}{d\mu}(x) = \frac{\beta e^{-\beta x}}{\frac{1}{\sqrt{2\pi\sigma}}\exp(-(x-m)^2/(2\sigma))}, \quad x \geq 0$$

and $\frac{d\nu}{d\mu}(x) = 0$ for $x < 0$.

Q8. Let $A_{i,j}$ be a system of disjoint events, with $\cup_{i,j} A_{i,j} = \Omega$. Let $A_i = \cup_j A_{i,j}$. Let $\mathcal{G}_2$ be generated by all $A_{i,j}$'s, and let $\mathcal{G}_1$ be generated by A_i's. Describe as precise as you can the random variables $\mathbb{E}[X|\mathcal{G}_1], \mathbb{E}[X|\mathcal{G}_2]$. Assuming $\mathbb{P}(A_{i,j}) > 0$, prove the tower property in this example.

The conditional expectations are simple random variables

$$\mathbb{E}[X|\mathcal{G}_2] = \sum_{i,j} x_{ij} 1_{A_{ij}}, \quad \mathbb{E}[X|\mathcal{G}_1] = \sum_i x_i 1_{A_i},$$

where

$$x_{ij} = \frac{\mathbb{E}[X 1_{A_{ij}}]}{\mathbb{P}(A_{ij})}, \quad x_i = \frac{\mathbb{E}[X 1_{A_i}]}{\mathbb{P}(A_i)} = \frac{\sum_j \mathbb{E}[X 1_{A_{ij}}]}{\mathbb{P}(A_i)} = \frac{\sum_j x_{ij}\, \mathbb{P}(A_{ij})}{\mathbb{P}(A_i)}.$$

In particular,

$$\mathbb{E}[1_{A_{ij}} | \mathcal{G}_2] = \sum_k \frac{\mathbb{E}[1_{A_{ij}} 1_{A_k}]}{\mathbb{P}(A_k)} 1_{A_k} = \frac{\mathbb{P}(A_{ij})}{\mathbb{P}(A_j)} 1_{A_j}.$$

Whence,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}_2]|\mathcal{G}_1] = \mathbb{E}\left[\sum_{i,j} x_{ij} 1_{A_{ij}} | \mathcal{G}_1\right] = \sum_{i,j} x_{ij}\, \mathbb{E}[1_{A_{ij}} | \mathcal{G}_1] = \sum_{i,j} x_{ij} \frac{\mathbb{P}(A_{ij})}{\mathbb{P}(A_i)} 1_{A_i} = \mathbb{E}[X|\mathcal{G}_1].$$