Ex sheet #5 Harmonic analysis

① Let $\varphi \in C^{m}(\mathbb{T})'$. Prove: $\hat{\varphi}(p) = O(|p|^{m})$, $|p| \to \infty$

② Let $a > 0$, and consider the space $\mathcal{B}_a$ of $f \in C^{\infty}(\mathbb{T})$ such that

$$\sum_{k=0}^{\infty} \frac{a^{2k}}{(k!)^2} \int_0^1 |f^{(k)}(x)|^2 \, dx < \infty .$$

(i) $\mathcal{B}_a$ is a homogeneous Hilbert space

(ii) If $|\operatorname{Im} z| < a$, the formula

$$\varphi_z(f) = \sum \frac{f^{(k)}(x)}{k!} (iy)^k \qquad (z = x + iy)$$

defines a continuous functional on $\mathcal{B}_a$.

(iii) Compute $\hat{\varphi}_z$

③ $(X_j)_{j \in \mathbb{Z}}$ - stationary Gaussian process, ρ - its spectral measure.

(a) $\rho\{0\} = 0 \iff \operatorname{Var} \frac{X_1 + \dots + X_N}{N} \to 0$

$\iff \frac{X_1 + \dots + X_N}{N} \to 0$ in distr.

(b) Compute ρ and check the cond-s of (a) for (i) iid std Gaussians, (ii) $X_j \equiv X_0 \sim N_{\mathbb{C}}(0,1)$

④ (a) $a = (a_p)_{p \in \mathbb{Z}}$ is the seq-ce of Fourier coef-s of $\varphi \in M(\mathbb{T})$ iff $\exists C > 0$ $\forall k \geq 1$, $\forall z_1, \dots, z_k \in \mathbb{C}$

$$\left| \sum_{p,q=1}^{k} a_{p-q} \bar{z}_p z_q \right| \leq C \sum_{p=1}^{k} |z_p|^2$$

(b) Find φ corresponding to

$$a_p = \begin{cases} 2, & p = 0 \\ -1, & p = \pm 1 \\ 0 \end{cases}$$