Ex sheet #3 — Harmonic analysis

① X, Y – $\mathbb{Z}$-valued random variables, assume: $\forall k \geq 0$ $\mathbb{E}X^k = \mathbb{E}Y^k = M_k$ where $M_{2k} \leq C^{2k}(2k)!$

Prove: $\mathbb{E}e^{2\pi i x X} = \mathbb{E}e^{2\pi i x Y}$, $x \in \mathbb{T}$ and consequently $X = Y$ in distribution

② Let $f \in L_1(\mathbb{T})$ be such that

- $f(\pm x) = (1+o(1)) A_\pm x^{-\alpha}$, $x \to +0$ for some $\alpha \in [0,1)$
- $f(\pm x)/x^{-\alpha}$ is in $C^1[0, 2/3]$

then $\hat{f}(p) = \left(A_+ (2\pi i p)^{\alpha-1} + A_- (-2\pi i p)^{\alpha-1}\right) \cdot \Gamma(\alpha+1) + \bar{o}(1/|p|)$.

③ $0<\alpha<1$, $f(x) = \sum_{p \geq 1} 3^{-p\alpha} \cos(2\pi 3^p x)$

Prove $f \in \mathrm{Lip}_\alpha$; $\limsup_{p \to \infty} |\hat{f}(p)| \cdot p^\alpha > 0$

④ (a) $\omega_{L_2(\mathbb{T})}(h; f) \leq \omega_{L_1(\mathbb{T})}(h; f) \cdot \omega_{C(\mathbb{T})}(h; f)$

(b) If $f' \in L_1(\mathbb{T})$ (ie f is of bounded variation) and

$$\int_0^1 \omega_{C(\mathbb{T})}(h; f) \frac{dh}{\sqrt{h}} < \infty$$

then $\sum_p |\hat{f}(p)| < \infty$.

⑤ If $\sum |\hat{f}(p)| < \infty$ and $\sum |\hat{g}(p)| < \infty$ then $\sum |(fg)^\wedge(p)| < \infty$ (ie. $A(\mathbb{T})$ is an algebra)