Ex sheet 1 Harmonic Analysis

① Let $u_t(j)$ $(t \in \mathbb{Z}_+,\ j \in \mathbb{Z}/N\mathbb{Z})$ be the sol-n to the heat eq-n

$$\begin{cases} u_{t+1}(j) - u_t(j) = \frac{1}{2}\left[u_t(j+1) + u_t(j-1) - 2u_t(j)\right] \\ u_0(j) = \begin{cases} N, & j=0 \\ 0 \end{cases} \end{cases}$$

Prove: (a) $\max_j |u_t(j) - 1| \leq 2\exp(-\frac{t}{N^2})$ for $t \geq N^2$

(b) $\max_j |u_t(j) - 1| \geq \frac{1}{2}$ for $t \leq \frac{N^2}{100}$

② $u_0 \in C(\mathbb{T})$, $u_t(x) = \int_0^1 u_0(y) P_t(x-y)\,dy$

where

$$P_t(x) = \sum_{p \in \mathbb{Z}} \exp(-2\pi^2 p^2 t)\, e_p(y)$$

Prove: (a) $\dot{u}_t = \frac{1}{2} u_t''$, $t > 0$

(b) $u_t(x) \rightrightarrows \overline{u_0} = \int_0^1 u_0(y)\,dy$, $t \to \infty$

③ Prove: $P_t(x) = \frac{1}{\sqrt{2\pi t}} \sum_{n \in \mathbb{Z}} \exp\left(-\frac{(x-n)^2}{2t}\right)$

Hint $\int_{-\infty}^{\infty} \exp(-A\xi^2 + iB\xi)\,d\xi$

$= \sqrt{\frac{\pi}{A}} \exp(-B^2/4A)$, $A > 0$.

④ Solve the equation $\dot{u}_t = \frac{1}{2} u_t''$ $x \in [0, \frac{1}{2}]$ with the boundary condition $u_t(0) = u_t(\frac{1}{2}) = 0$.

Hint Define $u_t(-x) = -u_t(x)$, $-\frac{1}{2} \leq x \leq 0$.

What happens to the sol-n as $t \to +\infty$?

⑤ Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

$\operatorname{dist}(p\alpha, \mathbb{Z}) \geq a|p|^{-\tau}$, $p \in \mathbb{Z}$

and let $\varepsilon_1, \dots, \varepsilon_K$ be iid random signs, $\mathbb{P}(\varepsilon_k = \pm 1) = \frac{1}{2}$. Prove:

$$\sup_{(a,b)} \left| \mathbb{P}\left\{ \left\{ \sum_{k=1}^{K} \varepsilon_k \alpha \right\} \in (a,b) \right\} - (b-a) \right| \leq \frac{C_{a,\tau}}{K^{1/2\tau}}.$$