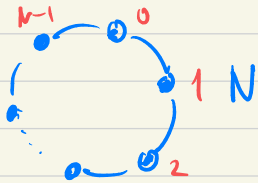
(I) Motivating example #1 (Fourier)

The heat equation



$u_t(j)$ - temperature of the j-th piece at time $t \in \mathbb{Z}_+$

$$\underbrace{u_{t+1}(j) - u_t(j)}_{\text{temp. increment at } j \text{, time } t} = \frac{\varkappa}{2}\Big[\underbrace{(u_t(j+1) - u_t(j))} + \underbrace{(u_t(j-1) - u_t(j))}\Big]$$

$0 < \varkappa < 1$

what happens as $t \to \infty$?

- $\overline{u_t} = \frac{1}{N} \sum u_t(j)$ is preserved $= \overline{u_0}$

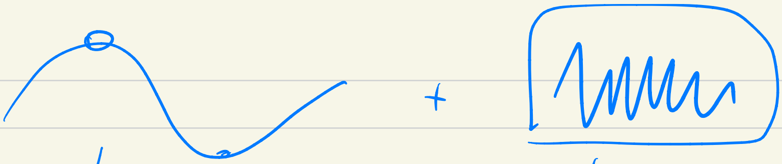
- $u_t \xrightarrow[t \to \infty]{} \overline{u_0}$??

Quantitative q-n: $u_0(j) = \begin{cases} N & j=0 \\ 0 & \end{cases}$ $N \gg 1$

after how many steps $u_t \approx 1$ everywhere ?

$(\cancel{\sqrt{N}}$? N? N^2? e^N?$)$

Idea: "fast" & "slow" fluctuations.



↳ take more time ↳ even out fast

Main trick: $e_p(j) = \exp\left(2\pi i \frac{jp}{N}\right)$.

$p \in \mathbb{Z}/N\mathbb{Z}$

* "wave of wavelength $\frac{N}{p}$"
* character prop. $e_p(j+k) = e_p(j) \cdot e_p(k)$

Ex there are no other characters

$$e : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^\times$$

Claim e_p form an orthogonal basis

$$\langle e_p, e_q \rangle = \frac{1}{N} \sum_{j=0}^{N-1} e_p(j)\, \overline{e_q(j)} = \begin{cases} 1, & p=q \\ 0, & \text{otherwise.} \end{cases}$$

Decompose: $u_t = \sum \underbrace{\hat{u}_t(p)} e_p$

$$\hat{u}_t(p) = \frac{1}{N} \sum_j u_t(j)\, \overline{e_p(j)} = \langle u_t, e_p \rangle.$$

Plug this into our heat equation:

$$u_t(j \pm 1) = \sum \hat{u}_t(p)\, e_p(j \pm 1)$$

$$u_{t+1} - u_t = \sum_j \hat{u}_t(p)\, e_p(\pm 1)\; e_p(j)$$

$$\sum_j \boxed{\left(\hat{u}_{t+1}(p) - \hat{u}_t(p)\right)} e_p =$$

$$= \frac{\varkappa}{2} \sum_j \boxed{\left[\hat{u}_t(p)\left(e_p(1) - 1\right) + \hat{u}_t(p)\left(e_p(-1) - 1\right)\right]} \cdot e_p \cdot$$

$$\boxed{\forall p \in \mathbb{Z}/N\mathbb{Z} \quad \hat{u}_{t+1}(p) - \hat{u}_t(p) = \frac{\varkappa}{2}\, \hat{u}_t(p)\left[e_p(1) + e_p(-1) - 2\right].}$$

$$\boxed{e_p(j) = e^{2\pi i p j / N}}$$

$$\hat{u}_{t+1}(p) = \hat{u}_t(p)\left[1 - \varkappa\left(1 - \cos\frac{2\pi p}{N}\right)\right]$$

$$\boxed{\hat{u}_t(p) = \left[1 - \varkappa\left(1 - \cos\frac{2\pi p}{N}\right)\right]^t \hat{u}_0(p)}$$

* $p = 0$ $\quad \hat{u}_t(0) = \overline{u_t}$

$$\hat{u}_t(0) = \hat{u}_0(0)$$

* $p \neq 0$ $\quad \hat{u}_t(p) \longrightarrow 0$ exponentially

the rate of exp decay is faster for large p

the slowest one is $p = 1$ (and -1).

$$\hat{u}_t(1) = \left(1 - x\left(1 - \cos\frac{2\pi}{N}\right)\right)^t \hat{u}_0(1)$$

$$\sim \left(1 - x\frac{(2\pi)^2}{2N^2}\right)^t \hat{u}_0(1)$$

$$\sim \exp\left(-\frac{2\pi^2 x}{N^2} t\right) \hat{u}_0(1).$$

becomes small when $t \gg N^2$.

Ex Let $u_0(j) = \begin{cases} N & j = 0 \\ 0 & j \neq 0 \end{cases}$

(1) For $t \geq CN^2$ $\max |u_t(j) - 1|$

$$\leq C\exp\left(-\frac{t}{CN^2}\right)$$

(2) For $t \leq \frac{1}{C}N^2$, $\max|u_t(j) - 1| \geq \frac{1}{2}$.

More realistically: continuous time, cont. space

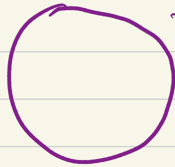
(a) instead of $t \in \mathbb{Z}_+$ $t \in \mathbb{R}_+$

$$\frac{\partial}{\partial t} u_t(j) = \frac{x}{2}\left[u_t(j+1) + u_t(j-1) - 2u_t(j)\right]$$

Ex develop this theory.

(b) continual space variables

$$\frac{\partial u_t(x)}{\partial t} = \frac{\varkappa}{2} \frac{\partial^2 u_t(x)}{\partial x^2}$$



$\mathbb{T} = \mathbb{R}/\mathbb{Z}$

"circular rod"

or $\boxed{\dot{u}_t = \frac{\varkappa}{2} u_t''}$

Characters : $e_p(x) = \exp(2\pi i p x)$

$p \in \mathbb{Z} \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$

$$e_p(x+y) = e_p(x) e_p(y)$$

$$L_2(\mathbb{T}) = \{ f : \mathbb{T} \to \mathbb{C} : \int |f(x)|^2 dx < \infty \}$$

<u>Claim</u> e_p form an orthonormal basis of $L_2(\mathbb{T})$.

* $\langle e_p, e_q \rangle = \int e_p(x) \overline{e_q(x)} \, dx = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}$
* linear combs of e_p are dense.

(Weierstrass $\Rightarrow$ dense in $C(\mathbb{T})$)

$$u_t(x) \overset{(?)}{=} \sum_{p \in \mathbb{Z}} \hat{u}_t(p) e_p(x)$$

$\hookrightarrow \hat{u}_t(p) = \langle u_t, e_p \rangle$

$$e_p''(x) = -4\pi^2 p^2 e_p(x)$$

$$\boxed{e_p(x) = e^{2\pi i p x}}$$

$$\boxed{\dot{\hat{u}}_t(p) = -2\pi^2 p^2 \hat{u}_t(p)}$$

$$\boxed{\dot{u}_t = \tfrac{1}{2} u_t''}$$

$$\boxed{\hat{u}_t(p) = \exp(-2\pi^2 p^2 t)\, \hat{u}_0(p)}$$

$$\hat{u}_t(0) = \hat{u}_0(0)$$

$$u_t(x) = \sum \hat{u}_t(p)\, e_p(x)$$

$$= \sum \exp(-2\pi^2 p^2 t) \cdot \overbrace{\int u_0(y)\, \overline{e_p(y)}\, dy}^{e}\; e_p(x)$$

$$= \int dy \left[\sum_p \exp(-2\pi^2 p^2 t)\, e_p(x-y) \right] u_0(y)$$

$$= e_p(x)\, e_p(-y) = e_p(x)\, \overline{e_p(y)}$$

Denote $\boxed{P_t(x) = \sum_p \exp(-2\pi^2 p^2 t)\, e_p(x)}$

$$u_t(x) = \int P_t(x-y)\, u_0(y)\, dy = (P_t * u_0)(x)$$

* Easy: $\dot{u}_t = \frac{1}{2} u_t''$, $t > 0$.
* Easy: $t \to \infty$ $\quad u_t \rightrightarrows \overline{u_0}$

$$\left(P_t \xrightarrow[t \to \infty]{} e_0 \equiv 1 \right)$$

* harder: $u_t \underset{t \to +0}{\rightrightarrows} u_0$!

Ex

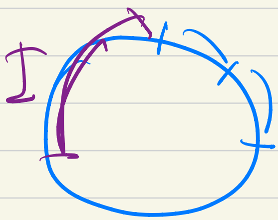
Magic formula:

$$P_t(x) = \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{(x-n)^2}{2t}\right)$$

Motivating example #2 (Weyl)

$\alpha \in \mathbb{R}\setminus\mathbb{Q}$ $\qquad \{k\alpha\}$

↪ fractional part



$$\#\{k\in\{1,\dots,K\} : \{k\alpha\}\in I\}$$
$$= |I|\cdot K + o(K).$$

What about $\{k^2\alpha\}$?

$\sum c_k x^k$ some $c_j \notin \mathbb{Q}$, $j\neq 0$

Theorem (Weyl) If $P(x)$ is a pol. of deg ≥ 1, and it has at least 1 (non-const) irrational coefficient, then $\{P(k)\}$ is equidistributed mod 1

$$\#\{k\in\{1,\dots,K\} : \{P(k)\}\in I\} = |I|\,|K| + o(K).$$

μ — prob. measure on $\mathbb{T}$

($\mu = \delta_0$)

μ_t - shift of μ by t

$$\int f(x)\, d\mu_t(x) = \int f(x+t)\, d\mu(x)$$

$$(\delta_0)_t = \delta_t$$

$$\boxed{T_K \mu = \frac{1}{K}\left(\mu + \mu_\alpha + \mu_{2\alpha} + \dots + \mu_{(K-1)\alpha}\right)}$$

ef $\boxed{\mu = \delta_0}$ this measure is $\int \ldots$

$$(T_K \mu)(I) = \frac{1}{K} \# \{1 \le k \le K : \{k\alpha\} \in I\}.$$

We expect: $T_K \mu \xrightarrow[K \to \infty]{\text{(weak)}}$ uniform measure

[reminder: $\nu_k \xrightarrow{\text{weakly}} \nu$

if $\forall f \in C(\mathbb{T})$ $\int f\, d\nu_k \to \int f\, d\nu$].

lin combs of e_p are dense in $C(\mathbb{T})$

suff: $\int e_p\, d\nu_k \to \int e_p\, d\nu$ $(\forall p)$

$$\hat{\nu}(p) = \int e_p \, d\nu = \int \exp(-2\pi i p x) \, d\nu(x)$$

$$\widehat{\mathrm{mes}}(p) = \int e_p(x)\,dx = \begin{cases} 1 & p=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\nu_k \xrightarrow{w} \nu \iff \forall p \neq 0 \quad \hat{\nu}_k(p) \xrightarrow[k\to\infty]{} 0}$$

$$\mu_K(I) = \#\{0 \le k \le K-1 : \{k\alpha\} \in I\} / K$$

$$= \frac{1}{K} \sum_{k=0}^{K-1} \delta_{\{k\alpha\}}.$$

$$\hat{\mu}_K(p) = \frac{1}{K} \sum_{k=0}^{K-1} e_{-p}(k\alpha)$$

$$= \frac{1}{K} \sum \exp(-2\pi i p k\alpha)$$

$$= \frac{1 - \exp(-2\pi i K p\alpha)}{K \underbrace{(1-\exp(-2\pi i p\alpha))}_{\neq 0}} \xrightarrow[K\to\infty]{} 0 \text{ for any } p!$$

Q-n How to bound

$$\sup\{|\mu_K(I) - |I|| : I\text{-interval}\}$$

Clean : $\gtrsim \frac{1}{K}$

hope: α is not well approx we get $\lesssim \frac{1}{K^{1-\varepsilon}}$

<u>Thm</u> (Erdős–Turán)

Let μ be a probability measure on $\mathbb{T}$ then for any $N \geq 1$

$$\sup_I \left| \mu(I) - |I| \right| \leq C \left\{ \frac{1}{N} + \sum_{p=1}^{N} \frac{|\hat{\mu}(p)|}{p} \right\}.$$

How to apply this?

assume : α is of "type τ"

$\operatorname{dist}(p\alpha, \mathbb{Z}) \geq a/|p|^{\tau}$

* for $\alpha = \sqrt{2}$ this holds with $\tau = 1$
* for typical α this holds with $\tau = 1+\varepsilon$

<u>Naive</u>
$$\sum_{p=1}^{N} \frac{|\hat{\mu}_K(p)|}{p} \leq \sum_{p=1}^{N} \frac{2}{K} \frac{1}{|1-e^{2\pi i p\alpha}|} \cdot \frac{1}{p} \circledast$$

$$\Big|1-\underbrace{e^{2\pi i p\alpha}}\Big| \gtrsim \mathrm{dist}(p\alpha, \mathbb{Z}) \geq \frac{a}{|p|^{\tau}}$$

$$\circledast \lesssim \frac{1}{aK}\sum_{p=1}^{N} \frac{1}{p^{1-\tau}} \lesssim \frac{1}{aK}\cdot N^{\tau}$$

Erdős–Turán:

$$\sup \Big| \mu_K(I) - |I| \Big| \lesssim \left(\frac{1}{N} + \frac{N^{\tau}}{aK}\right)$$

equate: $\frac{1}{N} = \frac{N^{\tau}}{aK}$

$$N = (aK)^{\frac{1}{1+\tau}}$$

$$\sup_I \Big| \mu_a(I) - |I| \Big| \lesssim \frac{1}{(aK)^{\frac{1}{1+\tau}}}$$

(not better than $K^{-1/2}$)

How to bound $\sum_{1}^{N} \frac{|\hat{\mu}_K(p)|}{p}$ more carefully ?

$$N = 2^M - 1$$

$$(*) \quad \sum_{m=1}^{M} \sum_{p=2^{m-1}}^{2^m - 1} \frac{|\hat{\mu}_k(p)|}{p} \le \sum_{m=1}^{M} \frac{1}{2^{m-1}} \boxed{\sum_{p=2^{m-1}}^{2^m-1} |\hat{\mu}_k(p)|}.$$

$$|\hat{\mu}_k(p)| \lesssim \frac{1}{K \operatorname{dist}(p\alpha, \mathbb{Z})}.$$

Clear: If $2^{m-1} \le p \le 2^m - 1$

$$\operatorname{dist}(p\alpha, \mathbb{Z}) \ge a p^{-\tau} \ge a 2^{-m\tau}.$$

assume: $\boxed{\operatorname{dist}(p\alpha, \mathbb{Z}) \le \frac{2^{l-1} a}{2^{m\tau}}}$

$$\boxed{\operatorname{dist}(p'\alpha, \mathbb{Z}) \le \frac{2^{l-1} a}{2^{m\tau}}}$$

$$\operatorname{dist}((p-p')\alpha, \mathbb{Z}) \le \frac{2^l a}{2^{m\tau}}$$

$\underline{a}$ $|p-p'|^{\le \tau}$

$$\boxed{|p-p'| \geq 2^{m-\ell/\tau}}$$

The number of such p's which lie in $[2^{m-1}, 2^m-1]$ is $\leq 2^{m-1}/2^{m-\ell/\tau} + 1 \leq 2^{\ell/\tau+1}$

$$(*) = \sum_{m=1}^{M} \frac{1}{2^m} \sum_{p=2^{m-1}}^{2^m} \frac{|\hat{\mu}_k(p)|}{p}$$

$$\leq \frac{1}{K}\sum_{m=1}^{M} \frac{1}{2^m} \sum_{\ell=1}^{m\tau} \frac{2^{m\tau}}{2^{\ell-1}a} \cdot \underbrace{2^{\frac{\ell}{\tau}+1}}_{\substack{\#\text{ of } p\text{-s} \\ \text{for which} \\ \operatorname{dist}(p\alpha,\mathbb{Z}) \leq \frac{2^{\ell-1}a}{2^{m\tau}}}}$$

$$\lesssim \frac{1}{Ka} \sum_{m=1}^{M} \frac{2^{m\tau}}{2^m} \cdot \underbrace{\sum_{\ell=1}^{m\tau} 2^{-\ell+\ell/\tau}}_{}$$

if $\tau=1$, this is m

if $\tau>1$ this is $\lesssim 1$

$$\frac{1}{Ka} \cdot \sum_{m=1}^{M} 2^{m(\tau-1)} \cdot \begin{cases} 1, & \tau > 1 \\ m & \tau = 1 \end{cases}$$

$$\le \begin{cases} M^2/Ka, & \tau = 1 \\ 2^{M(\tau-1)}/Ka, & \tau > 1. \end{cases}$$

$$\lesssim \begin{cases} \log^2 N/Ka & \tau = 1 \\ N^{\tau-1}/Ka & \tau > 1. \end{cases}$$

<u>E-T</u>:

$$\sup \left| \mu_K(I) - |I| \right| \lesssim$$

$$\lesssim \left\{ \frac{1}{N} + \begin{array}{ll} \log^2 N/Ka & \tau = 1 \\ N^{\tau-1}/Ka & \tau > 1 \end{array} \right\}$$

$$\lesssim \begin{cases} \log^2 K/K, & \tau = 1 \\ 1/K^{1/\tau} & \tau > 1. \end{cases}$$

Known: for $\sqrt{2}$ the true order of magnitude is $\log K/K$

Ex $\varepsilon_1, \dots, \varepsilon_n$ iid ± 1

$$\sup_I \left| \mathbb{P}\left\{ \left\{ \sum_{k=1}^{K} \varepsilon_k \cdot \alpha \right\} \in I \right\} - |I| \right| \lesssim K^{-\frac{1}{2\tau}} \quad (\tau \geq 1).$$