1 Lecture 4 : The Brownian motion.

4.1 Finite-dimensional distributions, Gaussian processes

$(X(t), t \geq 0)$ $\quad t \in \mathbb{R}_+$ time parameter

$X(t)$ a random variable on some $(\Omega, \mathcal{F}, \mathbb{P})$.

Characterized by finite-dimensional distributions

$$\mu_{t_1,\dots,t_k}(A) = \mathbb{P}((X(t_1),\dots,X(t_k)) \in A) \qquad A \in \mathcal{B}(\mathbb{R}^k)$$

$k \in \mathbb{N}, \qquad 0 \leq t_1 < t_2 < \dots < t_k$

$$\mu_{t_1,\dots,t_k}(A \times \mathbb{R}) = \mu_{t_1,\dots,t_{k-1}}(A) \qquad A \in \mathcal{B}(\mathbb{R}^{k-1})$$

— consistent

Conversely, consistent finite-dimensional distributions define a unique probability measure on space of $x : \mathbb{R}_+ \to \mathbb{R}$ (paths) endowed by the product σ-algebra $\mathcal{B}(\mathbb{R}^{[0,\infty)})$, generated by $\{x : x(t) \in A\}$ (cylinders with base $A \in \mathcal{B}(\mathbb{R})$).

<u>Definition</u> $(X(t), t \geq 0)$ and $(Y(t), t \geq 0)$ defined on same $(\Omega, \mathcal{F}, \mathbb{P})$ are <u>versions</u> of same process if $\mathbb{P}(X(t) = Y(t)) = 1$ for each $t \geq 0$.

Deep question: given finite-dim. distributions, is there a version of the process which has continuous (or smooth...) paths?

(2) <u>Definition</u> A random vector has multivariate normal distribution (or is Gaussian) if

$$\sum_{i=1}^{k} a_i \xi_i \qquad ((\xi_1, \dots, \xi_k) \sim MVN$$

is 1-dim. normal for all choices of $a_1, \dots, a_k \in \mathbb{R}$.

— need not have a density, e.g. (ξ, ξ) is bivariate normal for $\xi \sim N(0,1)$.

Then $(\xi_1, \dots, \xi_k)$ is characterized by the

mean vector $(m_1, \dots, m_k)$ $\quad m_j = \mathbb{E}\xi_j$

covariance matrix Σ, $\quad \Sigma_{ij} = Cov(\xi_i, \xi_j)$.

Properties:

(i) $(\xi_1, \dots, \xi_k) M$ is Gaussian for any matrix M

(ii) $\xi_1, \dots, \xi_k$ independent $\Leftrightarrow$ Σ diagonal

(iii) a density exists if $\det \Sigma \neq 0$

$$f(\vec{x}) = \frac{1}{(2\pi)^{k/2}\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{m})\, \Sigma^{-1} (\vec{x}-\vec{m})^*\right]$$

<u>Definition</u> $(X(t), t \geq 0)$ is called Gaussian process if every vector $(X(t_1), \dots, X(t_k))$ has a MVN distribution.

Characterized by mean function $m(t) = \mathbb{E}X(t)$ and the covariance function

$$c(s,t) = Cov(X(s), X(t)).$$

3 For every $m(t)$ and positive definite $c(s,t)$ there exists a Gaussian process.

The Brownian motion appears in the case

$$m(t) = 0, \quad c(s,t) = s \wedge t$$

4.2 Processes with stationary independent increments (Lévy processes)

Definition $(X(t), t \geq 0)$ has stationary increments if $X(t) - X(s)$ has distribution only depending on $t-s$ for $0 \leq s < t$.

Has independent increments, if for $k \in \mathbb{N}$ the increments $0 \leq t_0 < t_1 < \dots < t_k$

$$X(t_i) - X(t_{i-1}), \quad i = 1, \dots, k$$

are independent random variables.

Example Poisson process $(N(t), t \geq 0)$

$$N(t) - N(s) \sim \text{Poisson}(\lambda(t-s)).$$

Example Compound Poisson process

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad \text{where}$$

$(N(t), t \geq 0)$ Poisson process, independent of $Y_1, Y_2, \dots$ – i.i.d. random variables.

— Has discontinuous paths, with jumps.

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Proposition

(i) $(X(t), t \geq 0)$ a Lévy process with $X(t) \sim \mathcal{N}(0, t)$

$\iff$

(ii) $(X(t), t \geq 0)$ is Gaussian with $m(t) = 0$, $c(t, s) = s \wedge t$ $\quad s, t \geq 0$.

4.3 Scaled random walk.

Let $\xi_1, \xi_2, \dots$ iid, $\mathbb{E}\xi_i = 0$, $\operatorname{Var} \xi_i = 1$.

$(S_n, n \in \mathbb{N})$ $\quad S_n = \xi_1 + \dots + \xi_n$ a random walk

By Central Limit Theorem

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} B(1), \quad \text{a } \mathcal{N}(0,1) \text{ r.v.}$$

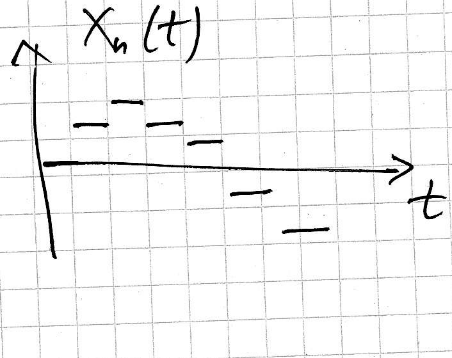
$$\left(\iff \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \leq x \right) \to \Phi(x), \quad x \in \mathbb{R} \right)$$

Define

$$X_n(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \qquad t \geq 0$$

$X(0) = 0$

– scaled random walk



(5) Fix $0 = t_0 < t_1 < \dots < t_k$

$$X_n(t_i) - X_n(t_{i-1}) = \sum_{j=\lfloor n t_{i-1} \rfloor + 1}^{\lfloor n t_i \rfloor} \xi_j$$

independent increments

$$(X_n(t_1), \dots, X_n(t_k)) \xrightarrow{d} (B(t_1), \dots, B(t_k))$$

$$\Leftrightarrow \quad X_n(t_i) - X_n(t_{i-1}) \xrightarrow{d} B(t_i) - B(t_{i-1})$$

where $\mathrm{Cov}(B(t_i), B(t_j)) = t_i$

$\mathbb{E} B(t_i) = 0$

$$\Leftrightarrow \quad B(t_i) - B(t_{i-1}) \quad i \le k \text{ independent}$$

$$\sim \mathcal{N}(0, t_i - t_{i-1}).$$

Thus finite-dimensional distributions of $(X_n(t), t \ge 0)$ converge to finite-dimensional distributions of a Gaussian process.

Q: Is there a limit for $\max_{t \in [0,1]} X_n(t)$?

(6) **Definition** A (standard) Brownian motion $(B(t), t \geq 0)$:

(i) $B(0) = 0$

(ii) continuous paths (a.s.)

(iii) stationary, independent increments

(iv) $B(t) \sim N(0,t)$, $t \geq 0$.

Covariance calculation: $s < t$

$\mathbb{E}\, B(s)\, B(t) =$

$\mathbb{E}\, B(s) \left(B(s) + [B(t) - B(s)] \right) =$

$= \mathbb{E}\, B(s)^2 + \mathbb{E}\, B(s) \left(B(t) - B(s) \right)$

$= s + \mathbb{E} B(s)\ \mathbb{E}\left(B(t) - B(s) \right) = s.$

(v) Gaussian with $c(s,t) = s \wedge t$.

Transformations BM $\to$ BM:

(a) $W(t) = B(t+c) - B(t)$, $c > 0$

(b) $W(t) = B(ct)/\sqrt{c}$

(c) $W(t) = B(1-t) - B(1)$ $\qquad t \in [0,1]$

(d) $W(t) = t\, B(1/t)$.

⑦ 4.5 Existence of the BM.

Q: Is there a continuous-path version?

By Kolmogorov's extension define Gaussian process $(B(t),\ t \in \mathbb{Q}_1)$ $\quad \mathbb{Q}_1 = \mathbb{Q} \cap [0,1]$ rationals

with $m(t) = 0$, ~~c(s,t)~~

$$c(s,t) = s \wedge t \qquad s,t \in [0,1].$$

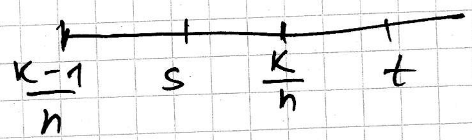
If the process is uniformly continuous on $\mathbb{Q}_1$, we may extend by continuity to $[0,1]$.

$$\Delta_n := \sup_{\substack{s,t \in \mathbb{Q}_1 \\ |s-t| < 1/n}} |B(t) - B(s)|$$

Need to show $\Delta_n \xrightarrow{a.s.} 0$ as $n \to \infty$.

Let
$$Y_{k,n} = \sup_{s,t \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \cap \mathbb{Q}} |B(t) - B(s)|$$

then
$$\Delta_n \le 3 \max_{1 \le k \le n} Y_{k,n}$$



$$|B(t) - B(s)| \le \left|B(t) - B\left(\tfrac{k-1}{n}\right)\right| + \left|B\left(\tfrac{k}{n}\right) - B\left(\tfrac{k-1}{n}\right)\right| + \left|B(s) - B\left(\tfrac{k}{n}\right)\right|$$

(8) $\mathbb{P}\left(\max_{1\le k\le n} Y_{k,n} \geq \varepsilon\right) \le$

$$\sum_{k=1}^{n} \mathbb{P}(Y_{k,n} \geq \varepsilon) = \text{(stationary increments)}$$

$$n\,\mathbb{P}(Y_{1,n} \geq \varepsilon)$$

<u>Proposition</u> $(B(t),\ t\in\mathbb{Q})$ is a martingale

$$\mathbb{E}(B(t) \mid \mathcal{F}_s) = B(s)$$

$$\mathcal{F}_s = \sigma(B(u);\ u\in\mathbb{Q},\ u\le s)$$

$\Rightarrow \left([B(t)]^4,\ t\in\mathbb{Q}_1\right)$ submartingale

since $x \mapsto x^4$ is convex.

Using a maximal inequality (Lecture 3)

$$\mathbb{P}(Y_{1,n} \geq \varepsilon) = \mathbb{P}\left(\max_{t\in\mathbb{Q}\cap[0,\frac{1}{n}]} |B(t)| \geq \varepsilon\right)$$

$$\le \frac{1}{\varepsilon^4}\,\mathbb{E}\left[B\left(\tfrac{1}{n}\right)\right]^4$$

$$= \frac{3}{n^2\varepsilon^4}$$

using
$\mathbb{E}B^4(t) = t^2\,\mathbb{E}B^4(1) =$
$= 3t^2$

$$\Rightarrow \mathbb{P}\left(\max_{1\le k\le n} Y_{k,n}\right) < \frac{3}{n\varepsilon^4} \longrightarrow 0$$

as $n\to\infty$

9 It follows, that $\Delta_n \xrightarrow{\mathbb{P}} 0$

But $\Delta_1 \geq \Delta_2 \geq \dots$ hence $\Delta_n \xrightarrow{a.s.} 0$,

and $B(t)$ continuous on $\mathbb{Q}_1$,

and hence on $[0,1]$.

For $\underline{0 < \alpha < 1/2}$ $\underline{\text{Hölder continuity}}$

holds:
$$\sup_{s,t \in [0,1]} |B(t) - B(s)| < C|t-s|^{\alpha}$$

$C > 0$ constant.

Brownian path is nowhere differentiable!

$$(B(t), t \geq 0) \stackrel{d}{=} (tB(1/t), t \geq 0)$$

$$\Rightarrow \left(\frac{B(t)}{t}, t \geq 0\right) \stackrel{d}{=} (B(1/t), t \geq 0)$$

But $B(1/t)$ does not have a limit as $t \to 0$.

$$\Rightarrow \frac{B(t)}{t} = \frac{B(t) - 0}{t - 0}$$ has no $t \to 0$ limit.

Moreover, let $\mathcal{Z} = \{t : B(t) = 0\}$ zero set of BM

$\mathcal{Z}$ has no isolated points, $\lambda(\mathcal{Z}) = 0$

(topological Cantor set)

"Local time" which measures time the path spends near 0 is similar to Cantor ladder.

(10) 4.6 BM as a random series

Consider the Hilbert space $L^2([0,1], \mathcal{B}[0,1], \lambda)$
with scalar product

$$\langle f, g \rangle = \int_0^1 f(s) g(s) ds$$

Take orthonormal basis $\varphi_j \in L^2$

$$\langle \varphi_i, \varphi_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

and complete:

$$\langle f, \varphi_j \rangle = 0 \implies f = 0 \text{ a.s.}$$

Possibility: $1, \sqrt{2} \sin(2\pi j s), \sqrt{2} \cos(2\pi j s)$
$j \in \mathbb{N}$

Let $\xi_0, \xi_1, \dots$ be i.i.d. $\mathcal{N}(0,1)$, then the series

$$B(t) = \sum_{j=0}^{\infty} \xi_j \int_0^t \varphi_j(s) ds \qquad t \in [0,1]$$

converges uniformly in t almost surely and
defines a BM.

(11) **4.7 BM as a Markov process and martingale**

$\mathcal{F}_t = \sigma(B(s), s \le t)$ $\quad (\mathcal{F}_t, t \ge 0)$ natural filtration of BM.

Markov property: for $s < t$

$$\mathbb{E}[g(B(t)) \mid \mathcal{F}_s] = \mathbb{E}[g(B(t)) \mid B(s)]$$

$\Leftrightarrow$ G any past-t event

H any event in terms of $B(s)$, $s \le t$

$$\mathbb{P}(G \mid B(t), H) = \mathbb{P}(G \mid B(t)).$$

$\Leftrightarrow$ given $B(t)$ "future" and "past" independent

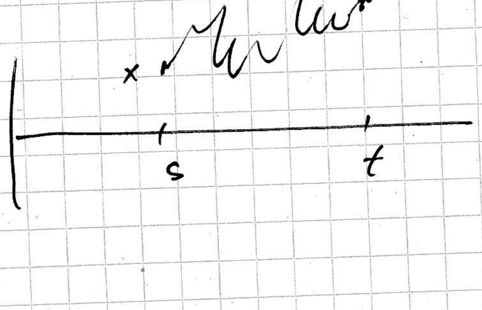
For $s < t$, conditional distribution of $B(t)$ given $B(s) = x$ is $\mathcal{N}(x, t-s)$, as

$$B(t) = \underbrace{B(s) + [B(t) - B(s)]}_{\text{indep.}}$$

$\tau = t - s$

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left[\frac{-(y-x)^2}{2\tau}\right]$$

transition density



(12) For $s < t$

$\mathbb{E}(B(t) \mid \mathcal{F}_s)$ = Markov property

$\mathbb{E}[B(t) \mid B(s)] = B(s)$

(because $\mathbb{E}[B(t) \mid B(s) = x] = x$)

$\Rightarrow$ $(B(t), t \geq 0)$ is a <u>martingale</u>.

<u>4.8</u> Finite-dim. distributions

$B(t_1), B(t_2), \dots, B(t_k)$ is MVN with mean 0,

$t_1 < \dots < t_k$ $\qquad \Sigma_{ij} = t_i \wedge t_j$

Joint density can be written as

$$f(x_1, \dots, x_k) = \prod_{j=1}^{k} p(t_j - t_{j-1}, x_{j-1}, x_j)$$

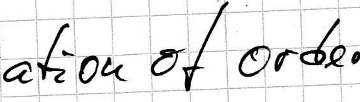
$x_0 = 0$

<u>4.9</u> The quadratic variation

<u>Def</u> For $f : [a,b] \to \mathbb{R}$ the variation of order β is

$$V_\beta(f; a, b) = \sup \sum_{i=1}^{k} |f(t_i) - f(t_{i-1})|^\beta$$

$\beta > 0$ $\qquad$ over partitions



$\beta = 1$ — variation

$\beta = 2$ — quadratic variation

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If f is smooth,

$$V_1(f;a,b) = \int_a^b |f'(t)|\,dt$$

$$V_\beta(f;a,b) = 0 \quad \text{for } \beta > 1$$

$$f(t+h) - f(t) = f'(t)\,h + o(h)$$

This is very different for the BM!

$[a,b] = [0,1]$ take $t_j = j/n$ $\quad j = 0, \dots, n$

$$\mathbb{E} \sum_{i=1}^{n} \left| B\left(\tfrac{i}{n}\right) - B\left(\tfrac{i-1}{n}\right) \right|^2 = n\, \mathbb{E}\left[B\left(\tfrac{1}{n}\right)\right]^2 = n \cdot \tfrac{1}{n} = 1$$

$$\operatorname{Var} \sum_{i=1}^{n} \left| B\left(\tfrac{i}{n}\right) - B\left(\tfrac{i-1}{n}\right) \right|^2 = n \operatorname{Var} B^2\left(\tfrac{1}{n}\right)$$

$$= \frac{n}{n^2} \operatorname{Var} B^2(1) = \frac{3}{n} \to 0.$$

$$\Longrightarrow \sum_{i=1}^{n} \left| B\left(\tfrac{i}{n}\right) - B\left(\tfrac{i-1}{n}\right) \right|^2 \xrightarrow{\mathbb{P}} 1$$

(choosing $n = 2^k$ increases, hence convergence almost surely).

$\langle B \rangle (t) = t$ quadratic variation of the BM

also written $\quad [dB(t)]^2 = dt$

leads to formulas

$$\int_0^t B(s)\, dB(s) = \frac{1}{2} B^2(t) - \frac{1}{2} t$$

due to quadr. variation

For smooth f with $f(0)=0$

$$\int_0^t f(s)\,df(s) = \int_0^t f(s)f'(s)\,ds = \frac{1}{2} f^2(t)$$

The variation of BM is infinite —
the Brownian path has infinite length on every time interval.

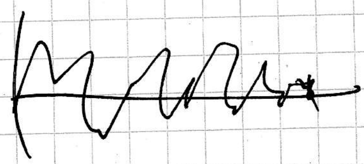
4.10 Related processes

BM with drift/volatility

$$B_{\mu,\sigma}(t) = \mu t + \sigma B(t)$$

Brownian bridge: BM conditioned on $B(1)=0$

$$B^{\circ}(t) = B(t) - t\,B(1)$$



Itô processes

$$X(t) = X(0) + \int_0^t \Delta(t)\,dB(t) + \int_0^t \Theta(t)\,dt$$

Δ, Θ processes adapted to $(\mathcal{F}_t,\ t \geq 0)$

$$dX(t) = \Delta(t)\,dB(t) + \Theta(t)\,dt$$

$$\langle X \rangle(t) = \int_0^t \Delta^2(s)\,ds \qquad (dX)^2 = \Delta^2(t)\,dt$$

15 **Example** Brownian bridge

$$B^0(t) = (1-t)\int_0^t \frac{1}{1-s}\, dB(s)$$

satisfies stochastic differential equation

$$dX(t) = -\frac{X(t)}{1-t}\, dt + dB(t).$$

Change of measure (Girsanov's Theorem)

<u>Example</u> Let $X \sim \mathcal{N}(0,1)$ standardnormal r.v. under $\mathbb{P}$

$$Z := \exp\left(\mu X - \frac{1}{2}\mu^2\right)$$

$\mu \in \mathbb{R}$

Then $Z > 0$, $\mathbb{E} Z = 1$ $\Rightarrow$ there exists a probability measure $\tilde{\mathbb{P}}$ such that

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z \qquad \text{(Radon-Nikodym derivative)}$$

What is distribution of X under $\tilde{\mathbb{P}}$?

We have $\tilde{\mathbb{E}} f(X) = \mathbb{E}\left[f(X) Z\right] =$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-x^2/2} e^{\mu x - \frac{1}{2}\mu^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-(x-\mu)^2/2} dx$$

$\Rightarrow$ $X \sim \mathcal{N}(\mu, 1)$ under $\tilde{\mathbb{P}}$

For densities

$$\tilde{\varphi}_X(x) = e^{\mu x - \frac{\mu^2}{2}} \varphi_X(x)$$

Let $(B(t),\ t\in[0,T])$ be a BM

$\tilde{B}(t) = B(t) + \mu t, \quad t\in[0,T]$ with drift

$Z := \exp\left(-\mu B(T) - \frac{1}{2}\mu^2 T\right).$

Define $\tilde{\mathbb{P}}$ by $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z.$

<u>Theorem</u> (Girsanov) Under probability measure $\tilde{\mathbb{P}}$ the process $(\tilde{B}(t),\ t\in[0,T])$ is a standard BM.

<u>Proof</u> Check finite-dimensional distributions

$$0 = t_0 < t_1 < \dots < t_n = T$$

Under $\mathbb{P}$ the density of vector

$$(\tilde{B}(t_1), \dots, \tilde{B}(t_n))$$

is
$$f_\mu(x_1,\dots,x_n) = c\prod_{i=1}^{n}\exp\left[-\frac{(x_i - x_{i-1} - \mu(t_i - t_{i-1}))^2}{2\ t_i - t_{i-1}}\right]$$

$$c = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}}$$

Expanding squares in gives

$$\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} - \mu(x_i - x_{i-1}) + \frac{(t_i - t_{i-1})\mu^2}{2}$$

Summing over $i=1,\dots,n$ the product becomes

$$\exp\left[-\sum_{i=1}^{n}\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}+\mu x_n-\frac{\mu^2 T}{2}\right]$$

$$\Leftrightarrow f_{\mu}(x_1,\dots,x_n)=f_0(x_1,\dots,x_n)\,e^{\mu x_n-\mu^2 T/2}$$

To obtain density of $(\tilde{B}(t_1),\dots,\tilde{B}(t_n))$ under $\tilde{\mathbb{P}}$ we multiply by $e^{-\mu x_n+\mu^2 T/2}$

because $\quad \tilde{B}(T)=B(T)+\mu T \Leftrightarrow B(T)=\tilde{B}(T)-\mu T$

$$\Rightarrow -\mu\tilde{B}(T)+\mu^2\frac{T}{2}=-\mu B(T)-\frac{\mu^2 T}{2}$$

$T=t_n$, x_n dummy variable for $\tilde{B}(T)$

This yields $f_0(x_1,\dots,x_n)$ which is the joint density of $(B(t_1),\dots,B(t_n))$ for BM $(B(t), t\in[0,T])$.