Q4 Let $\mathcal{A}$ be the family of sets $A \in \mathcal{B}(\mathbb{R})$ with the property that there exists a limit

$$\mu(A) = \lim_{n\to\infty} n^{-1}\lambda(A \cap [0, n]).$$

Show that $\mathcal{A}$ is an algebra. Is μ σ-additive on $\mathcal{A}$?

It is checked straightforwardly that $\mathcal{A}$ is closed under taking complements, disjoint union of finitely many sets, hence also under finite intersections. Both $\varnothing$ and $\mathbb{R}$ belong to $\mathcal{A}$. Actually $A \cap (-\infty, 0)$ does not impact $\mu(A)$ provided $\lambda(A \cap (-\infty, 0) < \infty$, and $\mu(A) = \infty$ if $\lambda(A \cap (-\infty, 0)) = \infty$.

Now let $A_n = [n - 1, n]$ for $n = 1, 2, \cdots$. Then $\mu(A_n) = 0$ but $\mu(\cup_n A_n) = 1$. Hence μ is not σ-additive.

Q6 Is there a probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ supported by $\mathbb{Q}$, i.e. such that $\mu(x) > 0$ for $x \in \mathbb{Q}$, $\sum_{x\in\mathbb{Q}} \mu(x) = 1$. Is this measure discrete or diffuse?

Since $\mathbb{Q}$ is countable we can enumerate the rational numbers $q_1, q_2, \cdots$. Let $\mu(q_n) = 2^{-n}$. The total measure is then

$$\sum_n \mu(q_n) = 1.$$

The measure is discrete, because it is supported by a countable set $\mathbb{Q}$ (although the closure of this set is the whole of $\mathbb{R}$).

Q7 For $A \subset \mathbb{R}$ define $x + A := \{x + a, a \in A\}$. Prove translation invariance of the Lebesgue measure: $\lambda(x + A) = \lambda(A), A \in \mathcal{B}(\mathbb{R})$. Extend the property to Lebesgue-measurable sets A.

Let $\lambda_x(A) := \lambda(x + A)$, where $x + A$ is the collection of points $x + a, a \in A$. For every interval $(a, b]$ it holds that $\lambda_x((a, b]) = \lambda((a, b])$, therefore the measures λ_x and λ coincide on the semiring of intervals. But then by the uniqueness of extension the measures coincide of $\mathcal{B}\mathbb{R}$, and further on the Lebesgue-measurable sets.

Q10 Let $\mu = \sum_{j=1}^{\infty} 2^{-j}\delta_j$. Is it a probability measure? Sketch the graph of its cumulative distribution function.

Yes, this is a discrete measure putting mass 2^{-j} on point j, with total measure equal to 1. The c.d.f. is piece-wise constant, taking value $1 - 2^{-j}$ on interval $[j - 1, j),\ j = 1, 2, \cdots$

Q13 Consider $\mathcal{S} := \{\{x\} : x \in \mathbb{R}\}$. Show that for $A \in \sigma(\mathcal{S})$, either A is countable (i.e. either finite or countably infinite) or A^c is countable. Now let $\mu(x) = 1$ for every $x \in \mathbb{R}$. What are possible values of $\mu(A)$? When $\mu(A) = \infty$?

Note that if $A^c = \{x_1, x_2, \cdots\}$ (finite or countably infinite set) then for any B we have

$$(A \cup B)^c = \{x_1, x_2, \cdots\} \setminus B$$

again a finite or countably infinite set.

Let $\mathcal{G}$ be the collection of sets A such that either A or A^c is countable. Clearly, $\mathcal{G}$ is closed under complements, and contains $\varnothing$. Suppose $A_1, A_2, \cdots \in \mathcal{G}$. If all A_n's countable then $\cup_n A_n$ is countable hence belongs to $\mathcal{G}$. If at least one, say A_1 has countable complement, say $\{x_1, x_2, \cdots\}$ (perhaps, finitely many points) then $(\cup_n A_n)^c \subset A_1^c \subset \{x_1, x_2, \cdots\}$ is again a countable set, thus also in this case $\cup_n A_n \in \mathcal{G}$. Thus $\mathcal{G}$ is closed under the countable unions. It follows that $\mathcal{G}$ is a σ-algebra, $\mathcal{G} \supset \mathcal{S}$ and in view of $\mathcal{G} \subset \sigma(\mathcal{S})$ we have $\mathcal{G} = \sigma(\mathcal{S})$.

The measure μ is a counting measure, with $\mu(A) = n$ if $\mathrm{card}(A) = n$, and $\mu(A) = \infty$ if the cardinality of A is infinite.