Bifurcation Exploration of Hindmarsh Rose Model

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Abstract

This dissertation is based on the analysis and exploration of the mathematical Hindmarsh-Rose model in terms of nonlinearity and bifurcation structure. The exploration of this project represents insights of the simulation in the 3D phase space as global picture and other mathematical plots produced by the author investigating the bifurcation with prospects to identify the macro and micro chaotic elements and features in the nonlinear structure of single neuron as visualised by the bifurcation diagrams and literature survey. This mathematical model is consisted of three ordinary differential equations containing eight parameters where are taken into the account for the mathematical analysis where extracted the dynamical behaviour of the mathematical model from the construction of plots. In the present study, the author spend time for familiarisation of the elements of nonlinear dynamics and bifurcation types during his reading, in order to create the strategy for the exploration of the model. The author, in order to start the exploration of bifurcation by identifying the fixed points, taking into the consideration the parameter ‘b’, ‘r’ to produce the bifurcation diagram as shown in parametric plots in chapter 3. Due to the complex dynamics that is produced during spiking and bursting in neurons further analysis will be required by using all the available computational tools for detailed explanation in future projects as recommended in future work section.

Declaration

No portion of this work referred in this dissertation has been submitted in support for the application of another degree or qualification of this or any other university or any other institute of higher learning.

Ioannis Karatzias

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Copyright Statement

Acknowledgements

I greatly appreciate the writings and teaching from people who I learned regarding the Mathematical background and computational methods which were used for the particular project. I would like to thank the Professors and my Lecturers from the Queen Mary University of London who provided me the relevant learning material of this course to work out this project.

I would like to thank my supervisor Dr Wolfram Just for the help and guidance during the progress and execution of the report and other any comments regarding the improvement of my project as it is concerned to provide a worthwhile body.

1.0 Introduction

1.1 Introduction of dissertation

This project is carried out as part of the fulfilling the requirements of a Master of Science in Data Analytics. The dissertation is undertaken in the Queen Mary University of London in the department of Mathematical Sciences.

The report is based on the exploration and analysis of the Hindmarsh-Rose mathematical model in order to investigate the non linear dynamics of single bursting neuron with qualitative bifurcation diagrams. Insights for quantitative analysis with associated spikes when the neuron is firing are shown from the literature survey but is not major part and not very detailed in this project. Upon completion of the analysis, comparison of the neuron dynamical behaviour carried out with other scientific paper presented in the literature survey as they were selected from the author for this project goal to find common elements in terms of dynamics.

This report is intended to investigate, analyse and highlight some different non linear phenomena of bursting neurons and considering the importance of the mathematical model parameters which they can influence the macro and micro dynamical behaviour which is initiated in the soma, developed and transmitted in the axon of neuron to the other connected neighbour neurons connected with the synapses and dendrites.

1.2 Background theory of neuron

Last decades has been obtained many researches and experimental applications in order to investigate the dynamics of biological neurons isolated from the rest or interacting with the others in order to understand and predict neurological diseases such as epilepsy, Parkison or to use these insights in information theory and other technological applications. The mathematical formulation of these coupled ordinary differential equations describes the neuronal activity of this dynamical system in time where will be analysed and represented in further sections more detailed. In this section will be explained the electro chemical function of neuron and will be visualised the neuron structure in following figure in order to understand where the activity occurs in this physical representation.

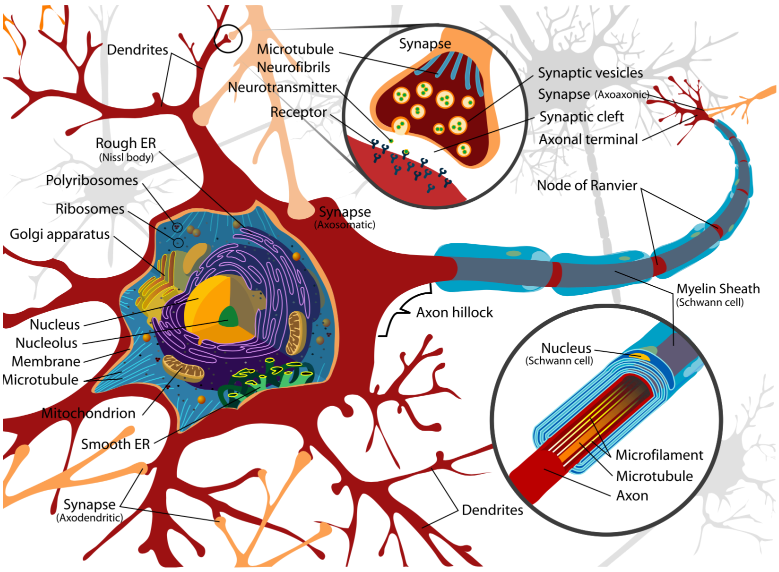


Figure 1 Neuron Structure [internet]

The neurons in human brain operate with electrical flow and the difference with the other cells where consists of the frequent changes in the electro-potential, which represented as 'oscillation patterns'. The potential acquired through synapses and spread through dendrites to the soma, called as cell body differently, where the electrochemical calculation occurs depend to the neuron type. Every neuron according to its type is carrying out different calculation in soma and produces different output signal. If the signal passing a certain threshold point then is allowed to send the signal to the output through the axon nerve by opening the Na+ channels and subsequently K+ ion channels where the oscillation patterns occurs.

The change of potential 'spike' is then actively spread through axon to the other neurons, in meantime the resting potential maintained in cell membrane. Most of the energy in cells is consumed for maintaining the gradient in concentrations in selected ions and then maintaining the difference in potentials to keep in balance the system. The main concept arise where the high sensitivity in initial input signal and small changes in inputs as deterministic values, can produce or transform the output signal where exhibit response similar to stochastic process. The next subsection will be set the aim and the objectives as techniques to investigate the above complicated neuronal dynamical activity using the selected mathematical model.

1.3 Aims and objectives of the dissertation

Aims

- ➢ To perform research in order to select a mathematical model in theoretical and computational neuroscience for exploration of neuron dynamics as of author’s interest in cooperation with his supervisor. The criterion for the selection of the model is based on the plethora of the chaotic phenomena covered by this model in comparison with other mathematical models in neuroscience.
- ➢ To perform a mathematical analysis of the model for exploring the dynamics of the Hindmarsh Rose model received insights from bifurcation produced by simulation and other bifurcation diagrams created from other mathematical software.
- ➢ To present as many possible nonlinear phenomena that this model can produce from author’s present work and other literature sources and also recommend future work opportunities for exploring the neuron dynamics.

Objectives

- ➢ To perform research in theoretical and computational neuroscience in order to select the Hindmarsh Rose mathematical model of interest.
- ➢ To enhance the author's knowledge in nonlinear dynamics and chaos by reading the learning material from the book written from Steven H. Strogatz and from the module 'complex systems' in order to understand the relevant steps for construction and interpretation of bifurcation diagrams.
- ➢ To construct and modify a Hindmarsh Rose code in Python and produce a general overview of the nonlinear dynamics in 3D phase space. To analyse further the dynamics of the model by analysing the ordinary differential equations and produce bifurcation parametric plots using the Wolfram Alpha mathematical software.
- ➢ To analyse and explain in qualitative manner the plots starting from identification of fixed points. Upon completion of the simulation and the other plots in order to understand the dynamical structure of the model.
- ➢ To write a report by mentioning the most important findings and conclusion after the research and recommend future work suggestions.

1.4 Review of contents

Chapter concepts are listed below as a quick introduction to the material, which is presented in the report and as an indication of the key topics that should be mentioned before the reader moving to the next chapter. This report is consisted of four chapters which includes the activities of this project.

Chapter one set the scene of the subject under the theoretical study. In addition, gives to reader the background of the neuron structure, and introduce the dynamics of the neuron function in biological form. The importance of the model parameters are examined in the next chapters more detailed. Furthermore introduces the aims and objectives of the project in order to determine simulation tools and techniques to achieve the goal of this project.

Chapter two outlines all the learning materials which have been discovered from the author in order to gain a general view for the discipline of the computational neuroscience in term of dynamics, and other theoretical neuronal models.The author has extracted from different scientific documents and from the recommended book the non linear dynamical behaviour which was used for the analysis with the his model.

Chapter three presents the mathematical analysis of the model in more detail and the re-arrangement of the equations in parametric forms to explore the dynamics in qualitative form.
Chapter four presents the way that the results from the mathematical analysis and computational studies produced. Conclusion, limitation and further recommended studies is mentioned for future improvement of the present study in this section as well.
The dissertation includes appendices, which shows the code from the model after the simulation in order to help the reader to understand the general complexity of the project with plot represented in 3D phase space similar to chapter 3.

2.0 Literature Survey

2.1 Introduction of literature survey

In this chapter will be discussed and introduced some other mathematical models performed in theoretical and computational neuroscience where analyse the neuron function taking into the account the action and importance of their parameters. In addition, in this chapter will be mentioned the analytical steps analysing the trajectory structure during spike or bursting of neuron or any other non linear behaviour where occurs in any dynamical system represented in the selected plots. Introduction of the theoretical background and slightly comparison between the neuroscience models and the mathematical formulations of the equations will be part of the following sections, starting from the selected Hindmarsh - Rose model as selected for this project.

2.2 Hindmarsh - Rose model

'The Hindmarsh - Rose model is consisted of the following three ordinary differential equations on the dimensionless dynamical variables x(t), y(t), z(t) where includes the (4) and (5) equations and also including the parameters as mentioned below.

$$dx/dt = y + \phi(x) - z + I \qquad (1)$$

$$dy/dt = \psi(x) - y \qquad (2)$$

$$dz/dt = r[s(x - x_{rest}) - z] \qquad (3)$$

where

$$\phi(x) = -\alpha x^3 + bx^2 \qquad (4)$$

$$\psi(x) = c - dx^2 \qquad (5)$$

Relating to the function of the equations, the first two ordinary differential equations can show the dynamics and the nonlinear behaviour of the slow subsystem and the third equation was used to describe fast subsystems.

The model contains eight parameters where some can affect significantly, the function of the mathematical model in terms of change of dynamics according with the change of their value and their specific operation in the equations. introducing the first parameter 'I' which is the current that enters in neuron and is considered as control parameter where takes values in range $-10 < I < 10$. The parameters α, b, c, and d are part of the fast ion channels and often used as fixed parameters with specific values, α=1, b=3, c=1, and d=5. There is also a range where some of the fixed parameter can have particular importance such as the parameter 'b' and experimented in certain range. The 'r' is also a control parameter and normally takes values within the range $0 < r < 1$ where works in slow ion channel and control the timescale of z variable. In the analysis section will be introduced some different values of r within the range and their response graphically. The parameters where are normally kept fixed are the s=4 and the rest $x_{rest}=-8/5$ where in some literature is changed experimentally as well.'[9].The below plot represent a signal from a typical simulation of the burst and spike activity of Hindmarsh-Rose model in neuron. The bifurcation diagram of this model will be presented in the next sections as it is the model of interest for this project.

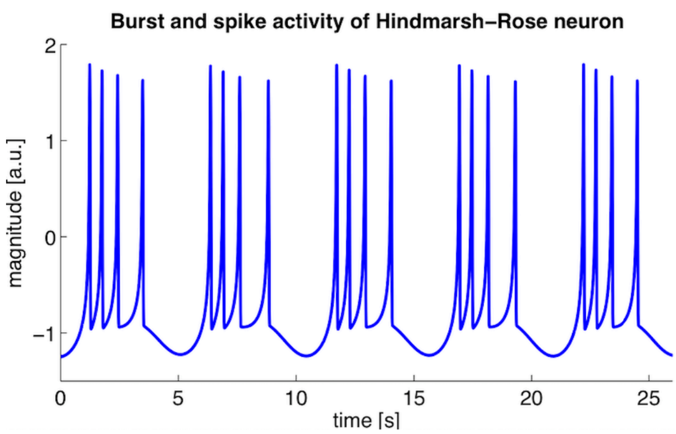


Figure 2 Simulation of HR model of a typical neuronal bursting.[9]

2.3 Hodgkin-Huxley model

The author will introduce the main informations in respect of interest for this mathematical model as is not the major part for the analysis of this project.The Hodgkin-Huxley model is used as one of the main model based on the realistic performance in biophysical applications and its projections can be interpreted in four-dimensional phase trajectories. Introducing the mathematical properties of this model where 'The Hodgkin-Huxley model which is a differential equation system contain four state variables, $V_m(t)$, n(t), m(t) and h(t) that change in respect of time (t).Is a nonlinear system and quite challenging to be solved analytically but there are numeric methods to analyse it.' [10]. Not only the limit cycles as one of the elements in its dynamic where can be proven with this model but also hopf bifurcation and canard phenomenon appeared as well. Also because of the four state variables is difficult to visualise the orbit in phase space, so two variables are chosen for better visualisation of limit cycles, qualitatively. 'The following equations describe the model where referenced' [10].

$$I = C_m (dV_m /dt) + g_K n^4 (V_m - V_K) + g_{Na} m^3 h(V_m - V_{Na}) + g_l (V_m - V_l), \tag{6}$$

$$dn/dt = \alpha_n (V_m)(1- n) - \beta_n (V_m)n \tag{7}$$

$$dm/dt = \alpha_m (V_m)(1- m) - \beta_m (V_m)m \tag{8}$$

$$dh/dt = \alpha_h (V_m)(1- h) - \beta_h (V_m)h \tag{9}$$

'where α_i and β_i are rate constants for the *i-th* ion channel, which depend on voltage but not in time. g_n is the maximum value of the conductance. Also n, m and h are dimensionless quantities within the range 0 and 1 that are associated with potassium channel activation, sodium channel activation, and sodium channel inactivation'.[10]

Modification of this model have been carried out over the years to incorporate different states in terms of dynamic such as transition state theory

2.4 Fitzhugh-Nagumo model

The Fitzhugh - Nagumo model is a two dimensional model simplified from Hodgkin-Huxley model for spike generation. ‘This model is consisted of the following two equations and the associated parameters:

$$dV=f(V) - W + I \tag{10}$$

$$dW=\alpha(bV - cW) \tag{11}$$

The V is the membrane potential and the ‘W’ is the recovery variable, where ‘I’ is the magnitude of stimulus current.

where the f(V) is the polynomial of third degree, and the parameters , α, b and c are constants.’[7]

The advantage of the Fitzhugh - Nagumo model based on its simplicity, it give us solution of nonlinear geometrical representation at once in the phase portrait of the biological phenomena related to neuronal excitability and spike generating mechanism. Refer in below in the figure representative phase portrait of the biophysical state diagram of Fitzhugh - Nagumo model.

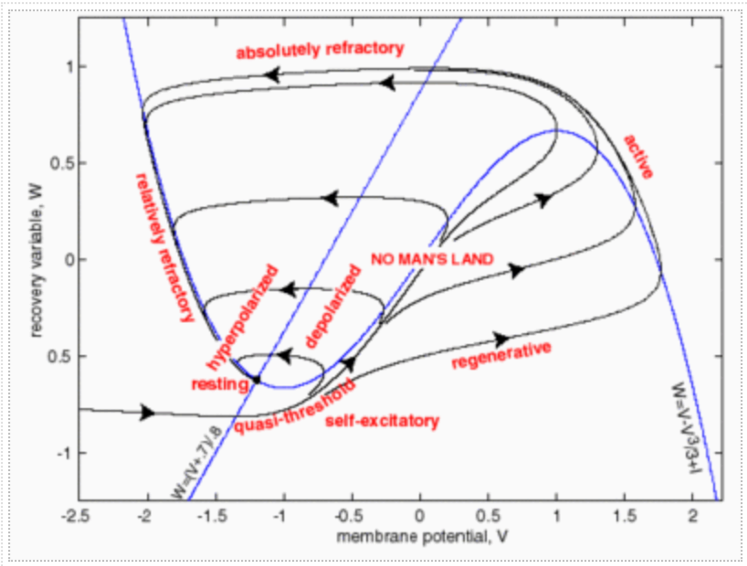


Figure 3 Fitzhugh Nagumo biophysical state diagram, [7]

2.5 Comparison and criteria of selecting the Hindmarsh Rose model.

The Hindmarsh Rose model is considered approachable in computational neuroscience and provides a variety and satisfactory description of trajectory patterns in terms of qualitative analysis, in comparison with the FitzHugh-Nagumo model where is used for less complex cases and is not producing self-sustained chaotic dynamics, no bursting and has only a few parameters where is difficult to adapt to neurons with specific properties. On the other hand the high complexity of the four variable Hodgkin Huxley model led the author to choose the Hindmarsh rose model for the analysis in this dissertation and future possible involvement in terms of exploring further the model.

2.6 Fixed points

From this section and in the next sections will be mentioned some of the important terminology and the way that this terminology and the calculation associated with this and is used in nonlinear dynamics. Some of these steps will be followed in this project.

A common strategy when nonlinearity is examined is analysing the flow of the dynamical system finding if and where the fixed points occur in the dynamics of the system. In order to understand the flow dynamics of any function, we can think a vector field of a fluid where flowing with some variation of the velocity

along an x - axis of coordinate system and imagine of creating a trajectory with a particle called as phase point following a 'sin' curve along the x-axis. 'The function of x=sinx in this example, and the phase status of the function where the the phase point is x > 0 and where is x < 0 then we say that exist a flow in this vector field, with direction from left to right as an example. On the other hand where the phase point is in the trajectory and there is no flow along the trajectory, such point is called 'fixed point'.[3] Refer to the below figure of the above function following periodic flow.

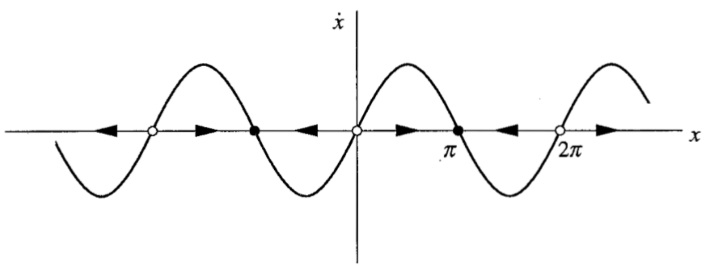


Figure 4. 1D representation of fixed points of sinx function [3]

The fixed points are separated from stable black dots on x-axis and unstable fixed points the white dots, or combined fixed points where are half stable and half unstable where is not presented in the above figure. Stable fixed called attractors or sinks and are the fixed points where the flow is toward them, and unstable fixed points are called differently as repellers or sources with direction in open circles.

More analysis of the fixed points are referred in the next section related to the linear analysis of fixed point to identify the stability status. Different calculation methods are used to solve the equations where they represent different functions in any nonlinear system in order to explore and present the appearance of fixed points graphically.

2.7 Stability and linear stability analysis.

To determine the stability of the fixed points have to be sketched graphically and let x^* be a fixed point and check the solution of the function, where if is greater or less than zero, in order to determine the status of the stability, if is stable or unstable. Linear stability analysis, is evaluating the stability for the fixed point considering a point x with very small distance far away from the fixed point and if the point x has a solution $x=x^*$ then is considered stable, not only initially, but also remains for all the time. In addition, an equilibrium solution is said to be stable when small disturbances damp out in time close to the fixed point. On the other hand when disturbances grow in time causing instability in the fixed points, so numerically will repelled out to infinity.

When we have one fixed point locally stable but another unstable then the system called locally stable but not globally stable.

With the linear stability analysis, we take a quantitative measure to show the rate of grow or decay occurs to a stable fixed point, and this measure can be obtained by linearising the fixed point x^*. The outcome of this calculation shows the magnitude of this change in time for this grow or decay and how it varies in the neighbourhood of x^*. Numerically, Taylor series expansion is one of the method of using this linearisation process.

2.8 Limit cycles

Another phenomenon which appeared in the phase portrait of a dynamical system is the limit cycles. 'The structure of limit cycle is an isolated closed trajectory and the neighbours trajectories are not closed, which can have spiral form or moving forward or away from the closed limit cycles.'[3]

Stable limit cycles structure appeared in many applications such as beating of heart, human body temperature, in neurons or exhibit self sustained oscillations. 'In different engineering or physics cases, self sustained vibrations have their own preferred period of operation but in some cases could be dangerous such as in bridges or airplane wings.[3].The following figure shows the stability of limit cycles in relation with the surrounding trajectories.

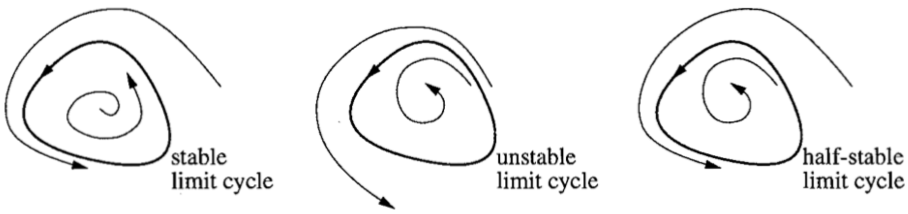


Figure 5 different structure of stability of limit cycles [3]

As it is observed in above picture, by looking the first picture on the left where all neighbour trajectories approaching the limit cycle, then the limit cycle called stable or attracting, in contrast with the middle picture where the all trajectories not approaching the limit cycle, so this limit cycle called unstable limit cycle, and lastly in the last right picture where both stable and unstable structure occurs.

2.9 Phase Portrait

As mentioned previously analysing the trajectory by solving some of the explicit formulas is a challenging task to determine the qualitative behaviour of the solutions, and sometimes because of the set of different trajectories belonging in

one phenomenon, finding the phase portrait describes better the behaviour of the dynamical system. An example is shown in following sketch in order to observe different trajectories and phase point status in particular phase portrait.

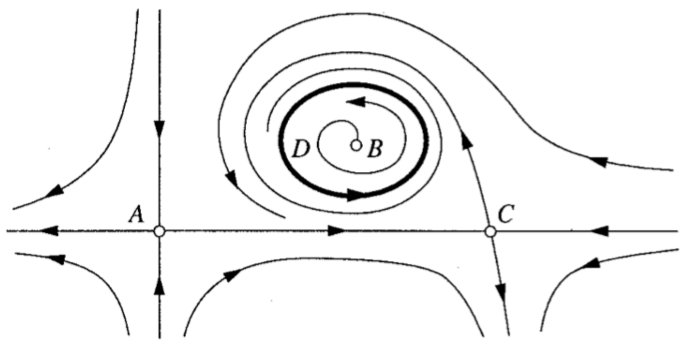


Figure 6. Phase portait [3]

‘Explaining with a short description the above phase portrait, starting with the fixed points A, B, and C where satisfy the status of function at that point f(x*)=0, and where are located in different trajectories and are unstable fixed points as per white circle where the close trajectories move away from the fixed points as shown in the vector field. In middle of the picture we see closed orbit ‘’D” with the thick black line and is stable where is corresponded to a periodic solution. In addition, is observed the trajectory starting from fixed point B is enclosed in the D closed cycle’[3].

A numerical computation of phase portraits can be done by applying the Runge - Kutta method in vector points in the trajectory field as can be applied when computing the Hindmarsh rose model in python code in 3D phase plane. A representation of a plot of the directions fields could be segmented by null clines to separate the vector arrows in corresponding directions of each trajectories.

2.10 Introduction to Bifurcation of dynamical systems

In previous sections introduced some elements describing the flow of the trajectory in a dynamical system.In this section is introduced the main elements of the anatomy of the dynamical system and how these elements change the system is vital to describe the bifurcation. Any dynamical system is consisted of the function where describes the dynamics of the system, the vector of values of state of the system, the time where the system involves in time, or with a specific example, how the brain changes in time. In mathematical terminology, there are variables and some control parameters or not control parameters where we want to understand the dependence of them in dynamics in the system. Defining the term bifurcation with simple way, is any change of the dynamics in any dynamical system is called bifurcation. Small disturbances in the dynamical system can change the dynamics and the prediction of our model in near future states. In order to examine the nonlinear phenomena of the Hindmarsh Rose dynamical system, we recall the qualitative changes in the flow and dynamics as called bifurcation and is adopted for analysis and understanding of the model using the geometrical representation of the mathematical function. When we talk for qualitative, we mean a geometrical change or sometimes a topological change of the phase portrait, or more specifically the change of fixed points, when they are created, disappeared, change their stability status or further changes in closed orbits or saddle connections and other various trajectory structures with the parameters changes. In the following section will be mentioned some different kind of bifurcation structures where some of them could be appeared and be part of the present study.

2.11 Saddle node bifurcation

The saddle node bifurcation or fold bifurcation is the basic type of bifurcation for the construction or destroy of the fixed points, where also can be seen in limit cycles as well. It worth to mention that any kind of bifurcation can occur to any dimension from one dimensional to two or three dimensional system depend on the motion of the dynamical system. The following two dimensional example shows the status of saddle node bifurcation.

$dx=m-x^2$ equation in x-direction

$dy=-y$ equation in y-direction

refer to the following graphical representation of the above equations.

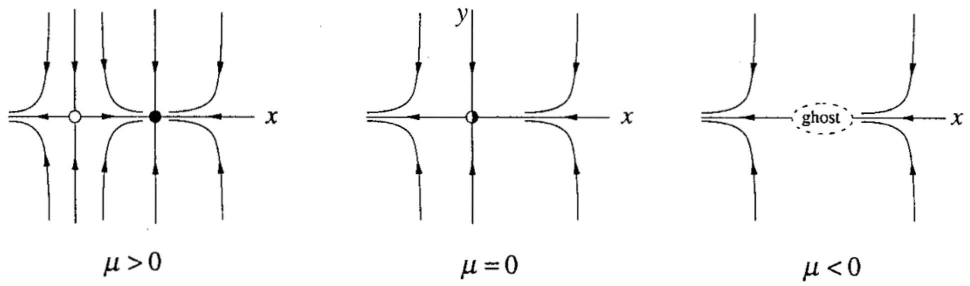


Figure 7 Saddle node bifurcation stages [3]

The above phase portraits occurs when parameter '*μ*' changes. In more detailed explanation 'when *μ>0* then is observed two fixed points, due to the equation status as well, one stable $(x^*, y^*)=(\sqrt{\mu}, 0)$ and an unstable saddle at $(-\sqrt{\mu}, 0)$. In order to see the progress of change as parameter '*μ*' increases we observe the middle portrait a collision between the saddle point and the fixed point creating a half stable node when μ=0. When parameter '*μ*' continue decreasing examining the last picture at μ<0 then is observed a disappearance of the saddle node, which they leave a 'ghost' where actually sucked some trajectories.'[3]. There is actually a critical value of parameter μ_c where the bifurcation occurs.

2.12 Transcritical and Pitchfork Bifurcations

There are case where the difference between the saddle node bifurcation and the transcritical bifurcation is that the two fixed points do not disappeared but is changing their stability, which is element of transcritical bifurcation.

'The pitchfork bifurcation or pitchfork trifurcation is applied to physical cases where there are cases with spatial left and right symmetry in diagram, and in these circumstances the fixed points appeared or disappeared in symmetrical pairs'.[3] There are two kind of pitchfork bifurcations, the supercritical and subcritical where are presented graphically in following vector fields and the bifurcation diagrams.

Looking first the supercritical case, in the example below is observed the change of vector field when parameter 'r' is changing on particular function and further represented in the bifurcation diagram.

The corresponding function is dx=rx-x^3 represented in the vector fields below.

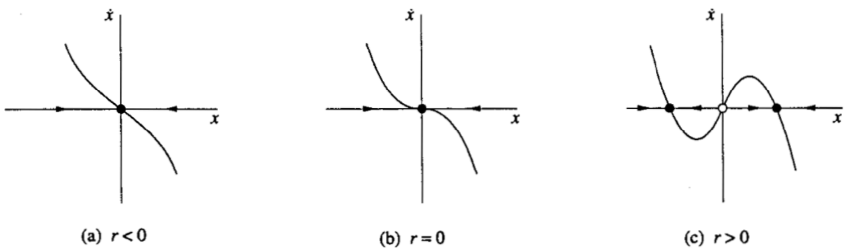


Figure 8 Vector field of different values of r. [3]

'This example introduces the invariant wording, where the equation is invariant where that means when change the x to -x, even with the change of signs we achieve the same equation and also because of the invariance the symmetry is achieved left and right direction. In addition, observing the vector field and the fixed point status, when r<0 then is observed a stable fixed point, or when r=0 then is still stable but weakly, and when r>0 then we see the origin unstable fixed points and another symmetrical stable fixed points left and right, at $x^* = \pm\sqrt{r}$.'[3]

The pitchfork bifurcation diagram between x and parameter r is visualised in the following figure for the case above.

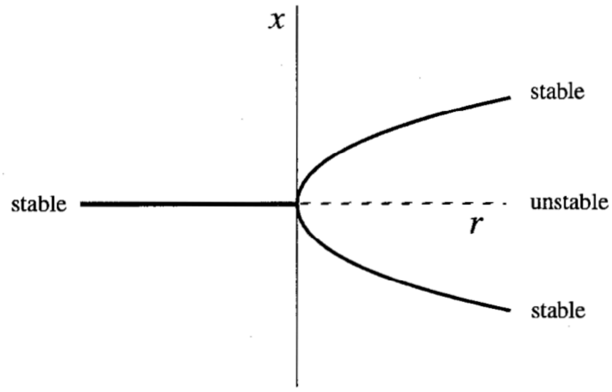


Figure 9 Supercritical pitchfork bifurcation diagram.[3]

Referring to the following paragraph in this section, we will focus on subcritical pitchfork bifurcation where is more challenging in terms of dynamics. In opposite of stabilisation of the above function, now we have in this function $dx=rx+x^3$ destabilising of fixed points then we have a subcritical pitchfork bifurcation as the digram shows as follow.

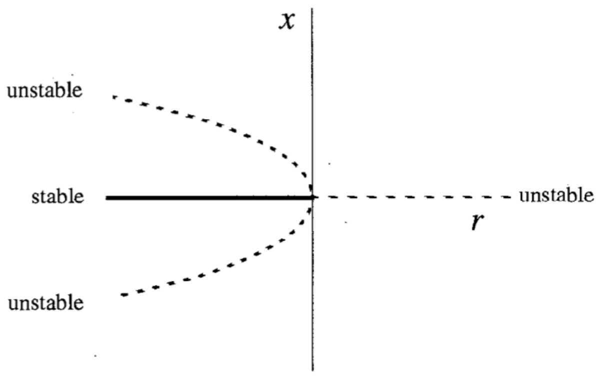


Figure 10 Subcritical pitchfork bifurcation diagram.[3]

From the above diagram, the non zero fixed points x*= ±√-r, are unstable as we see from the dotted curve lines and appeared when r<0 where defines the subcritical terms. The origin is stable and the unstable fixed points bifurcate from the origin when r=0. In the case when r > 0 then the cubic term of the equation leading the trajectories to infinity as involves with the time.

‘In real physical systems the stabilising term x^5 added to the previous equation, where results the equation $dx=rx+x^3-x^5$ shows that the unstable curves with dotted lines turns to stable fixed points when $r=r_s$ where r_s is a saddle node bifurcation and is $r_s<0$ at x^5 as it can be observed from the following figure.’[3]

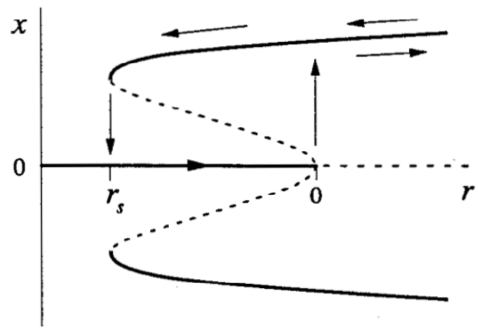


Figure 11 Hysteresis pitchfork bifurcation [3]

As we can see in the above figure where shows the bifurcation states when r is compared with r_s and varies along the function.

From the above figure within the ‘r’ there are two stable states, one at the origin where is stable and another stable state starting at r_s, as mentioned previously. Because of these two stable states interrupted with the unstable dotted line when r varies, leads to the possibility of jumps and the phenomenon of hysteresis.

Explaining more, the above figure when the stable state starting from the origin and keeps its stability until the r=0. Further to zero, when r increases loosing its stability with a small disturbance where will cause this jump to one of the large amplitude branches. on the other hand when r is decreased and reach let’s say the value of r_s

or below is observed a jump back to stable state as it was in the origin, this phenomenon is called hysteresis.

2.13 Hopf bifurcation

The hopf bifurcation is very important and appeared as common phenomenon especially in the subcritical hopf bifurcation in nerve cells, aeroelastic flutter in turbine blades and airplane wings.

As a general definition, 'the appearance and disappearance of a periodic orbit through a local change in the stability properties of a fixed point is called a hopf bifurcation'.[8] introducing further the mathematical explanation is when occurs 'local bifurcation in which the fixed point in the system loses stability, as a pair of complex conjugate eigenvalues of the linearisation around the fixed point and passes the complex plane in imaginary axis.'[8]

In the hopf bifurcation in phase plane the limit cycle observed an elliptical shape and its shape is changing as the parameter '*μ'* changing where '*μ'* defines its distance from the fixed point. So, is observed a change in topological form of the limit cycle. The hopf bifurcation as the pitchfork bifurcation has a subcritical and supercritical form as well.

Firstly, we introduce below an example for supercritical hopf bifurcation case in phase plane as shown in the figure.

Consider the following system of equations.

$dr = \mu r - r^3$

$d\theta = \omega + br^2$

'In the above equations there are three parameters, the '*μ'* where is a control parameter and controls the stability of the fixed point at the origin, the 'ω' where introduce the infinitesimal oscillations and the 'b' where determines the frequency on amplitude for larger amplitude oscillations.'[3]

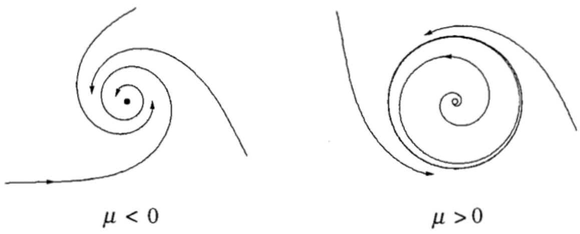


Figure 12 Supercritical hopf bifurcation. [3]

In the above graph is shown the change in phase portraits where '*μ*' is below or above zero in combination with the other two parameters. 'when $\mu < 0$ and the origin r=0 then is observed a stable spiral, where the parameter 'ω' gives the rotation according the sign. For $\mu = 0$ the origin is still stable spiral, but then when $\mu < 0$, is observed an unstable spiral form at the origin as shown from notation with open circle dot, and a stable circular stable cycle at $r= \sqrt{\mu}$ in the right picture'.[3]

To investigate how the eigenvalues behave during the bifurcation a conversion of the equations to cartesian form and write the Jacobian calculation is necessary. Is important to mention that the conversion is omitted at this stage and is introduced below only the expected eigenvalues formula as final result.

$$\lambda = \mu \pm i\omega$$

where *'i'* is an imaginary term and *'λ'* the eigenvalues.

From the above formula is shown that the eigenvalues crosses the imaginary axis from left to right as '*μ*' increases from negative to positive numbers.'[3]

The subcritical hopf bifurcation case is more dangerous in several applications in nature and engineering where can result a fatal damage. In general, 'the physical evolution is after the subcritical hopf bifurcation the trajectories must jump to a distant attractor, which may be a fixed point, a limit cycle, infinity or in a higher dimensions, a chaotic attractor.'[3]

'Keeping the same example as above with the additional term r^5 in the first equation of the system and refer to the following plots in order to explain the bifurcation graphically.

$dr = \mu r + r^3 - r^5$

$d\theta = \omega + br^2$

The difference in the subcritical case with the supercritical is that the term r^3 is destabilising where leads the trajectories far from the origin. The following plots describe the evolution of the subcritical hopf bifurcation.'[3]

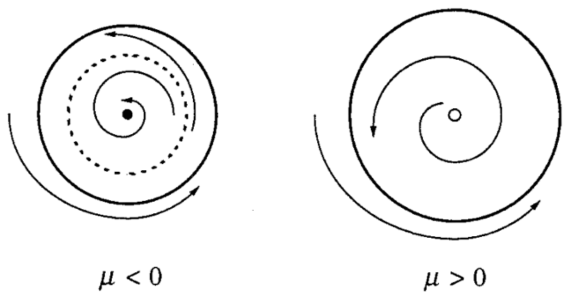


Figure 13 Subcritical hopf bifurcation [3]

In the first plot when $\mu < 0$, is observed a stable limit cycle, an unstable limit cycle with the dotted line and a stable fixed point in the center of the phase portrait. 'As the parameter 'μ' the unstable limit cycle tightens around the fixed point. The subcritical hopf bifurcation occurs when $\mu = 0$, and the unstable limit cycle reduced the amplitude to zero and surround completely the origin and convert it to unstable fixed point. In second picture of the above plot is when $\mu > 0$ is observed the unstable fixed point at the origin where now forced to grow into large amplitude oscillations and the phenomenon of hysteresis is shown in this case, where a large oscillation begun and cannot return back when $\mu = 0$'.[3]

Another distinction form of hopf bifurcation is the degenerate hopf bifurcation.An example where shows that is, in the case lets say, of a damped pendulum equation, when we change the μ from positive to negative, the fixed point at the origin is changed from stable to unstable spiral and becomes a nonlinear center at μ = 0 rather a weak spiral as per requirement of hopf bifurcation resulting not having limit cycles, but closed orbits surrounding the origin.

2.14 Homoclinic bifurcation

Homoclinic bifurcation is another type of bifurcation where appeared in nonlinear problems. In the following figure will be described the homoclinic bifurcation in combination with other trajectories structures created in the phase portraits. 'The main element of the homoclinic bifurcation is the bifurcation orbit where the bifurcation orbit is a trajectory of a flow in a nonlinear system where starts from that saddle node and returns back again to that point'.[11] In order to understand graphically one of the possible scenarios of homoclinic bifurcation construction, refer to the following graphical example produced by the below equation system.

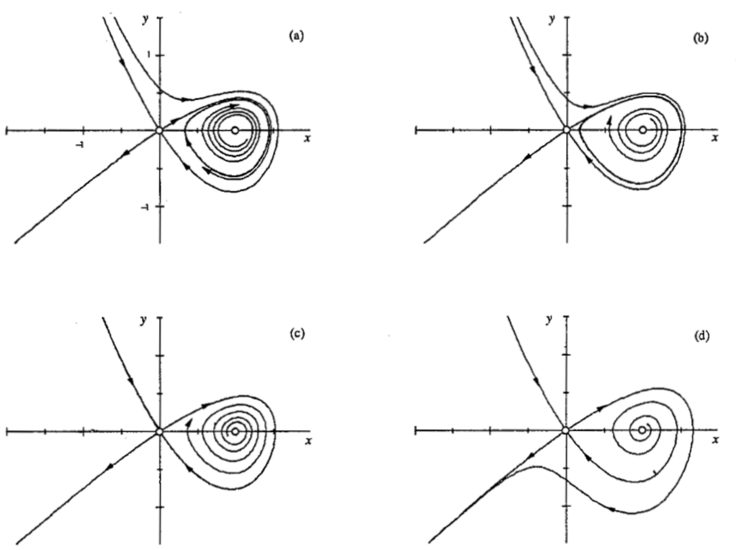


Figure 14 Interpretation of homoclinic bifurcation [3]

In the following system of the equations produced the above figures and with the changes of parameter μ , the changes in phase portrait presented where the progress of bifurcation occurs.

System of Equations

$dx = y$
$dy = \mu y + x - x^2 + xy$

'Numerically the bifurcation occurs when the value is $\mu_c = -0.8645$. According the value of μ, in different case comparing with μ_c the phase portrait involves as shown above. when $\mu < \mu_c$ for example $\mu = -0.92$ is observed stable limit cycles encircle the saddle point but is close to the saddle point at the origin in picture (a), and when the value of μ increases close to $\mu_{c,}$ then is observed an expansion of limit cycles as shown in (b), then where touch the saddle node in (c) is created a homoclinic orbit, where lastly, when the $\mu > \mu_c$ then the saddle connection breaks and the loop is destroyed as shown in figure (d)'.[3]

2.15 Introduction to Bifurcation of Hindmarsh Rose model

In this section is introduced elements from the analysis upon completion of simulation from the selected article of *'Macro and micro-chaotic structures in the Hindmarsh Rose model of bursting neurons'* , in order to describe and introduce a few types of bifurcation as mentioned previously in this chapter and some other new elements due to the nonlinearity and chaotic behaviour where arise during the neuron activation as included in referenced document. Due to the complexity and variety of nonlinear phenomena arise from neurons function, the author will mention a few nonlinear forms selectively and will split them in two major categories of interest as macro and micro chaotic structures, where described further in more detail resulting from the particular reference.

2.16 Macro Chaotic structure of Hindmarsh Rose model

Prior introducing the elements of this structures is important to mention that as part of neuronal dynamics, is to see how robust to perturbation or any change in terms of transformation of dynamics the system is, and where is depending on the parameters which is a major part of the bifurcation analysis.

‘The macro structure due to neuron spike adding cascade is characterised by fold bifurcation and period doubling bifurcation curves originating in codimension-two homoclinic bifurcations. In this section will be reported the global organisation of the bifurcation structure in selected parameters where constructed upon the specific activity of neurons, concerning the spike -quantification approach as represented in following parameter plane in detailed.’[1]

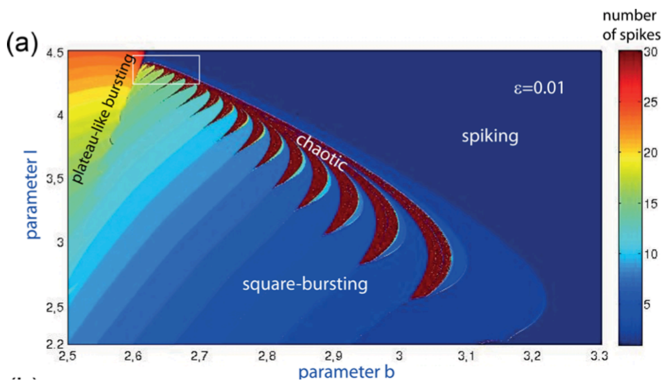


Figure 15 (b-I) parameter sweep of HR model.[1]

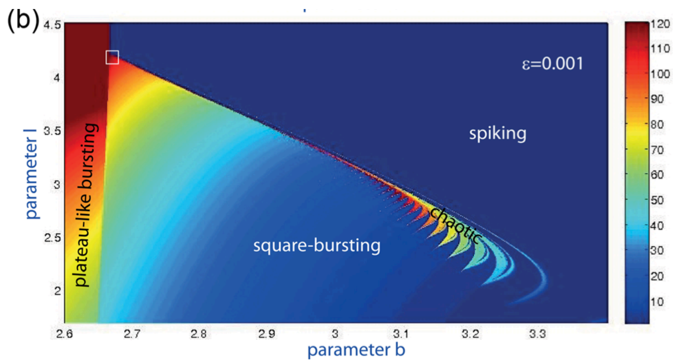
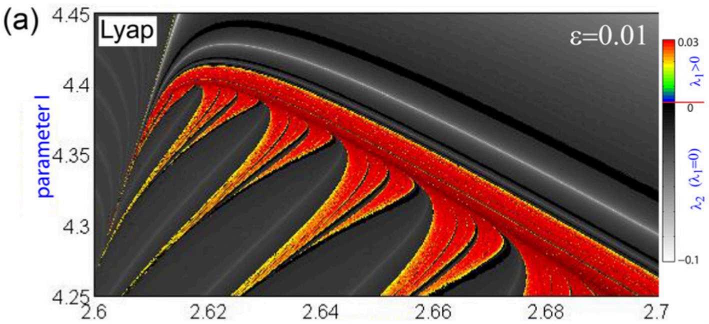


Figure 16 (b-I) parameter sweep of HR model[1]

In the above large scale diagrams is investigated the regions and the transitions, ‘corresponding to periodic tonic spiking, chaotic and regular bursting of the square-wave and plateau-like type’ [1].

Referring to the spike quantification approach, is when the number of spikes within a complete revolution of burst orbit around the spiking manifold is taken into the account. ‘When there is a fixed number of spikes per bursts is an indication of regular bursting but when there is unpredictable number of spikes associated with the chaotic dynamics, this is examined computationally with evaluation of Lyapunov exponent’[1]. ‘Also when there is an increasing inter-spike interval at the end of the burst, this is a signature of square wave bursting where indicates a homoclinic bifurcation’[1].

The dark blue region considered a stable single spiking activity. The ‘gradual change of colour in stripes where this correspond to bursting, showing an incremental number of spikes due to spike adding cascade’[1]. Bursting become a chaotic near the transition to tonic spiking in a chain of ‘’onion” like regions as is observed in figure a the dark red stripes and the change of colours in picture b) above. In addition, the sudden change in the number of spikes per bursts is associated with the transition from square-wave to plateau-like bursting.



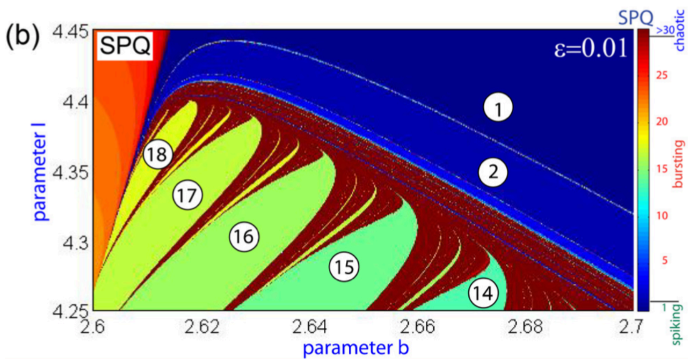


Fig 17. Present the 1st and 2nd Lyapunov ex[1] *Fig18. Shows SPQ in same region. [1]*

In the figure above is shown the typical structure for all the regions by evaluating the range of the Lyapunov exponent based on the range of parameter b and parameter ‘I’. ‘In picture a) represents the bi-parametric sweep using the Lyapunov exponents,

$\lambda_2 \leq \lambda_1$.’[1]. ‘The increasing value of the first, maximum Lyapunov exponent, $\lambda_1>0$ indicate the chaotic dynamics and quantifying the disorder degree and is represented by change of colour from blue to red. when there is $\lambda_1 = 0$ on a periodic orbit within its existence region, is evaluated the second Lyapunov exponent, λ_2 that determined its stability’ [1]. ‘Negative values of λ_2 are coloured in grey and with black colour means is close to zero, where means that corresponding multiplier of a periodic orbit is close to +1 or -1 and is about to go to period doubling or saddle node bifurcation. So a single black line passing through symmetrically grey area in the diagram should be interpreted as stable periodic orbit loosing stability through period doubling bifurcation'[1]. 'From picture a) above is shown that the transition from tonic spiking to square-wave bursting must pass through the a strip of chaotic dynamics.'[1]

Interpreting the picture b) where representing the bi-parametric sweep using the spike qualification, 'it can be observed from the diagram that the blue colour shows the tonic spiking or single-spike bursting. In this picture is shown the increase of spikes where numbered in circles and represents the number of spikes for bursts where the spike numbers fluctuating over the time in this phase plane and when

reaching a threshold limit for some single parameter values where is associated with red black, thus indicating a chaotic bursting'[1]. In both pictures above observed the change in these region related to chaotic dynamics.

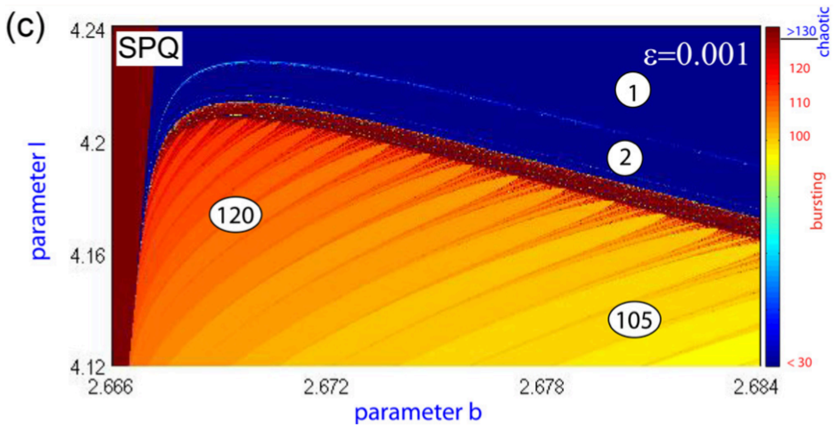


Figure 19 spike quantification sweep for ε=0.001.[1]

In the picture 19 c) is represented the spike quantification sweep with the parameter ε=0.001 where is the same notation with parameter 'r' in previous examples, 'where decreasing the value of 'ε' that results an proportional increase of spikes per regular bursts, so is observed a condensed structure'.[1] The numbers in the circle represent the spikes in this region as mentioned in previous pictures as well.

2.17 Micro Chaotic structure of Hindmarsh Rose model

In this part of the chaotic structure will be examined the micro structure where includes the co-existence of the chaotic bursting and the periodic orbits as can been seen from the magnification of the below picture as extracted from figure 4 of the selected reference [1].

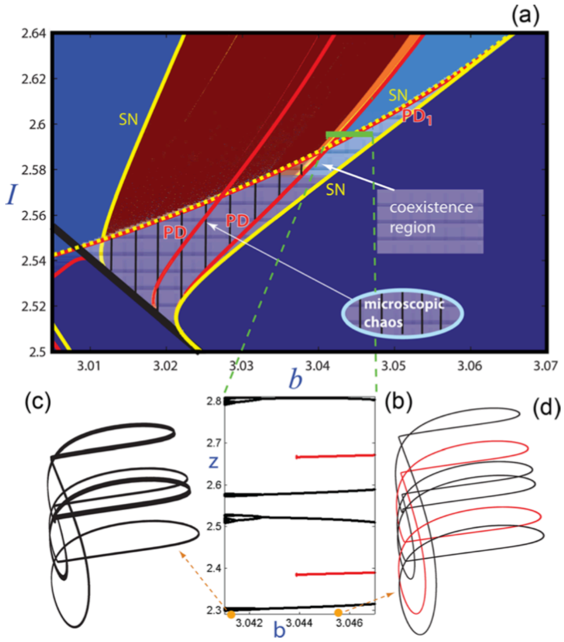


Figure 20 Magnification of micro chaos and coexistence region .[1]

In the above picture is observed the notation PD_1 where comes from the orbit flip bifurcation points with lower b values in the homoclinic curve.

In the magnified region picture 20 a) where extracted from figure 4 of the document in reference one, ‘is shown the crossing of the first period doubling bifurcation curve with the rest of the bifurcation curves of the chaotic layer. From the green line of picture a) as shown a projection below in picture b) the bifurcation diagram represent the co-existence of two attractors in respect to the values of parameter b higher than the corresponding value of first period doubling bifurcation as shown in picture d). Where this is due to fold bifurcation (yellow dotted) curve where touching the period doubling bifurcation and resulting one the attractor unstable.’[1] As is observed in picture d) the attractors with the red colour has fewer spikes in comparison with the attractor with black colour where becomes chaotic due to the bifurcations.

‘The coexistence of the chaotic attractors with the small basin of attraction as shown in picture 21 c) and d) , defined as the micro chaotic structure.’[1]

The second phenomenon that mentioned before is examined is the case of the bi -stability, is the hysteresis loop and will be described and visualised in the following picture.

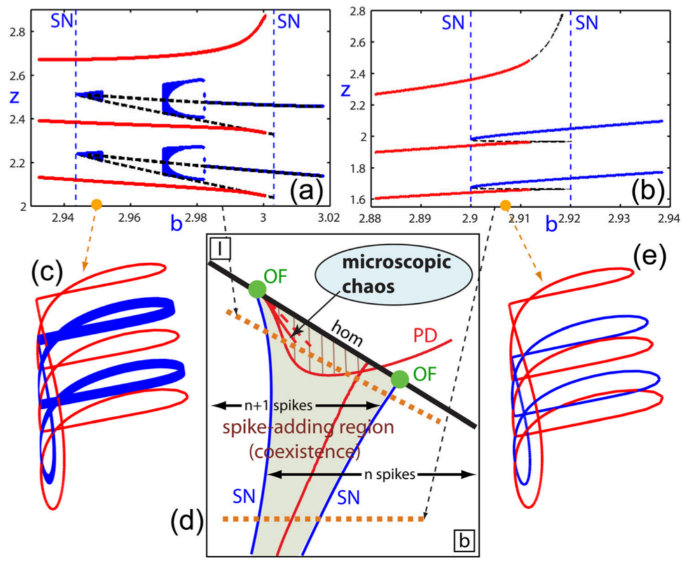


Figure 21 Structure of different bifurcation structures in micro chaos.[1]

In the above picture b) it can be seen the two hysteresis loops where the vertical dotted line shows the two saddle node bifurcation points, and looking the lines we can see the two solid lines representing the stable case and the dotted line represent the unstable, so resulting this bi stability of two attractors where separated by an unstable threshold. ‘There are different types of bi-stability in Hindmarsh Rose model but a typical one is the coexistence oft stable periodic orbits, one of which has an extra spike, as shown in picture d) above, results every spike adding bifurcations.’[1]

As a global picture of the existence between the micro chaos and macro chaos structure we refer to the following picture.

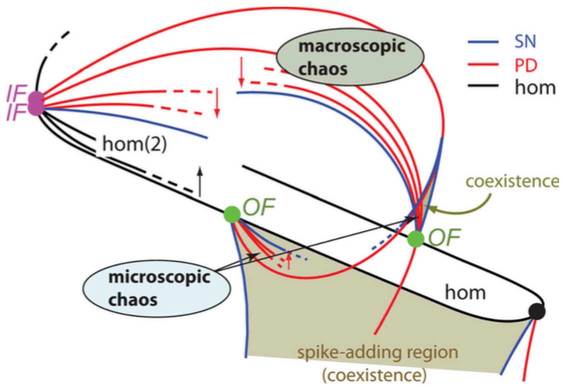


Figure 22 Global organisation of bifurcation diagram.[1]

In the above picture is represented a variety of the features, 'such as islands of micro and macroscopic chaos, and also including saddle node, period doubling, homoclinic bifurcation, spike addition regions and other elements describing the complex dynamics.'[1]

The strategy for exploration and bifurcation analysis of the model will be discussed in chapter 3 starting initially from the exploration and identification of fixed points as part of bifurcation process.

3.0 PRESENTATION OF RESULTS

3.1 Insights in 3D phase space due to parameters change

The first attempt from the author was to explore the changes in the trajectories during the simulation where the values of the several different parameters changing and producing bifurcation in the dynamical system. The author has adopted and modified a code of the Hindmarsh Rose model in python software by changing the values of parameters in order to observe the changes in the trajectory in the three dimensional space. This computational attempt was carried out in order to understand the sensitivity of the model in global and general form in 3D phase space due to the changes and the importance of the parameters in bifurcation and in general the nonlinear structure. Typical representative plots as shown below showing a global picture of the model.

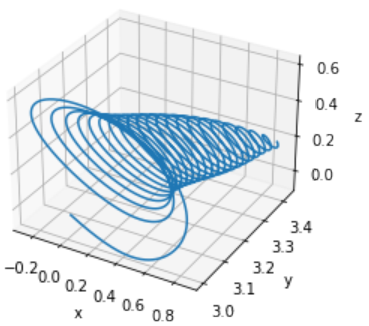


Figure 23, b=3.05, r=0.001,I=2.5

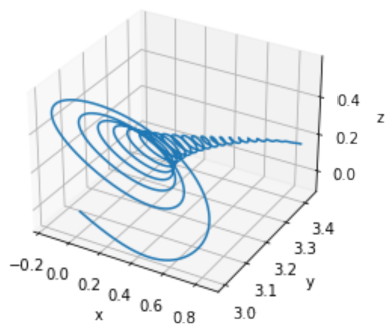


Figure 24, b=2.87, r=0.001,I=2.5

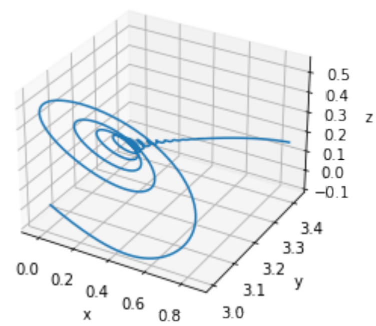
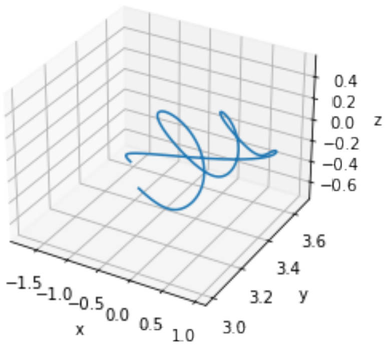
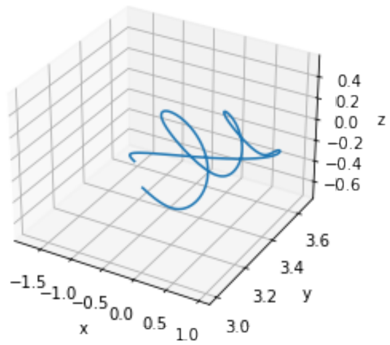


Figure 25, b=2.626, r=0.001,I=2.5

According with the author reading and the outcome from this simulation, the pictures above interpreted as an indication of the chaotic attractors in 3D space and when involves during the time, is observed such an initiation of periodic orbits with parameter value b=2.626 looking from the right picture to the left with the increase of parameter value ‘b’ value from 2. 626 to 2.87 and finally ‘b=3.05 where is observed an increasing number of turns (spikes) of bursting orbit. The current remained fixed and the parameter r, and only the parameter b, changed for this comparison.In addition, is observed a high amplitude of outer spiking manifold reducing to lower amplitude in the next turns further inside in parameter space. In this picture was not quantified the number of spikes or examined the inter-spike interval or explained other further bifurcation states.





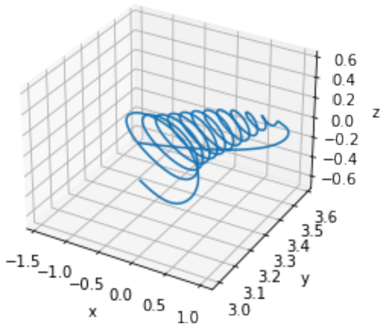
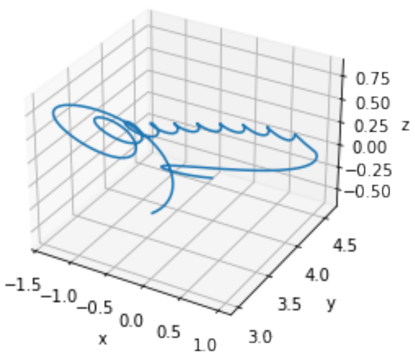
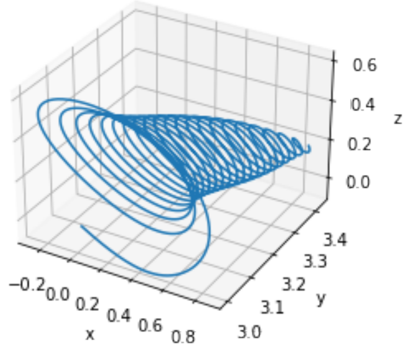


Figure 26, r=0.1, b=3.05, I=2.5 Figure 27, r=0.01, b=3.05, I=2.5 Figure 28, r=0.002, b=3.05, I=2.5

From the literature it was used the range of r between 0<r<1, and from the above outcome of the simulation, looking the figure 26 with r=0.1, b=3.05 and current I=2.5 is observed only two turns with large inter-spike interval between them and as compared with the middle picture where the r=0.01 we do not see any significant change in the non-linear structure geometrically and topologically. On the other hand when the r=0.002 then is observed a lot of turns with a structure similar to previous case but with the difference that at the end of the last turn the line comes back along all the way to the homoclinic orbit.





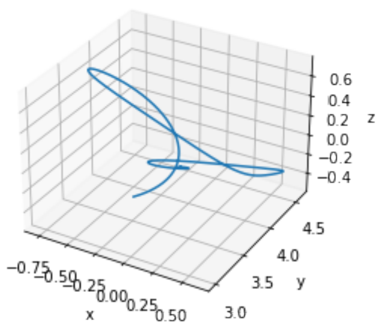


Figure 29, I=3.5, b=3.05, r=0.01, Figure 30, I=2.5, b=3.05, r=0.001   Figure 31, I=3.5, b=3.05, r=0.1

From the figure 29, is observed periodic orbit with inclined turns and at the end of the last turn the line of the orbit reruns back to the first turn where this phenomenon seems that undergoes a chaotic basin but not so complete. In this first picture of the simulation the values set with the current I=3.5, b=3.05 and r=0.01. In the middle figure 30 there is a combination of parameter setting with current I=2.5, b=3.05 same as before and change of r=0.001. With those settings the pattern that we receive is similar to the figure 29 from the first simulation. In order to highlight the complex dynamics that occurs from the different combination of parameters values resulting variations of orbits, we can look the last picture with I=3.5, b=3.05 and r=0.1, where we get completely different orbit in the 3D phase plane.

In order to investigate and explore more detailed the nonlinear structure of this continues dynamical system of the above computational results where related to the periodic orbits stability, fixed points or any other element, a mathematical analysis required which this important part of the next section.The following section introduces the first initial step by re- arranging the equations and use them for producing plots for exploring the Hindmarsh Rose mathematical model.

3.2 Mathematical re-arrangement of HR model equations

In order to start for the existence of the fixed points, the author used the HR equations by substituting the second and third equation to the first one and then solving the first equation in respect to x-variable where plotted graphically in next section. Refer below the re arrangement steps and representative plots in next section at exploring stage of the dynamics by changing as well certain values in the variables equations.

The final equation system of HR model that was used upon substitution of equation (4) and (5) in to the equation (1), (2) and (3) are as follow:

$$dx/dt = y - x^3 + bx^2 - z + I \quad (12)$$

$$dy/dt = 1 - 5x^2 - y \quad (13)$$

$dz/dt = r[s(x - x_{rest}) - z]$ (14)

setting the above equations to zero and re -arranging the equations (13) and (14) in order to use them in equation (12) and create one polynomial equation including only one variable x.

$0 = y - x^3 + bx^2 - z + I$

$0 = 1 - 5x^2 - y$

$0 = r[s(x - x_{rest}) - z]$

re- arrangement of the above equations results

$0 = y - x^3 + bx^2 - z + I$

$y = 1 - 5x^2$

$z = r[s(x - x_{rest})]$

Further re -arrangement and set the last two equations to the first results the final equation as follow. which was used for the exploration of the fixed points for the HR model.

$0 = 1 - 5x^2 - x^3 + bx^2 - (r[s(x - x_{rest})]) - I$

The following function in respect to x variable and in combination with the rest parameters was used for the exploration of the fixed points for the HR model.

$\mathbf{f(x) = 1 - 5x^2 - x^3 + bx^2 - (r[s(x - x_{rest})]) - I}$ (15)

Some values of the fixed parameters were substituted in above equation as per as mentioned in section 2.1, such as b=3.

In the following section are represented the plots based on the above equation in combination with the change of selected parameter values, in order to view the change in function dynamics.

3.3 Computational Insights for exploration of fixed points

The exploration of the fixed points by using the equation (15) from section 3.2 which was derived as represented in the following plots. The plots were separated according to the arbitrary selection of the parameters values within the previous ranges as mentioned in the literature.

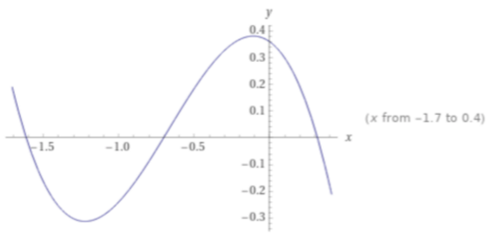


Figure 32 Plot with r=0.1 and I=0

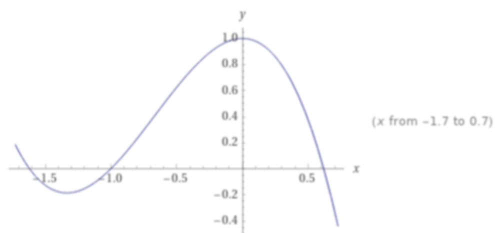


Figure 33 Plot with r=0 and I=0

In figure 32 the control parameter r=0.1 and parameter I=0 selected and upon completion of plotting, is observed that the function crossing three times the x axis where represent three fixed points as it is in figure 33 with r=0, and I=0, where also three fixed points are shown.

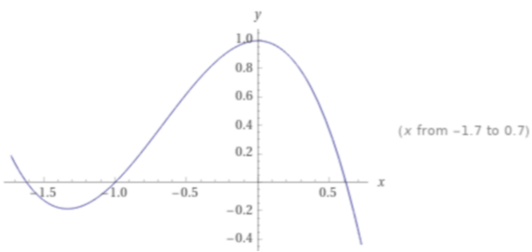


Figure 34 Plot with r=0.001 and I=0

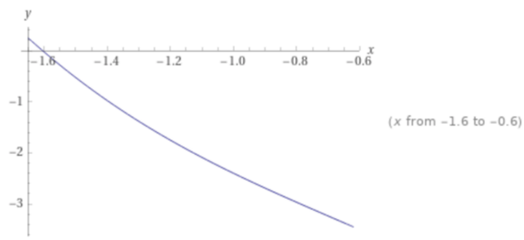


Figure 35 Plot with r=1 and I=0

In figure 34 with parameter r=0.001 and I=0, still is appeared three fixed points in comparison with the figure 35, with values r=1 and I=0 is observed one fixed point, as the function line crosses only one time the x-axis.

Figure 36 Plot with I=-10 and r =0.01

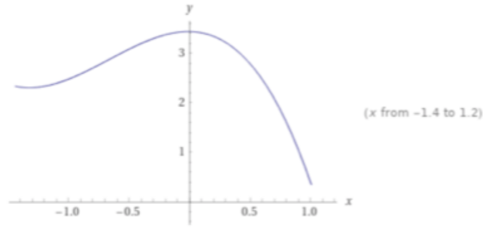


Figure 37 Plot with I=-2.5 and r=0.01

In figure 36, was used the highest negative current value from the range, I=-10 and r=0.01, where the plot shows one fixed point in contrast with the figure 37 where the current negative value I=-2.5 and with same r=0.01, the function tend to cross the x-axis in order to create a fixed point. In both figures, the function lines tend to have similar trajectory.

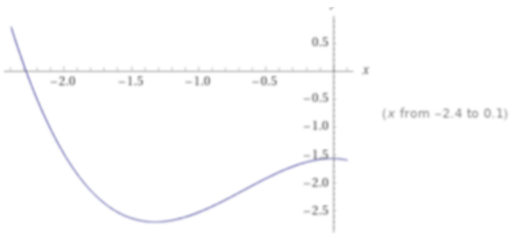


Figure 38 Plot with I=2.5 and r=0.01

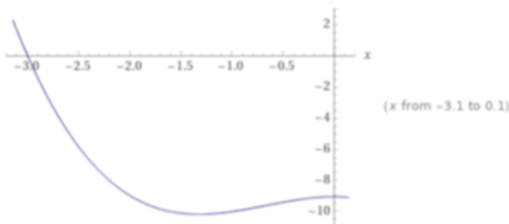


Figure 39 Plot with I=10 and r=0.01

In figure 38 is observed one fixed point crossing the negative part of x axis, where the trajectory belongs in the negative partition of coordinate system with value I=2.5 and r=0.01. Similar trajectory pattern is observed in figure 39 with I=10, and r=0.01 resulting again one fixed point.

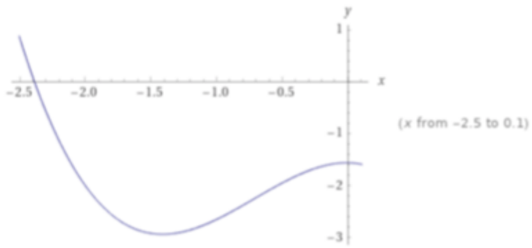


Figure 40 Plot with b=2.867, I=2.5, r=0.01

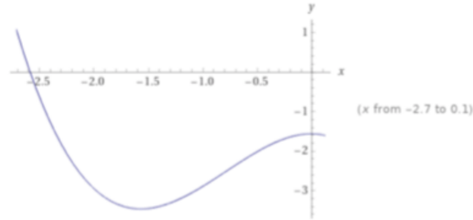


Figure 41 Plot with b=2.635, I=2.5, r=0.01

A combination of parameter values in figure 40, with b=2.867, I=2.5 and r=0.01 the trajectory of the function appeared in negative lower left part of coordinate system with one fixed point. On the other hand in figure 41 is observed similar trajectory pattern from the function with one fixed point with only change in parameter value b=2.867.

The reason of the above exploratory plots is to observe not only the existence of fixed points from the polynomial function and the possible number but also with the change of control parameter values to see if there is any change in trajectory pattern and in which partition of coordinate system could be appeared. In further section, will be explored the fixed point for the specific bifurcation diagram as produced from parametric plot.

3.4 Mathematical formulation of the equations in parametric form.

In this section the following equation that was produced previously will be solved and re-arranged in term of two parameters in order to construct the equations for the parametric plot.

$$\mathbf{f(x) = 1 - 5x^2 - x^3 + bx^2 - (r[s(x - x_{rest})]) - I}$$

The parameter values of b=3, s=4, x_{rest} = -1.6 and I=0 will be substituted in later stage of re-arrangement.

$0 = 1 - 5x^2 - x^3 + bx^2 - (r[s(x- x_{rest})]) - I$

$(r[s(x- x_{rest})]) - I = 1 - 5x^2 - x^3 + bx^2$

$(r[4(x- (-1.6)]) - I =1 - 5x^2 - x^3 + 3x^2$

The following equation is the first equation solved for the first parameter.

$$\boldsymbol{r = (1 - 5x^2 - x^3 + 3x^2 / 4x+6.4 - 0)} \quad (16)$$

Taking the derivative in respect of x the following equation and solving in respect to the second selected parameter 'b' we will produce the second parametric equation.

$f'(x) =1 - 5x^2 - x^3 + bx^2 - (r[s(x- x_{rest})]) - I$

re-arranging below, substitute the values from above lead to the following

$f'(x) = 1 - 5x^2 - x^3 + bx^2 - (r[4(x- (-1.6)]) - I$

The derivative of f(x) produces the below

$f(x) = -10x -3x^2 + 2bx -4r$

$0 = -10x -3x^2 + 2bx -4r$

$2bx = 10x + 3x^2 + 4r$

$$\boldsymbol{b = (10x + 3x^2 + 4r / 2x)} \quad (17)$$

where 'r' can take any arbitrary value within the specified range.

The author solved the below equation in respect of the another parameter 'I' in order to use it for another two equation combination of parametric plot.

$f(x) =1 - 5x^2 - x^3 + bx^2 - (r[s(x- x_{rest})]) - I$

$0 = 1 - 5x^2 - x^3 + bx^2 - (r[4(x- (-1.6)]) - I$

$I = 1 - 5x^2 - x^3 + bx^2 - 4rx+1.6r$

using r=1 and b=3 we have below by re-arranging

$$\boldsymbol{I = 1 - 5x^2 - x^3 + 3x^2 - 4x+1.6} \quad (18)$$

3.5 Parametric bifurcation diagram based on selected parametric equations.

The following bifurcation diagram was constructed from the parametric phase plane by selecting the above parametric equations. (16) and (17). The two parametric equations are as follow.

The first parametric equation as follow used for the plot. The selected arbitrary parameters values, b=2.626, and l=3.4 within the range -10 to 10.

$r = (1 - 5x^2 - x^3 + 3x^2 / 4x+6.4 - 3.4) = 1-5x^2 - x^3 + 3x^2 / 4x+3$

The second parametric equation used for the plot is as follow with r=0.01 inside.

$b = (10x + 3x^2 + 4r / 2x) = (10x + 3x^2 + 4 * 0.01/ 2x) = 10x + 3x^2 + 0.04/ 2x$

Refer below to the parametric bifurcation diagram constructed from the above equations. Is important to mention that the scaling of variable x was selected carefully in order to capture detailed bifurcation diagram.

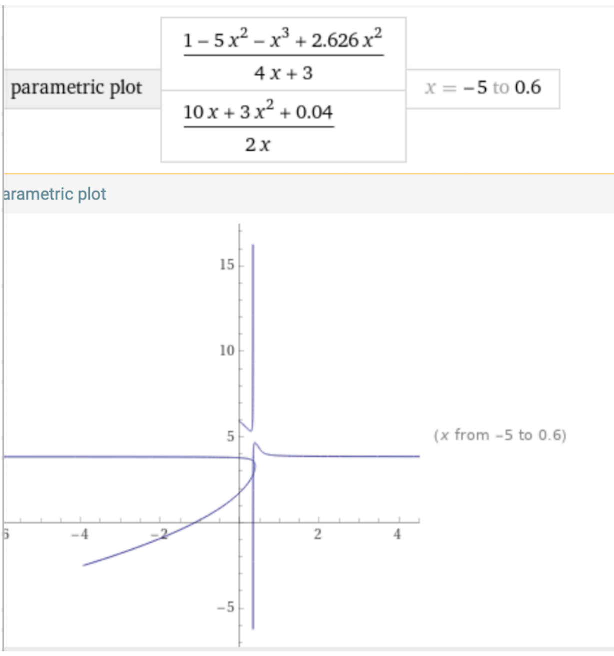


Figure 42 Bifurcation parametric diagram between parameter b and r.

The fold bifurcation occurs in the partition of the phase plane above and more analytically, in terms of lines orientation is shown an intersection almost tangentially, between a parabola and the asymptotic line in upper right positive part of the parametric plane. The vertical axis is the parameter ‘b’ and the horizontal axis is the parameter ‘r’ in the phase plane. Due to the cubic term is expected to have fixed points from one to three in different partitions of the parametric diagram above. The identification of the fixed points are shown from the following plots constructed from the initial polynomial cubic equation from section 3.2 which is used in the selected partitions from the bifurcation diagram.

To initialise the identification of fixed points the author selected the lower left partition, and selected arbitrary the values b, r, of (-3, -1) from the vertical and horizontal axis accordingly, from that partition in phase plane.
The above values belong to the lower left partition and located below to the lower curve line from parabolic shape from the line.

f(x) $= 1 - 5x^2 - x^3 + bx^2 - r[s(x - x_{rest})] - I$

entering the values **b=-3, r=-1**, s=4, I=0, $x_{rest} = -1.6$, below:

f(x) $1 - 5x^2 - x^3 + (-3)x^2 - (-1)[4(x - (-1.6)]) - 0$

f(x) $= 1 - 5x^2 - x^3 - 3x^2 + 1[4x + 6.4)]) - 0$

Plotting the resulting equation below for this function for b and r values:

f(x) $= -x^3 - 2x^2 + 4x + 7.4$ (19)

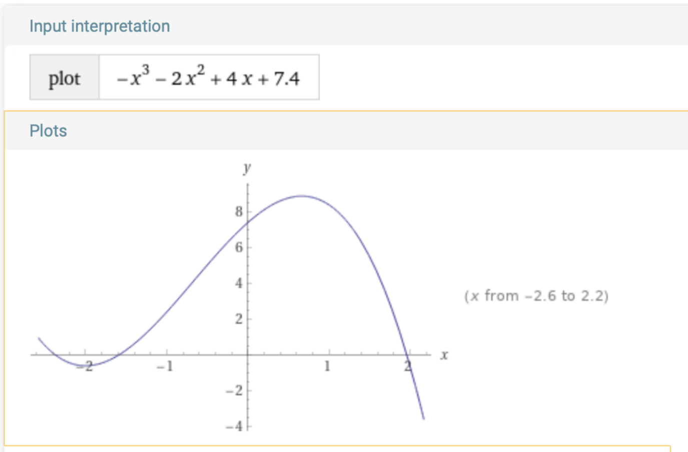


Figure 43 Fixed point from polynomial equation with b=-3, r=-1

From the above figure is identified three fixed points as we see the function line crossing three times the x - axis, where this result from figure 43, resulting that there are three fixed points in bifurcation diagram in this partition of the phase plane in figure 42 with the selected values b=-3 and r=-1.

In the next paragraph the same polynomial equation and the plot will be repeated in the neighbour partition choosing other parameter values b and r from the phase plane of that partition in order to prove that the fixed point change by two, resulting an alternate behaviour between one and three fixed points in neighbour partitions and this is happening in the entire phase plane due to cubic term of the equation.

The b and r following parameters values in this case selected for this point in the neighbour boundary curve line from previous case in left side of parametric plane from bifurcation diagram in figure 42, in order to be used in following equation and extract the number of fixed point in this neighbour partition.

$f(x) = 1 - 5x^2 - x^3 + bx^2 - r[s(x - x_{rest})] - I$

Entering the values **b=4, r=-2.5,** s=4, I=0, x_{rest} =-1.6, below:

f(x) $= 1 - 5x^2 - x^3 + 4x^2 - (-2.5)[4(x - (-1.6)] - 0$

f(x) $=1 - x^3 - x^2 + 2.5(4x+6.4)$

f(x) $=1 - x^3 - x^2 + 10x+16$

f(x) $= - x^3 - x^2 + 10x+17$

Plotting the resulting equation below for this function for b and r values:

$$\mathbf{f(x) = - x^3 - x^2 + 10x+17} \qquad (20)$$

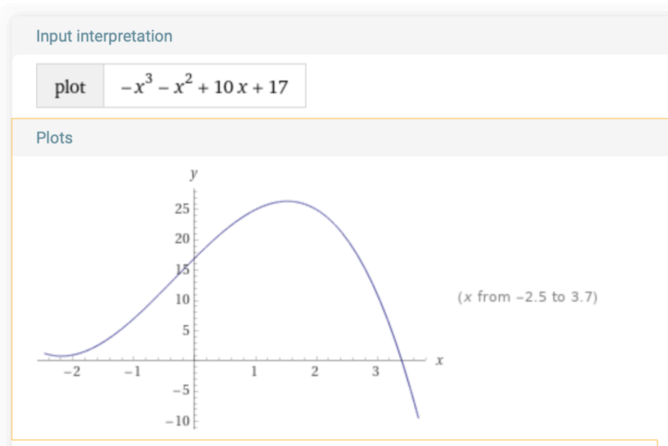


Figure 44 Fixed point from polynomial equation with b=4, r=-2.5

From the figure 44 is identified one fixed point as we see the function line crossing the x - axis, so connecting the result from figure 44, with the bifurcation diagram resulting that there is one fixed point in this partition in the phase plane in figure 42.

4.0 Conclusion

This dissertation is based on the mathematical exploration of Hindmarsh Rose model where considers the analysis of the three ordinary differential equations describing the bifurcation under operational neuron activity. The author had to cover learning material such as biological neuron physical operation, non-linear dynamics and bifurcation types in order to set a strategy for exploring and analysing the nonlinear dynamics where occurs in single neuron. The strategy of exploring the model is through the construction of the equations in the form in such a way, where is suitable to produce analytical plots with the support from literature survey and nonlinear dynamics material. The author at the beginning adopted and modified a code in python in order to make simulation and visualise the bifurcation of the model in 3D phase space. Upon completion of the simulation identified similar patterns and nonlinear elements such as periodic cycles, homoclinic pattern and basin attractor, comparing with the literature. The important outcome from this simulation was that when there is change of the parameter values for example in r, b and I, is noticed a change in the trajectory patterns. Since the above simulation produced a global picture of the model, the author formulated the HR equations in order, first to identify the existence of fixed points by combining the equation and producing one polynomial equation by plotting the fixed points. Secondly, the author rearranged the equations in parametric form that were used to plot the bifurcation diagram in respect to parameter b and r. Lastly, in order to start analysing the partitions of bifurcation diagram where the fixed points had to be identified in specific bifurcation diagram. The selection of parameter values b and r from one partition as arbitrary values in 2D plane from bifurcation diagram used in polynomial equation in order to plot and identify the number of fixed point. The same attempted again in a neighbour partition to identify the number of fixed points as well, where concluding that the number of fixed points change from one to three alternate in neighbour partitions separated by the boundary of trajectory lines an d this due to cubic term of the polynomial equation and the model.

4.1 Limitations

Specific bifurcation software had to be used for further analysis where this can be recommended for future work of the author for analysing the bifurcation numerically in more structured way. In addition, several computational methods, required to analyse in further detail the nonlinear structure in parametric space. Due to the variety of model trajectories and different combination of parameters, the author selected specific parameters and selected arbitrary values of the parameters around the specified range for the equations in order to produce representative plots. Additional time was required to produce several diagrams from simulation for big range or different combination of parameters in order to analyse the behaviour of the dynamics of this model further.

4.2 Further work suggestion

Due to the extensive nonlinear phenomena of the model, several computational tools, required to be adopted and used in future in order to explore the variety of the nonlinearity in terms of mathematical analysis and other parametric bifurcation diagram for interpretation. The author, formulated the equations in parametric form in respect of current 'I' and could be used in combination with another parameter to produce a bifurcation diagram. Further bifurcation types for exploration could be very interesting part for future work, where occurs in the bifurcation skeleton of the Hindmarsh Rose model under different neuronal activities. In addition, elements such as stability, limit cycles, period doubling or different phase portraits arise from trajectory patterns can be under investigation by calculations, analysis or simulations, providing all those results, as part of bifurcation process can be further subject for the author extending the scope of the present study. Different selection and combination of parameters could be part for further parametric future analysis and production of additional parametric bifurcation diagrams. To work on the above mentioned items, suitable training could be part for future improvement for the author with particular involvement of learning specific bifurcation softwares or any possible relevant future work opportunities arise.

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APPENDICES

APPENDIX A

A.1 Simulation of Hindmarsh Rose model in 3D please space

In [5]:
```
# Hindmarsh-rose model
import numpy as np
import matplotlib.pyplot as plt
```

In [ ]:
```
def RK4(f, x0, t):

    dt  =t[2] -t[1]  # time span
    N   = len(t)
    X   =np.empty((len(t), len(x0)))
    X[0]=x0

    for i in range(1, N):

        k1 =f(X[i-1], t[i-1])
        k2 =f(X[i-1] + dt/2*k1, t[i-1] +dt/2)
        k3 =f(X[i-1] + dt/2*k2, t[i-1] +dt/2)
        k4 =f(X[i-1] + dt*k3, t[i-1] + dt)

        X[i] =X[i-1] + dt/6*(k1 + 2*k2 +2*k3 +k4)

    return X
```

In [ ]:
```
def hindmarshRose(X, t):

    a=3.0         #fixed parameter a
    c=1.0         #fixed parameter c
    d=5.0         #fixed parameter d
    s=4.0         #fixed parameter s
    x0=-1.6       #-1.6, rest point

    #bifurcation parameters (control parameters)

    b=3.05      #3.05, 2.69,2.87,2.635, 2.626   # parameter b
    I= 3.5       #3.2, 5.0                      # parameter current rang
    eps=0.001     #0.01, 0.02, 0.001            # parameter equivalent t

    x,y,z=X

    dxdt=y-(a*x**3)+(b*x**2)+I-z
    dydt=c-(d*x**2)-y
    dzdt=eps*((s*(x-x0))-z)

    return np.array([dxdt, dydt,dzdt])
T=np.linspace(0, 100, 10000)

Y=[0.03, 0.03, 3]

param=RK4(hindmarshRose, Y, T)

ax=plt.axes(projection='3d')

zline=param[:,0]
xline=param[:,1]
yline=param[:,2]

ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')

ax.plot3D(xline, yline, zline)
```

Out[11]: [<mpl_toolkits.mplot3d.art3d.Line3D at 0x7ff485a67d00>]