Synchronization of Phase Coupled Oscillators

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1.0 Introduction

Time plays important role in the life of living things. The individual and social behavior of living things are determined and governed by cycles of different periods. In biological life, these cycles are a mechanism for survival and require precise timing. The occurrence of these cycles is hard to understand and can be only described as synchronization of individual actions in a population. Some of these cycles occur systematically while some occur spontaneously. Heartbeat is an example of synchronized activity and this can be felt when listening to instrumental music where your heartbeat tends to follow the music rhythm that is heartbeats can accelerate or decelerate because of music rhythm (Menen, 2004). The applause of people in a hall tends to be incoherent at the beginning and transformed into coherent applause, that is synchronized applause. Fireflies is also a display a natural synchronous flashing during the night, one firefly starts to emit a flash of light and suddenly many fireflies start to flash. The incoherent flashing at the start ends up being uniform among the swarm of fireflies over some time (Acebrón et al., 2005). The understanding of the synchronization phenomena is inevitable to know how nature and manmade things operate.

The paper structure is as follows. Section two describes the Kuramoto model that will be used to show the synchronization of the coupled oscillators and some empirical findings of the model were discussed. Section 3 provides the analysis of the differential equations of the model. The fourth section examines the linear stability of the model equation and the last section provides a summary and conclusion of the findings.

2.0 The Concept of Synchronization

The concept of synchronization in the dynamic systems of coupled oscillators is an important aspect considered when dealing with nonlinear and complex systems. By nonlinear systems in mathematics, we are referring to the mathematical models whose change in output does not reflect the exact change in input. The nonlinear equations are very interesting and found in almost every field like engineering, biology, physics, mathematics, and social sciences. The area of synchronization of phase coupled oscillators has a wide scope and its applications in many fields have drawn the attention of many researchers trying to understand and simplify the phenomenon that most people find difficult to understand. Grosu et al. (2016) used to phase in-plane and proposed a new general method of designing coupling between oscillators that exhibit phase synchronization. Their numerical results suggested that for Lorenz systems, the phase synchronization equals antiphase synchronization depending on the conditions in place when designing the model. Therefore, they developed a new Lorenz phase synchronized system using the new network designed.

Definition of coupling between oscillators in-phase coupled oscillators model is very important for one to understand the general concept of synchronization. Synchronization is a phenomenon in which a group of moving things try to relate or match. The synchronization behavior was first observed and recorded by Christian Huygens who observed the two pendulum clocks' motion in early 1665 (Huygens, 1967). Huygens stated that the moving pendulum clocks tend to adjust their time to be the same by trying to move at the same speed and pattern. Since then many kinds of research have been conducted with the main objective of determining when the moving array of systems tend to stabilize. According to Scafetta et al. (2016), coupled oscillations happen where two or more objects moving in rhythmic motions are related such that motion energy can be exchanged between them. The coupled oscillation objects occur naturally like the moon and earth that orbit each other and some coupled oscillators are man-made for example pacemakers.

Therefore, the main objective of this paper was to examine the synchronization of phase-coupled oscillators using the Kuramoto model. We examined the Kuramoto synchronization model and carried out an analysis on the two oscillators to show when the systems are stabilized and when the systems are unstable.

3.0 The General Kuramoto Model

Yoshiki Kuramoto a Japanese nonlinear dynamics physician proposed the Kuramoto model in 1975. The Kuramoto model is a mathematical model developed to explain the synchronization phenomenon. The model specifically describes the behavior of repetitive movements or coupled oscillators over time (Wang, 2020). Yoshiki Kuramoto's motivation was how biological and chemical oscillators behave when in the group. He later learned that the Josephson junctions followed his model principle. Josephson junctions are a system of coupled arrays through which current continuous flow without any voltage being applied. The physical system consists of two or more superconductors coupled by a weak link.

The general Kuramoto model consists of a population of N coupled phase oscillators, each oscillator believed to have its intrinsic natural frequency that is the oscillations are independent of other frequencies but coupling tries to be coherent or synchronize the oscillators to form a stable movement pattern. The Kuramoto model is stated as

$$\theta_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \text{ Where i = 1,…,N …… ….. (1)}$$

The equation one above can be transformed to an array of phase oscillators with zero means of its natural frequencies by making a perfect choice of rotating frame $\theta i \rightarrow \theta i - \Omega t$ where Ω is the first moment of $g(\omega)$. The transformed equation natural frequencies of oscillators are characterized with zero means where the linked system will be stable. The phase coupled oscillator's exhibit incoherent movement when the coupling is weak enough but the synchronization occurs beyond a certain threshold in coupling.

K is a coupling that has been used in various models for coupling like random long-range coupling, hierarchical coupling, nearest-neighbor coupling, and dependent interactions state. This section of the paper introduces the Kuramoto model that has mean-field coupling among its phase oscillators. Kuramoto model with mean-field coupled oscillators is the simplest model used to explain the synchronization phenomenon in both natural and manmade systems. The synchronization in the Kuramoto model is simply measured by parameter order. The number of oscillators in the coupled system can be up to infinite, that is $N = \infty$, the maximum displacement of the order of parameter diminishes as the oscillators move out of synchrony and the order parameter of oscillators is positive in synchronized states. The partial synchronization of phased coupled oscillators and bifurcation from incoherence systems is calculated on the closed interval between $-\pi$ and π with the same probability. Then analyze the stability of incoherence using the limit $N = \infty$. In the case of coupling constants a critical value of coupling is used, for example, $k < k_c$. Since the spectrum of its linear stability falls on the imaginary axis, incoherence is said to be neutrally stable.

According to Strogatz et al. (1992), the displacement movement from incoherence oscillators diminishes similarly to the Landau damping in plasmas. Landau damping is a phenomenon observed in plasma where exponential decay in the oscillations of the density electrons of plasma achieves a stable state in a certain area in phase space. A positive eigenvalue is observed from the spectrum in situations where unimodal natural frequency distribution and K > Kc are considered. In the case of K = Kc, the partially synchronized state bifurcates is observed from incoherence oscillators, there are no proofs that stability exists. The major assumption made in the Kuramoto model is that coupling strength K is positive in relation to the interaction between the oscillators.

Gushchin et al. (2015) carried out a study on the synchronization of the system of homogeneous phase-coupled with plastic coupling strength and arbitrary underlying topology. They pointed out that the phase difference between oscillators determines the coupling strength between two moving oscillators. In their work they show that two oscillating systems are gradient and always achieve synchronization in their frequency. Furthermore, sufficient stability and instability states are based on algebraic graph theory. Using the topology tree, they formulated stability for equilibria. Finally, they illustrated differences in the behavior of systems that has constant and plastic coupling strengths.

Acebrón et al. (2005) explained the synchronization phenomena in large populations of interacting objects that are subjected to intense research efforts in different fields. The fields identified to exhibit such researches include biology, chemical, physical and social systems. The study suggests that the best approach to synchronization problems is population modeling as a phase oscillator and the Kuramoto model is a useful model in analyzing the synchronization in coupled phase oscillators. The research analyzed the Kuramoto model and concluded that the model is applicable in different contexts.

Moioli et al. (2010) investigated the neural dynamics of simulation used in robots that perform cognitive tasks. The researchers employed the Kuramoto model of coupled oscillators as the nervous system of the automated robots. The main purpose of their study was to give an understanding of the new application of neuronal synchronization and generation towards cognitive technology. The study investigated the concept of restricting robot movements to a certain area, and the way robots adopt both scenarios by approaching their tasks in both inverted and normal. The researchers used community behaviors as benchmarks in the development and evolution of robots to adapt human behavior. Using the Kuramoto model they concluded that robots are embodied with cognitive behaviors.

3.1 Kuramoto Model with two Coupled Oscillators

The Kuramoto model with two oscillators refers to a system of two linked objects in movement. The best example that will be used in this paper is two clocks that are linked (Moioli et al., 2010). The similarity between the system of two coupled oscillators and an overdamped particle is that there exist two possibilities in their life namely synchronized state and unsynchronized phase. In the synchronized state, the particle is zero, and if the particle drifts then the system is said to be unsynchronized. To state the Kuramoto model for two coupled oscillators we start from the general model where the population is N.

General N model

$$\theta_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad \text{……… …… … …… ………………} \; (1)$$

Where, i = 1,…,N

For a two coupled oscillator model N = 2 and equations of the Kuramoto model are given as

$$\frac{d\theta_1}{dt} = \omega_1 + \frac{K}{2}\sin(\theta_2(t) - \theta_1(t) + a) \; \text{… ……………} \; (2)$$

$$\frac{d\theta_2}{dt} = \omega_2 + \frac{K}{2}\sin(\theta_1(t) - \theta_2(t) + a) \ldots \ldots\ldots\ldots\ldots \ldots (3)$$

Equations (2) and (3) are the Kuramoto model with N = 2, the model can be used to explore and explain the concept of synchronization between two oscillating objects. To explain the parameters of the model we illustrated by using two clocks denote as A and B.

The parameters of our model are ω, K and α where ω is the frequency, K refers to the coupling and α is the phase shift or drift between the two clocks. Neutral frequency (ω) is also known as Eigen frequency is the frequency at which a system oscillates in absence of external force induced on the system. Frequency is measured as the number of occurrences of repeated events per unit of time. For the two oscillators' model, ω_1 is the frequency of clock A and ω_2 is the frequency of clock B. The frequencies of the clocks show the speed at which each clock is moving and the angle θ increases with time. This means that if the clock moves at a higher speed then its angle increases faster. K is an important parameter known as coupling. Coupling refers to the interaction between two objects that can be induced by electromagnetic forces. Therefore, in this study when we talk about coupling we mean force resulting from the interaction between clock A and clock B. Naturally, some objects do not interact hence in such cases coupling will be equal to zero (K = 0). Some interactions may have negative or positive interactions meaning that K can be either negative or positive. Simply K lies between negative infinite and positive infinite. Finally, we have α, which is phase drift and it shows unintended frequency due to the speed of interaction between clocks.

3.2 Analysis of Kuramoto Model without Coupling

As we discussed earlier, coupling in the Kuramoto model represents the force that results from the interaction between objects in the system. This force of interaction may not exist between two objects which naturally possible. Some systems either natural or manmade can interact and synchronize without generating energy between them. In such cases coupling is equal to zero, that is, K = 0. If k = 0 then the Kuramoto model is given as

$$\frac{d\theta_1}{dt} = \omega_1 \text{ ……… …………. (4)}$$

$$\frac{d\theta_2}{dt} = \omega_2 \text{ ……… ……….…. (5)}$$

Equations 4 and 5 show that the oscillators have only frequencies ω_1 and ω_2, which are constants. In the model without coupling, it means that the only factors affecting the angle θ of the clocks are the time and frequency ω of the clock. It is simple to tell if the system synchronizes in the Kuramoto model without coupling because we just take the difference between two equations (4) and (5) then compare the frequencies. For example, if $\omega_1 = \omega_2$ then it is said that the system synchronizes and when $\omega_1 \neq \omega_2$ then system do not synchronize. When $\omega_1 = \omega_2$ then the clocks move at the same rate and show the same time implying that they have synchronized. If $\omega_1 \neq \omega_2$ simply means that one clock is moving at a higher frequency than the other is hence displays a different time depending on the angle θ of the clock.

3.3 Kuramoto Model with Coupling

According to Simonović (2013), the coupling coefficient in the system of oscillators because synchronization of the system depends on coupling. He examined the effect of coupling on the phase coupled oscillators and concluded that the coefficient of coupling determines if the synchronization is high or low. To examine the effect of coupling on synchronization we adopted the Kuramoto model with coupling takes the general model format. This model shows that the interaction between the two clocks or systems results in a force that triggers synchronization. The model with coupling analyses how external interaction between oscillators can lead to coherent activity between interacting objects. The synchronization occurs due to internal or external stimuli. Some of the examples include synchronization between heartbeat and music rhythm, the fireflies flashing synchronization, brain cells synchronization, and applause. The external forces can either be negative or positive but not zero, meaning that k ≠ 0. The model is given as

When K ≠ 0 and α = 0 then,

$$\frac{d\theta_1}{dt} = \omega_1 + \frac{K}{2}\sin(\theta_2(t) - \theta_1(t)) \ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots \ldots\ldots\ldots \quad (6)$$

$$\frac{d\theta_2}{dt} = \omega_2 + \frac{K}{2}\sin(\theta_1(t) - \theta_2(t)) \ldots\ldots \ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots \quad (7)$$

Equations (6) and (7) represent the Kuramoto model with coupling. The problem is to determine if there is a solution where the two clocks synchronize. To find the solutions of the model, the two differential equations (6) and (7) are reduced to one equation as follows

The difference between equation (6) and (7),

$$(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}) = (\omega_1 - \omega_2) + (\frac{K}{2}\sin(\theta_2(t) - \theta_1(t)) - (\frac{K}{2}\sin(\theta_1(t) - \theta_2(t))$$
$$\frac{d(\theta_1 - \theta_2)}{dt} = (\omega_1 - \omega_2) - K\sin(\theta_1 - \theta_2)$$
$$\frac{d\alpha(t)}{dt} = \delta - K\sin(\alpha(t))$$

The reduced differential equation is

$$\frac{d\alpha(t)}{dt} = \delta - K\sin(\alpha(t)) \ldots \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots \quad (8)$$

Where α (t) represents the difference between angles of the oscillators, δ is the difference between frequencies of clock A and B and K is the force resulting from the interaction between the two clocks. The reduced equation (8) shows the interaction model between the two clocks A and B. The equation (8) was analyzed in two different approaches by considering values of K. First we analyzed the reduced equation when K = 0 and second section K ≠ 0

When K = 0

The reduced differential is

$$\frac{d\alpha(t)}{dt} = \delta - 0\sin(\alpha(t))$$

$$\frac{d\alpha(t)}{dt} = \delta \quad \text{...... .... (9)}$$

$$\alpha(t) = \delta.t + \alpha(0) \quad \text{.............. .... (10)}$$

If δ = 0 from equation (10), then

$$\alpha(t) = \alpha(0)$$

This shows that if δ = 0 the difference between angles of oscillators is constant at the time (t). The constant difference between angles implies that the two oscillators will be moving at equal frequencies, that is $\omega_1 = \omega_2$. As we discussed earlier if the system is moving at equal frequency then the system is at a synchronized solution. The comparison between frequencies is only applicable in a situation system that has no forces resulting from their interaction and only time and frequency determine the angle difference of the two oscillators.

The second part is the analysis of the reduced differential equation (8) when δ ≠ 0. The δ quantity is the difference between frequencies of oscillators and if the quantity is not equal to zero then the two clocks are moving at a different speed. The solution when K = 0 and δ ≠ 0 is not constant and is given as

$$\alpha(t) = \delta.t + \alpha(0)$$

The solution shows that the difference between two-oscillator angles depends on the function of time that is the frequencies of two oscillators vary with time. Since the difference in frequency depends on time, then the system is unsynchronized.

When K ≠ 0, then when we assume that the two-phased oscillators will be at a synchronized state when the solution for equation (8) is a constant. Therefore, we need to solve for the constant solution of

$$\frac{d\alpha(t)}{dt} = \delta - K\sin(\alpha(t))$$

Such that α (t) = α (0) a constant.

$$0 = \delta - K\sin(\alpha_c)$$

$$\delta = K\sin(\alpha_c)$$

$$\frac{\delta}{K} = \sin(\alpha_c) \quad \text{……… …… ……………} \; (12)$$

The solution for equation (12) exists only if the absolute value of |δ/K| ≤ 1. If the ratio is greater than one the solution lies in the imaginary part of the system meaning that solution does not exist. The results show that coupling parameter K has a significant impact on the synchronization of the system compared to the difference between the frequencies. For example, given any value for frequency difference, the system can still synchronize as long as the coupling is greater than the difference in frequencies that is |δ| ≤ |K| for the synchronization point to existing.

4.0 Fixed Points and Linear Stability Analysis

4.1 Fixed Points

Fixed points refer to the points where the differential equation is constant that is its derivative equals zero. At fixed points in our case, the clocks move towards each other or away from each other at a constant speed. These points sometimes are referred to as stagnation points (Strogatz et al., 1992). In a stable state, the solution for the equation is

$$\delta = K\sin(\alpha_c)$$
$$\frac{\delta}{K} = \sin(\alpha_c)$$

Now the task is to determine the two exact points where the solution is linear stable. The exact values for α that the system will be stable are

$$\alpha_1 = \arcsin\left(\frac{\delta}{K}\right)$$
$$\alpha_2 = \pi - \alpha_1 \qquad \text{…… ……………… ……………….. (13)}$$

At α_1 and α_2 the two clocks synchronize that is, the clocks have an equal angle and at other points, the system does not synchronize. The two points are the fixed points only if $|\delta| \leq |K|$.

4.2 Linear Stability Analysis

Linear stability analysis is a technique used to determine whether the fixed points are stable or unstable. Linear stability analysis of linear dynamic equations formulated from observed experimental dynamics flow. The linear stability analysis depends on the nature of the observed experimental equation. The fixed point α_c is stable if $f^1(\alpha_c) < 0$ and the point is unstable if $f^1(\alpha_c) > 0$.

At fixed point α_1 linear stability analysis is

$$f^1(\alpha) = -K\cos(\alpha)$$
$$f^1(\alpha_1) = -K\cos(\alpha_1)$$
$$f^1(\alpha_1) = -K\cos(\arcsin(\frac{\delta}{K}))$$
$$f^1(\alpha_1) = -K\sqrt{1-\sin^2(\arcsin(\frac{\delta}{K}))}$$
$$f^1(\alpha_1) = -K\sqrt{1-\frac{\delta^2}{K^2}}$$

Where

$f^1(\alpha_1) < 0$ if $K > 0$ and

$f^1(\alpha_1) > 0$ if $K < 0$

The stability of the fixed point depends on the quantity of coupling K.

At fixed point α_2 linear stability analysis is

$$f^1(\alpha_2) = -K\cos(\pi - \arcsin(\frac{\delta}{K}))$$
use
$$\cos(\pi - z) = -\cos z$$
$$f^1(\alpha_2) = K\cos(\arcsin(\frac{\delta}{K}))$$
$$f^1(\alpha_2) = K\sqrt{1-\frac{\delta^2}{K^2}}$$

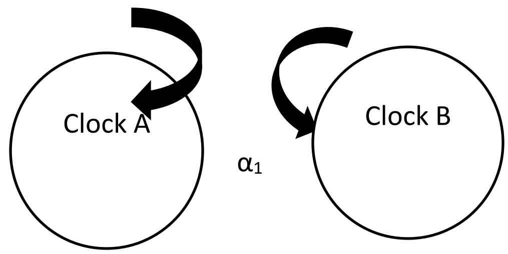
Where,

$f^1(\alpha_2) < 0$ if $K < 0$ and

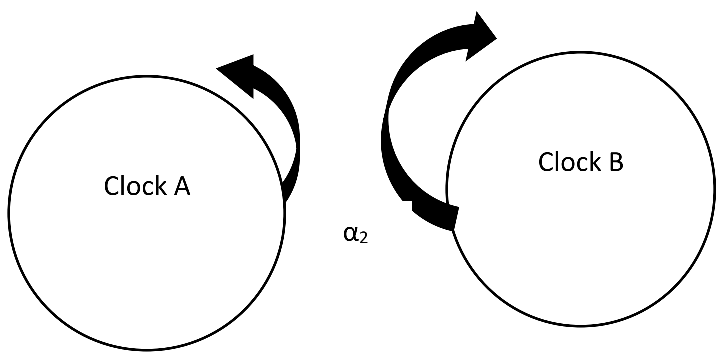
$f^1(\alpha_2) > 0$ if $K > 0$

The linear stability analysis shows that the fixed point will only be stable if and only if the derivative of equation (8) is less than zero and unstable if the derivative of equation (8) is greater than zero. The quantity of derivative depends on the strength of coupling K. Therefore, at fixed point α_1 the system is stable if $K>0$, and at fixed point α_2 the system is stable if $K<0$.

At fixed point α_1 clock A moves clockwise while clock B moves in an anti-clockwise direction.



At fixed point α_2 clock A move in an anti-clockwise direction while clock B move in a clockwise direction as shown below



5.0 Summary and Conclusion

The purpose of this paper was to examine the synchronization in the phase coupled oscillators. Synchronization is a phenomenon in which vibrating or moving systems tend to move at the same rate due to interaction between them or the time effect. Synchronization occurs in either natural or manmade systems like fireflies, heartbeat, applause, and moon and sun interaction. These systems tend to interact and tend to move at the same speed. The Kuramoto model of Japanese physicist Yoshiki Kuramoto (1975) of order two was used to examine synchronization of phase coupled oscillators. Kuramoto model has two parameters namely frequency and coupling. Frequency is the number of movements a system makes per period and on the other hand, the coupling is the force that emerges from the interaction between the two oscillators. Coupling is in the two oscillators triggers synchronization in most systems. From the analysis, we identified that coupling can be absent in some systems that no force originates from the interaction between the oscillating clocks.

In absence of coupling, the synchronization depends on the frequencies of the two oscillating clocks, and the system synchronizes only if the frequencies of the two clocks are equal that is $\omega_1 = \omega_2$. It was also noted that in the case in presence of coupling synchronization is triggered making it could be either negative or positive; K takes values from $-\infty$ to ∞ In case of coupling is not equal to zero ($K \neq 0$), the reduced equation (8) was analyzed. Equation (8) was obtained by finding the difference between the two differential equations of the system (6) – (7). In the analysis of equation (8), it was found that coupling is a significant determinant of synchronization in the oscillating system because for the synchronization solution to exist then the following condition must hold $|\delta/K| \leq 1$ or $|\delta| \leq |K|$. The synchronization point does not exist in case $|\delta/K| > 1$. Therefore, the quantity of coupling is very important in the Kuramoto model and it should be greater than the frequency of the oscillators. The linear stability analysis of the coupling oscillators shows that the synchronization at two fixed points given in equation (13). The condition for linear stability analysis that a fixed point if stable if $f^1 (\alpha_c) < 0$ and unstable if $f^1 (\alpha_c) > 0$. Linear stability shows that the system is stable at fixed point α_1 if $K > 0$ and stable at point α_2 if $K < 0$ this means that the stability of the system depends on the coupling strength at fixed point. Elsewhere the system is unstable.

In conclusion, the Kuramoto model is a very effective tool in explaining the synchronization phenomena between phase coupling oscillators but the computation of equations involved in the model is very cumbersome.

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