On the relation between sensitivity and correlation decay in piecewise linear maps

MSc Project

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Abstract

Chaotic motion shows two characteristic features, sensitive dependence on initial conditions (sometimes referred to as the butterfly effect) and a decay of correlations, despite of its deterministic character. Surprisingly, it is quite difficult to establish a relation between the two quantities which characterise these two behaviours, namely the Lyapunov exponent and the decorrelation rate. Based on a study by Badii et al. published in Physical Review A [1], we explore such a relation for the simple case of piecewise linear full branch maps.

Contents

1 Introduction

The main purpose of this project is to establish a relationship between the Lyapunov exponent and the Correlation decay rate. The motivation for doing this project was primarily an interest in dynamical systems from the Masters course and also to undertake a challenging project in an interesting area of research. The opportunity to gain a deeper understanding of dynamical systems not covered in lectures was appealing.

Dynamics can be either "simple" or complicated. In the case of "simple", we can have something which is periodic, for example, a clock. This has very simple dynamics since if the clock differs slightly from the actual time, this difference does not increase over time i.e. the next day the mismatch is similar, provided it has not stopped. There is no sensitive dependence on initial conditions and if we compare what the clock shows now with what it will show in 12 hours the results are pretty much the same and we see that the outcomes are correlated.

Complicated dynamics is just the opposite, it is chaotic. It normally shows sensitive dependence on initial conditions, for example the so called butterfly effect, where a small change at one place in a non linear system can result in large differences to a later state. Another example would be the decay of correlations, for instance, the correlation decay observed in weather forecasting. If you make the simplest possible prediction, say tomorrows weather is the same as today's, you may have a 70% success rate but that rate drops drastically if the following days are predicted in the same way i.e. the correlation decays.

The purpose of the study is to relate these two phenomena, and to establish a relation between the two characteristic quantities, the Lyapunov exponent and the correlation decay rate. We will study one dimensional expanding maps, specifically the Bernoulli shift map to derive an inequality between the decay rate and the Lyapunov exponent for this particular model.

In Chapter 2 we give a brief overview of Markov maps on which we would base our analysis. The following two Chapters introduces the map and discusses the computation of the Lyapunov exponent and the correlation decay. Finally in Chapter 5 we will establish the inequality and see that it is related to the concept of convex functions.

2 General Piecewise Linear Markov maps and Markov Partitions

This Chapter will be mostly a revision of textbook material from reference[2], that is useful in the subsequent development.

The term "Markov" means "memoryless". In other words the probability of each outcome conditioned on all previous history is equal to conditioning on only the current state; no previous history is necessary.

2.1 Partition of an interval

This subsection is appropriate here to introduce some notation.

Let $I = [a, b]$ be a closed interval. $|I| = |x_2 - x_1|$ denotes the size of the interval, and $int(I) = (a, b)$ the interior of I.

A collection of closed intervals $I_0, I_1, ..., I_N - 1$ is called partition of I if $I = \cup_{k=0}^{N-1} I_k$, and $int(I_k) \cap int(I_l) = \emptyset$ if $k \neq l$.

Example 2.1. *Consider the following two partitions of the interval* $I = [-1, 1]$.

1. $I_0 = [-1, -\frac{1}{3}], I_1 = [-\frac{1}{3}, \frac{2}{3}], I_2 = [\frac{2}{3}, 1]$, *so* $[I_0, I_1, I_2]$ *is a partition because the union of* I_0, I_1, I_2 *gives the interval* I *and all the intersections between* I_0, I_1, I_2 *gives the empty set.*

2. $I_0 = [-1, \frac{1}{2}], I_1 = [\frac{1}{4}, 1]$. *This is not a partition because the intervals overlap in more than their end points.*

2.2 General Piecewise Linear Markov maps and Markov Partitions

The General Piecewise Linear[1] Markov maps are one-dimensional chaotic systems which are amenable to analysis. They are defined as follows:

[1] A piecewise linear function is a piecewise-defined function whose pieces are linear. For example, since the graph of a linear function is a line, the graph of a piecewise linear function consists of line segments and rays.

A map $f : I \to I$ in $\mathbb{R}$, is called a Markov map if there exists a partition $\{I_0, I_1, ..., I_{N-1}\}$, (a Markov partition) such that for all $k, l = 0, ..., N-1$:

1. either $int(I_l) \cap f(int(I_k)) = \emptyset$
2. or $int(I_l) \subseteq f(int(I_k))$.

We will now consider the Bernoulli Shift map (also called Shift map, doubling map, dyadic transformation), and show that it is an example of a Piecewise Linear Markov map.

Example 2.2. *Bernoulli Shift map with two branches.*

$$f(x) = \begin{cases} 2x+1 & \text{if} \quad -1 \leq x < 0, \\ 2x-1 & \text{if} \quad 0 \leq x < 1, \end{cases} \tag{1}$$

Using the interval $I = [-1, 1]$ we split it into two partitions $I_0 = [-1, 0]$ and $I_1 = [0, 1]$, and get $f(int(I_k))$ for $k = 0, 1$.

$$\begin{aligned} int(I_0) = (-1, 0) \quad &\to \quad f(int(I_0)) = (-1, 1) \\ int(I_1) = (0, 1) \quad &\to \quad f(int(I_1)) = (-1, 1) \end{aligned}$$

Now using the statement for a Markov map, we check whether the Bernoulli Shift map satisfies either of the conditions.

1. $int(I_0) \subseteq f(int(I_0))$ ✓
2. $int(I_0) \subseteq f(int(I_1))$ ✓
3. $int(I_1) \subseteq f(int(I_0))$ ✓
4. $int(I_1) \subseteq f(int(I_1))$ ✓

All of these cases satifies the condition, so we conclude that the Bernoulli Shift map is a Markov map.

An important property of Markov maps is that many of their statistics (which we call correlation statistics) can be calculated in closed form. The broad definition of correlation statistics makes them potentially applicable in a number of chaotic data analysis. Applications will be presented throughout the next Chapters of this Project.

The statistics of a chaotic system depend on its Frobenius-Perron equation and it will be shown next.

3 Frobenius-Perron Equation and Lyapunov Exponents

This Chapter will explain the Frobenius-Perron equation and the Lyapunov Exponent. For convinience we are going to consider our maps on the interval $[-1, 1]$. Bernoulli Shift maps are going to be used. They are often considered on the interval $[0, 1]$, but here for our purpose the interval $[-1, 1]$ is more adequate. Two examples will be explained in each subsection. The first example we will consider equal slopes, whereas the second example will show different slopes.

3.1 Frobenius-Perron Equation

In a dynamical system, we are often interested in the overall behavior of the map, in other words, the evolution of an ensemble of initial conditions. The Frobenius-Perron equation is used to describe this evolution.

Let $f : I \to I$ be a (continuous) map and $h(x)$ be an integrable function such that $h : I \to \mathbb{R}$. The non-negative integrable function $\rho : I \to \mathbb{R}$ is called Invariant density if:

$$\int_I \rho(x)\mathrm{d}x = 1, \quad \text{and} \tag{2}$$

$$\int_I h(x)\rho(x)\mathrm{d}x = \int_I h(f(x))\rho(x)\mathrm{d}x \quad \text{for any function.} \tag{3}$$

The Frobenius-Perron equation acts on probability densities of dynamical systems, these are non-negative real functions whose integral over the whole phase space is unity. The image of a density represents it's time-evolution under the dynamics, which is obtained by summing the density over all pre-images of the point being considered. Each summand is weighted by the reciprocal of the absolute value of the derivative of the map. A fixed-point of the Frobenius-Perron equation is an invariant density. If the system is ergodic (in other words, it can not be decomposed into two invariant set of positive measures), the integral of an observable with respect to an invariant density gives the time average of the observable (integrable function). For more details the reader may consult references [3, 6].

Let $f : I \to I$ denote a (piecewise) smooth map with invariant density ρ. Then the density obeys the so called Frobenius-Perron equation:

$$\rho_\star(x) = \sum_{y \in f^{-1}(x)} \frac{1}{|f'(y)|} \rho_\star(y) \tag{4}$$

Example 3.1. *Consider the Bernoulli Shift map with same slopes as we have shown in equation (1):*

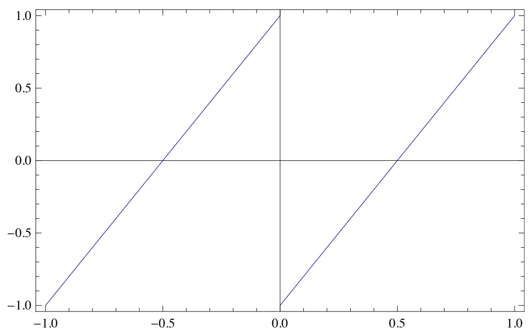

Figure 1: Bernoulli Shift map with same slopes.

$$f(x) = \begin{cases} 2x+1 & \textit{if} \quad -1 \leq x < 0, \\ 2x-1 & \textit{if} \quad 0 \leq x < 1, \end{cases}$$

This is a map defined on the interval $[-1, 1]$*, which has two branches, each branch having slope 2. We call the two branches* f_0 *and* f_1*, and the inverses of these branches are denoted by* f_0^{-1} *and* f_1^{-1}*.*

$$f^{-1}(x) = \left\{ f_0^{-1}(x), f_1^{-1}(x) \right\}$$

The explicity formulas for these branches are given by:

$$f_0(x) = y = 2x+1 \qquad f_0^{-1}(x) = \frac{x-1}{2}$$

$$f_1(x) = y = 2x-1 \qquad f_1^{-1}(x) = \frac{x+1}{2}$$

Finally, if we have a point x, the two pre images are given by:

$$f^{-1}(x) = \left\{ \frac{x-1}{2}, \frac{x+1}{2} \right\}$$

$$\begin{aligned} \textit{Now let } f^{-1}(x) &= \Phi(x) \\ \textit{Then } |\Phi_0'(x)| &= \frac{1}{2} \\ \textit{and } |\Phi_1'(x)| &= \frac{1}{2} \end{aligned}$$

Now we will use equation (4), which is the Frobenius-Perron equation, stated earlier.

$$\rho_\star(x) = \sum_{y \in f^{-1}(x)} \frac{1}{|f'(y)|} \rho_\star(y)$$

To explain in a simple way the meaning of $\rho_\star$, it is a density function that describes the distribution of all points. This is an equation which is difficult to solve, but let's try for the solution:

First we substitute $|\Phi'(x)|$ and $\rho_\star$ for the two branches, as follows:

$$\rho_\star(x) = \sum_{y \in \Phi(x)} |\Phi'(x)| \rho_\star(y), \qquad \textit{where } \Phi(x) = \left\{ \frac{x-1}{2}, \frac{x+1}{2} \right\} \tag{5}$$

$$\begin{aligned} &= \Phi_0'(x)\rho_\star(\Phi_0(x)) + \Phi_1'(x)\rho_\star(\Phi_1(x)) \\ &= \frac{1}{2} \underbrace{\rho_\star\left(\frac{x-1}{2}\right)}_{\frac{1}{2}} + \frac{1}{2} \underbrace{\rho_\star\left(\frac{x+1}{2}\right)}_{\frac{1}{2}} \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ \rho_\star(x) &= \frac{1}{2} \end{aligned}$$

So we can see here that the equation solves and is even normalised, in other words it's an Invariant density.

Example 3.2. *The Bernoulli Shift map with different slopes.*
The map is given by:

$$f(x) = \begin{cases} \gamma_0(x+1)-1 & \text{if} \quad -1 \leq x \leq -1+\frac{2}{\gamma_0}, \\ \gamma_1(x-1)+1 & \text{if} \quad -1+\frac{2}{\gamma_0} \leq x \leq 1, \end{cases} \tag{6}$$

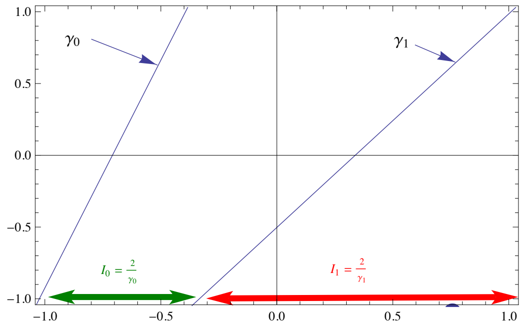

Figure 2: Bernoulli Shift map with different slopes.

This is again a map defined on the interval $[-1,1]$, *with two branches, each branch having different slopes. The branches are called* f_0 *and* f_1, *and the inverses are denoted by* f_0^{-1} *and* f_1^{-1}.

$$f^{-1}(x) = \left\{f_0^{-1}(x), f_1^{-1}(x)\right\}$$

Observing the graph, we note that:

$$\frac{2}{\gamma_0}+\frac{2}{\gamma_1}=2 \qquad \rightarrow \qquad \frac{1}{\gamma_0}+\frac{1}{\gamma_1}=1 \tag{7}$$

Multiplying out the equations in both branches and relabling the constants we get:

$$\begin{aligned} f_0(x) &= y = \gamma_0 x + \underbrace{\gamma_0 - 1}_{a_0} \\ f_0^{-1}(x) &= \frac{1}{\gamma_0}x + \underbrace{\frac{1}{\gamma_0} - 1}_{b_0} \\ f_1(x) &= y = \gamma_1 x + \underbrace{1 - \gamma_1}_{a_1} \\ f_1^{-1}(x) &= \frac{1}{\gamma_1}x + \underbrace{1 - \frac{1}{\gamma_1}}_{b_1} \end{aligned}$$

Note that a_0, a_1, b_0 and b_1 are all constants.
Finally, if we have a point x, the two pre images are given by:

$$f^{-1}(x) = \left\{\frac{1}{\gamma_0}x + b_0 \ , \ \frac{1}{\gamma_1}x + b_1\right\}$$

$$\begin{aligned} \textit{Let } f^{-1}(x) &= \Phi(x) \\ \textit{Then } |\Phi_0'(x)| &= \frac{1}{\gamma_0} \\ \textit{and } |\Phi_1'(x)| &= \frac{1}{\gamma_1} \end{aligned}$$

Now using the Frobenius-Perron equation (4), stated earlier, we substitute $|\Phi'(x)|$ and $\rho_\star$, for the two branches, as follows:

$$\rho_\star(x) = \sum_{y\in\Phi(x)} |\Phi'(x)|\rho_\star(y), \qquad \textit{where } \Phi(x) = \left\{\frac{1}{\gamma_0}x + b_0 \ , \ \frac{1}{\gamma_1}x + b_1\right\}$$

$$\begin{aligned} &= \Phi_0'(x)\rho_\star(\Phi_0(x)) + \Phi_1'(x)\rho_\star(\Phi_1(x)) \\ &= \frac{1}{\gamma_0}\underbrace{\rho_\star\left(\frac{1}{\gamma_0}x + b_0\right)}_{\frac{1}{2}} + \frac{1}{\gamma_1}\underbrace{\rho_\star\left(\frac{1}{\gamma_1}x + b_1\right)}_{\frac{1}{2}} \\ &= \frac{1}{\gamma_0}\cdot\frac{1}{2} + \frac{1}{\gamma_1}\cdot\frac{1}{2} \\ &= \frac{1}{2}\cdot\left(\frac{1}{\gamma_0} + \frac{1}{\gamma_1}\right) \\ &= \frac{1}{2}\cdot(1) \\ \rho_\star(x) &= \frac{1}{2} \end{aligned}$$

As the previous example (3.1), the equation solves again here for the map with different slopes, and is normalised, in other words it's an Invariant density.

3.2 Lyapunov Exponent

Consider two nearby initial conditions x_0, x_0' and the corresponding orbits $(x_0, x_1, x_2, ...)$, $(x_0', x_1', x_2', ...)$. Then $f(x_k') \approx f(x_k) + f'(x_k)(x_k' - x_k)$ and,

$$\begin{aligned}
|x_1' - x_1| &= |f(x_0') - f(x_0)| \approx |f'(x_0)||x_0' - x_0| \\
|x_1' - x_1| &\approx |f'(x_1)||x_1' - x_1| \approx |f'(x_1)||f'(x_0)||x_0' - x_0| \\
|x_n' - x_n| &\approx |f'(x_{n-1})||f'(x_{n-2})|...|f'(x_0)||x_0' - x_0| \\
&= e^{\sum_{k=0}^{n-1} ln|f'(x_k)|}|x_0' - x_0| = e^{n\Lambda}|x_0' - x_0|
\end{aligned}$$

Thus the distance grows at an exponential rate

$$\Lambda = \frac{1}{n}\sum_{k=0}^{n-1} \ln|f'(x_k)|, \qquad (n \gg 1) \tag{8}$$

Let $f : I \to I$ be a piecewise smooth map and let $(x_0, x_1, x_2, ...)$ denote the orbit with initial condition x_0. If the limit exists, the value Λ is called Lyapunov exponent of the orbit.

$$\Lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln|f'(x_k)|, \qquad (\Lambda = \Lambda(x_0)) \tag{9}$$

Important:

1. If $\rho(x)$ denotes an Invariant density then (choose $h(x) = \ln|f'(x)|$):

$$\Lambda = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln|f'(x_k)| = \int_I \ln|f'(x)|\rho(x)\mathrm{d}x \tag{10}$$

2. If $f : I \to$ I is a piecewise linear Markov map with slopes γ_l then:

$$\int_I \ln|f'(x)| \sum_l \rho_l \chi_l(x)\mathrm{d}x = \sum_l \ln|\gamma_l|\rho_l|I_l| \tag{11}$$

3. Λ also measures the average loss of information about the position of a point in [-1,1] after one iteration.

4. Λ measures sensitivity!

For more details the reader may consult reference [2].

Example 3.3. *Considering the Bernoulli Shift map with the same slopes as we have shown in equation (1), we will show the Lyapunov exponent:*

$$f(x) \quad = \quad \begin{cases} 2x+1 & \textit{if} \quad -1 \leq x < 0, \\ 2x-1 & \textit{if} \quad 0 \leq x < 1, \end{cases}$$

We find that the derivative is given by: $|f'(x)| = 2$

Substituting $|f'(x)|$ *and* $\rho_\star(x)$ *on the equation (10) we get:*

$$\begin{aligned}
\Lambda &= \int_{-1}^{1} \ln 2 \cdot \frac{1}{2} \mathrm{d}x \\
&= \left[\frac{x}{2} \ln 2\right]_{-1}^{1} \\
&= \left(\frac{1}{2} \ln 2\right) - \left(-\frac{1}{2} \ln 2\right) \\
\Lambda &= \ln 2 > 0
\end{aligned}$$

The Lyapunov exponent is positive, therefore the map is chaotic.

Example 3.4. *In this example we will consider the Bernoulli Shift map with different slopes, in other words the equation (6), that was introduced on the example (3.2), to find the Lyapunov exponent:*

The equation (6), is given by:

$$f(x) = \begin{cases} \gamma_0(x+1) - 1 & \textit{if} \quad -1 \leq x \leq -1 + \frac{2}{\gamma_0}, \\ \gamma_1(x-1) + 1 & \textit{if} \quad -1 + \frac{2}{\gamma_0} \leq x \leq 1, \end{cases}$$

Observing the graph, Figure (2), we have shown equations (7), and we can also note that the intervals I_0, I_1 *are given by:*

$$I_0 = -1 + \frac{2}{\gamma_0} - (-1) = \frac{2}{\gamma_0}$$

$$I_1 = 1 - (-1 + \frac{2}{\gamma_0}) = 2 - \frac{2}{\gamma_0}$$

The derivatives are as follows:

$$f'(x) = \gamma_k \begin{cases} \gamma_0 = f_0'(x) \\ \gamma_1 = f_1'(x) \end{cases}$$

Using the formula for the Lyapunov exponent, given by equation (11):

$$\begin{aligned}
\Lambda &= \sum_{k=0}^{n=1} \rho_k \ln |\gamma_k||I_k| \\
&= \frac{1}{2}(\ln \gamma_0) \cdot \left(\frac{2}{\gamma_0}\right) + \frac{1}{2}(\ln \gamma_1) \cdot \underbrace{\left(2 - \frac{2}{\gamma_0}\right)}_{\frac{2}{\gamma_1}} \\
&= \frac{1}{2}\left[\frac{2}{\gamma_0}\ln \gamma_0 + \frac{2}{\gamma_1}\ln \gamma_1\right]
\end{aligned}$$

$$\Lambda = \frac{1}{\gamma_0}\ln \gamma_0 + \frac{1}{\gamma_1}\ln \gamma_1 > 0 \tag{12}$$

$$\gamma_0 > 0, \quad \gamma_1 > 0 \quad \textit{and} \quad \ln \gamma_0 > 0, \quad \ln \gamma_1 > 0$$

The Lyapunov exponent is positive, therefore the map is chaotic.

If you check the graph, Figure (2), we can see how the Lyapunov exponent depends on the slope γ_0 or $\frac{1}{\gamma_0}$.
Using $\frac{1}{\gamma_0}$ gives the advantage that it is always between $[0, 1]$ and γ_0 is between $[1, \infty]$, the maximum approximately 2 when plotted as a function of the slope. It becomes smaller as the slope goes to 1 and also becomes smaller as the other slope goes to ∞.

4 Correlation functions

Given a time series the mean value tells you the time average of the quantity h. The Correlation function tells you how, on average, the value x_n after n time steps depends on the value x_0. The main purpose of the Chapter is the computation of a Correlation function in the particular case, "The Bernoulli Shift map with different slopes". This map was introduced in example (3.2), equation (6).

The Correlation function $C(m)$ for a Bernoulli Shift map is defined by:

$$C(m) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{x}_k \hat{x}_{k+m} \tag{13}$$

where

$$\hat{x}_k = f^k(x_0) - \bar{x} \; ; \quad \bar{x} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f^k(x_0) \tag{14}$$

The arguments follow the lines of reference [2, 4].

From this definition, it follows that $C(m)$ yields another measure for the irregularity of the sequence of iterates $x_0, f(x_0), f^2(x_0), ...$ It tells us how much the deviations of the iterates from their average value, $\hat{x}_k = x_k - \bar{x}$ that are m steps apart (i.e., $\hat{x}_{k+m}$ and $\hat{x}_k$) "know" about each other, on the average.

The computation of mean values is given by:

$$\langle h(x) \rangle = \int h(x)\rho(x)\mathrm{d}(x) \tag{15}$$

If the invariant density $\rho(x)$ for the Bernoulli map with different slopes is known, $C(m)$ can be written in the form:

$$C(m) = \int_{-1}^{1} f^{(m)}(x)\; x\; \rho(x)\mathrm{d}x - \left[\int_{-1}^{1} x\; \rho(x)\mathrm{d}x\right]^2 \tag{16}$$

Where m is the m^{th} iterate.

Note that only the first integral for $C(m)$ is going to be useful, because the other two are equal to zero, as shown:

$$\int_{-1}^{1} x\; \rho(x)\mathrm{d}x = \rho(x)\int_{-1}^{1} x\; \mathrm{d}x = \rho(x)\left[\frac{x^2}{2}\right]_{-1}^{1} = \rho(x)\cdot 0 = 0 \tag{17}$$

The commutative property of iterates is used:

$$x_{k+m} = f^{k+m}(x_0) = f^k f^m(x_0) = f^m f^k(x_0) \tag{18}$$

In the next subsections we are going to work out the Correlation function for three examples, by considering the cases when $m = 1$, $m = 2$, and $m = 3$. We will notice that using an inductive argument we can show that there is a General Formula for $C(m)$.

4.1 Computation of $C(1)$

Next we will compute the Correlation function for the Bernoulli Shift map with different slopes, when $m = 1$ and I=[-1,1], example (3.2). See figure (2).

The map given by equation (6):

$$f(x) \;=\; \begin{cases} \gamma_0(x+1)-1 & \text{if} \quad -1 \le x \le -1+\frac{2}{\gamma_0}, \\ \gamma_1(x-1)+1 & \text{if} \quad -1+\frac{2}{\gamma_0} \le x \le 1, \end{cases}$$

The map has two branches, f_0 defined on the interval I_0, and f_1 defined on the interval I_1.

Using the Correlation function equation (16) we have:

$$C(1) = \int_{-1}^{1} f^{(1)}(x)\; x\; \rho(x)\mathrm{d}x \;\; - \;\; \underbrace{\int_{-1}^{1} x\; \rho(x)\mathrm{d}x \int_{-1}^{1} x\; \rho(x)\mathrm{d}x}_{0} \tag{19}$$

The first iterative is derived as:

$$\begin{aligned} f_0(x) &= y = \gamma_0 x + a_0 \quad \rightarrow \quad \text{defined on the interval } I_0 = \left[1, -1+\frac{2}{\gamma_0}\right]. \\ f_0^{-1}(x) &= \frac{1}{\gamma_0}x + b_0 \\ f_1(x) &= y = \gamma_1 x + a_1 \quad \rightarrow \quad \text{defined on the interval } I_1 = \left[-1+\frac{2}{\gamma_0}, 1\right]. \\ f_1^{-1}(x) &= \frac{1}{\gamma_1}x + b_1 \end{aligned}$$

Note that $f_0(I_0) = I$ and $f_1(I_1) = I$.

Differentiating this gives:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= f_0'(x) = \gamma_0 & \qquad \frac{\mathrm{d}y}{\mathrm{d}x} &= f_1'(x) = \gamma_1 \\ \mathrm{d}x &= \frac{\mathrm{d}y}{\gamma_0} & \qquad \mathrm{d}x &= \frac{\mathrm{d}y}{\gamma_1} \end{aligned}$$

We split the range of integration into two intervals, such that I_0, I_1, is a partition of $I = [-1, 1]$. Substituting these into the equation (19) we get:

$$= \underbrace{\int_{I_0} f_0(x)\; x\; \frac{1}{2}\mathrm{d}x}_{f_0(x)=y\;,\; I\;,\; dx.\underbrace{f_0'(x)}_{\gamma_0}=dy} \quad + \quad \underbrace{\int_{I_1} f_1(x)\; x\; \frac{1}{2}\mathrm{d}x}_{f_1(x)=y\;,\; I\;,\; dx.\underbrace{f_1'(x)}_{\gamma_1}=dy}$$

When we replace the x values for y values the integration takes over the whole interval $I = [-1, 1]$.

$$= \int_I y \left(\frac{1}{\gamma_0} y + b_0\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_0} \;+\; \int_I y \left(\frac{1}{\gamma_1} y + b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_1}$$

Multiplying out the brackets and collecting the terms together we get:

$$\begin{aligned} &= \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_0}\right)^2 \frac{1}{2} \,\mathrm{d}y \;+\; \underbrace{\int_{-1}^{1} y\, b_0 \,\frac{1}{2} \left(\frac{1}{\gamma_0}\right) \mathrm{d}y}_{0} \\ &+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_1}\right)^2 \frac{1}{2} \,\mathrm{d}y \;+\; \underbrace{\int_{-1}^{1} y\, b_1 \,\frac{1}{2} \left(\frac{1}{\gamma_1}\right) \mathrm{d}y}_{0} \end{aligned}$$

Canceling down all the zero terms and factorizing it gives:

$$\begin{aligned} &= \left\{ \left(\frac{1}{\gamma_0}\right)^2 + \left(\frac{1}{\gamma_1}\right)^2 \right\} \frac{1}{2} \int_{-1}^{1} y^2 \,\mathrm{d}y \\ &= \left\{ \left(\frac{1}{\gamma_0}\right)^2 + \left(\frac{1}{\gamma_1}\right)^2 \right\} \frac{1}{2} \cdot \frac{2}{3} \end{aligned}$$

Finally, simplifying we have:

$$C(1) = \left\{ \left(\frac{1}{\gamma_0}\right)^2 + \left(\frac{1}{\gamma_1}\right)^2 \right\} \frac{1}{3} \tag{20}$$

This is the Correlation function for $m = 1$.

4.2 Computation of $C(2)$

Next we will compute the Correlation function for the Bernoulli Shift map with different slopes, when $m = 2$ and I=[-1,1], example (3.2), equation (6). This map has 4 branches.

Observing Figure (3) we know that:

$$\frac{2}{\gamma_0\gamma_0} + \frac{2}{\gamma_0\gamma_1} + \frac{2}{\gamma_1\gamma_0} + \frac{2}{\gamma_1\gamma_1} = 2$$

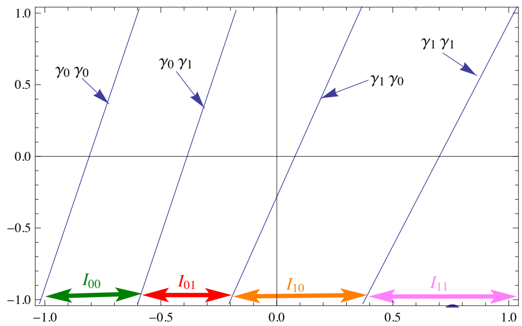

Figure 3: Bernoulli Shift map with different slopes and $m = 2$.

Now, when $m = 2$ we have four intervals, such that $I_{00}, I_{01}, I_{10}, I_{11}$, is a partition of $I = [-1, 1]$.

$$
\begin{aligned}
f^2_{00}(x) = f_0(f_0(x)) &= \gamma_0\,(\gamma_0 x + a_0) + a_0 &\rightarrow\quad I_{00} &= \left[-1, -1 + \frac{2}{\gamma_0\gamma_0}\right].\\
f^2_{01}(x) = f_0(f_1(x)) &= \gamma_0\,(\gamma_1 x + a_1) + a_0 &\rightarrow\quad I_{01} &= \left[-1 + \frac{2}{\gamma_0\gamma_0}, \frac{2}{\gamma_0\gamma_1}\right].\\
f^2_{10}(x) = f_1(f_0(x)) &= \gamma_1\,(\gamma_0 x + a_0) + a_1 &\rightarrow\quad I_{10} &= \left[\frac{2}{\gamma_0\gamma_1}, 1 - \frac{2}{\gamma_0\gamma_1}\right].\\
f^2_{11}(x) = f_1(f_1(x)) &= \gamma_1\,(\gamma_1 x + a_1) + a_1 &\rightarrow\quad I_{11} &= \left[1 - \frac{2}{\gamma_1\gamma_0}, 1\right].
\end{aligned}
$$

and the derivatives are given by:

$$
\begin{aligned}
f^{2'}_{00}(x) &= {\gamma_0}^2 = \gamma_0\gamma_0\\
f^{2'}_{01}(x) &= \gamma_0\gamma_1\\
f^{2'}_{10}(x) &= \gamma_1\gamma_0\\
f^{2'}_{11}(x) &= {\gamma_1}^2 = \gamma_1\gamma_1
\end{aligned}
$$

Using the Correlation function equation (16) we get:

$$
C(2) = \int_{-1}^{1} f^{(2)}(x)\; x\; \rho(x)\mathrm{d}x \;-\; \underbrace{\int_{-1}^{1} x\; \rho(x)\mathrm{d}x\; \int_{-1}^{1} x\; \rho(x)\mathrm{d}x}_{0} \tag{21}
$$

The second iterative is derived as:

$$\begin{aligned} y = f_{00}^2(x) &= {\gamma_0}^2 x + \gamma_0 a_0 + a_0 \\ x &= \frac{y}{{\gamma_0}^2} - \underbrace{\frac{a_0}{{\gamma_0}^2} - \frac{a_0}{\gamma_0}}_{b_1} \\ x &= \frac{y}{{\gamma_0}^2} - b_1 \end{aligned}$$

Differentiating this gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f_{00}^{2'}(x) = {\gamma_0}^2$$

$$\mathrm{d}x = \frac{\mathrm{d}y}{{\gamma_0}^2}$$

Using the same method we can get the second iterative of the intervals $f_{01}^2(x), f_{10}^2(x), f_{11}^2(x)$. Note that the derivatives of the intervals were given on the previous page.

We split the range of integration into the four intervals. Substituting these into the equation (21) we have:

$$\begin{aligned} &= \int_{I_{00}} f_{00}^2(x)\ x\ \frac{1}{2}\mathrm{d}x \ + \int_{I_{01}} f_{01}^2(x)\ x\ \frac{1}{2}\mathrm{d}x \\ &+ \int_{I_{10}} f_{10}^2(x)\ x\ \frac{1}{2}\mathrm{d}x \ + \int_{I_{11}} f_{11}^2(x)\ x\ \frac{1}{2}\mathrm{d}x \end{aligned}$$

When we replace the x values for y values the integration takes over the whole interval $I = [-1, 1]$.

$$\begin{aligned} &= \int_I y \left(\frac{1}{\gamma_0\gamma_0} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_0\gamma_0} + \int_I y \left(\frac{1}{\gamma_0\gamma_1} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_0\gamma_1} \\ &+ \int_I y \left(\frac{1}{\gamma_1\gamma_0} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_1\gamma_0} + \int_I y \left(\frac{1}{\gamma_1\gamma_1} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_1\gamma_1} \end{aligned}$$

Multiplying out the brackets and collecting the terms together we get:

$$= \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_0\gamma_0}\right)^2 \frac{1}{2}\,\mathrm{d}y \ - \underbrace{\int_{-1}^{1} y\, b_1\, \frac{1}{2} \left(\frac{1}{\gamma_0\gamma_0}\right) \mathrm{d}y}_{0} +$$

$$+\int_{-1}^{1} y^2 \left(\frac{1}{\gamma_0\gamma_1}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1\, \frac{1}{2} \left(\frac{1}{\gamma_0\gamma_1}\right) \mathrm{d}y}_{0}$$

$$+\int_{-1}^{1} y^2 \left(\frac{1}{\gamma_1\gamma_0}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1\, \frac{1}{2} \left(\frac{1}{\gamma_1\gamma_0}\right) \mathrm{d}y}_{0}$$

$$+\int_{-1}^{1} y^2 \left(\frac{1}{\gamma_1\gamma_1}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1\, \frac{1}{2} \left(\frac{1}{\gamma_1\gamma_1}\right) \mathrm{d}y}_{0}$$

Canceling down all the zero terms gives:

$$= \frac{1}{3(\gamma_0\gamma_0)^2} + \frac{1}{3(\gamma_0\gamma_1)^2} + \frac{1}{3(\gamma_1\gamma_0)^2} + \frac{1}{3(\gamma_1\gamma_1)^2}$$

Then we put the term $\frac{1}{3}$ in evidence:

$$= \frac{1}{3}\left[\frac{1}{(\gamma_0\gamma_0)^2} + \frac{1}{(\gamma_0\gamma_1)^2} + \frac{1}{(\gamma_1\gamma_0)^2} + \frac{1}{(\gamma_1\gamma_1)^2}\right]$$

Factorizing it gives:

$$C(2) = \left\{\left(\frac{1}{\gamma_0}\right)^2 + \left(\frac{1}{\gamma_1}\right)^2\right\}^2 \frac{1}{3} \tag{22}$$

This is the Correlation function for $m = 2$.

4.3 Computation of $C(3)$

Next we will compute the Correlation function for the Bernoulli Shift map with different slopes, when $m = 3$ and I=[-1,1], example (3.2), equation (6). This map has 8 branches.

Observing Figure (4) we know that:

$$\frac{2}{\gamma_0\gamma_0\gamma_0} + \frac{2}{\gamma_0\gamma_0\gamma_1} + \frac{2}{\gamma_0\gamma_1\gamma_0} + \frac{2}{\gamma_0\gamma_1\gamma_1} + \frac{2}{\gamma_1\gamma_0\gamma_0} + \frac{2}{\gamma_1\gamma_1\gamma_0} + \frac{2}{\gamma_1\gamma_0\gamma_1} + \frac{2}{\gamma_1\gamma_1\gamma_1} = 2$$

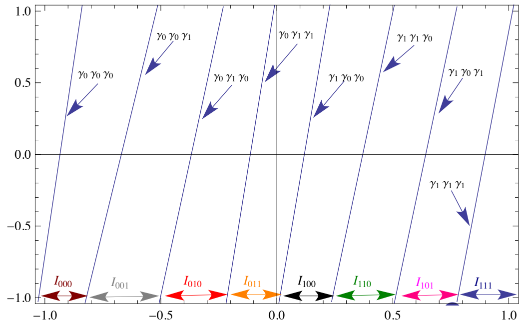

Figure 4: Bernoulli Shift map with different slopes and $m = 3$.

Now, when $m = 3$ we have eight intervals, such that $I_{000}, I_{001}, I_{010}, I_{011}, I_{100}$, $I_{110}, I_{101}, I_{111}$, is a partition of $I = [-1, 1]$.

$$\begin{aligned}
f^3_{000}(x) = f_0(f_0(f_0(x))) &= \gamma_0\left(\gamma_0\left(\gamma_0 x + a_0\right) + a_0\right) + a_0 \to I_{000} = \left[-1, -1 + \frac{2}{\gamma_0\gamma_0\gamma_0}\right],\\
f^3_{001}(x) = f_0(f_0(f_1(x))) &= \gamma_0\left(\gamma_0\left(\gamma_1 x + a_1\right) + a_0\right) + a_0 \to I_{001} = \left[-1 + \frac{2}{\gamma_0\gamma_0\gamma_0}, \frac{2}{\gamma_0\gamma_0\gamma_1}\right],\\
f^3_{010}(x) = f_0(f_1(f_0(x))) &= \gamma_0\left(\gamma_1\left(\gamma_0 x + a_0\right) + a_1\right) + a_0 \to I_{010} = \left[\frac{2}{\gamma_0\gamma_0\gamma_1}, \frac{2}{\gamma_0\gamma_1\gamma_0}\right],\\
f^3_{011}(x) = f_0(f_1(f_1(x))) &= \gamma_0\left(\gamma_1\left(\gamma_1 x + a_1\right) + a_1\right) + a_0 \to I_{011} = \left[\frac{2}{\gamma_0\gamma_1\gamma_0}, \frac{2}{\gamma_0\gamma_1\gamma_1}\right],\\
f^3_{100}(x) = f_1(f_0(f_0(x))) &= \gamma_1\left(\gamma_0\left(\gamma_0 x + a_0\right) + a_0\right) + a_1 \to I_{100} = \left[\frac{2}{\gamma_0\gamma_1\gamma_1}, \frac{2}{\gamma_1\gamma_0\gamma_0}\right],\\
f^3_{110}(x) = f_1(f_1(f_0(x))) &= \gamma_1\left(\gamma_1\left(\gamma_0 x + a_0\right) + a_1\right) + a_1 \to I_{110} = \left[\frac{2}{\gamma_1\gamma_0\gamma_0}, \frac{2}{\gamma_1\gamma_0\gamma_1}\right],\\
f^3_{101}(x) = f_1(f_0(f_1(x))) &= \gamma_1\left(\gamma_0\left(\gamma_1 x + a_1\right) + a_0\right) + a_1 \to I_{101} = \left[\frac{2}{\gamma_1\gamma_0\gamma_1}, 1 - \frac{2}{\gamma_1\gamma_1\gamma_1}\right],\\
f^3_{111}(x) = f_1(f_1(f_1(x))) &= \gamma_1\left(\gamma_1\left(\gamma_1 x + a_1\right) + a_1\right) + a_1 \to I_{111} = \left[1 - \frac{2}{\gamma_1\gamma_1\gamma_1}, 1\right].
\end{aligned}$$

and the derivatives are given by:

$$\begin{array}{rclcrcl}
f^{3'}_{000}(x) &=& \gamma_0\gamma_0\gamma_0 & \quad & f^{3'}_{100}(x) &=& \gamma_1\gamma_0\gamma_0\\
f^{3'}_{001}(x) &=& \gamma_0\gamma_0\gamma_1 & & f^{3'}_{110}(x) &=& \gamma_1\gamma_1\gamma_0\\
f^{3'}_{010}(x) &=& \gamma_0\gamma_1\gamma_0 & & f^{3'}_{101}(x) &=& \gamma_1\gamma_0\gamma_1\\
f^{3'}_{011}(x) &=& \gamma_0\gamma_1\gamma_1 & & f^{3'}_{111}(x) &=& \gamma_1\gamma_1\gamma_1
\end{array}$$

Using the Correlation function equation (16) we get:

$$C(3) = \int_{-1}^{1} f^{(3)}(x)\ x\ \rho(x)\mathrm{d}x \ \ - \ \ \underbrace{\int_{-1}^{1} x\ \rho(x)\mathrm{d}x \ \int_{-1}^{1} x\ \rho(x)\mathrm{d}x}_{0} \qquad (23)$$

The third iterative is defined as:

$$\begin{aligned} y = f_{000}^{3}(x) &= {\gamma_0}^3 x + {\gamma_0}^2 a_0 + \gamma_0 a_0 + a_0 \\ x &= \frac{y}{{\gamma_0}^3} - \underbrace{\frac{a_0}{\gamma_0} - \frac{a_0}{{\gamma_0}^2} - \frac{a_0}{{\gamma_0}^3}}_{=\ b_1} \\ x &= \frac{y}{{\gamma_0}^3} - b_1 \end{aligned}$$

Differentiating this gives:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= f_{000}^{3'}(x) = {\gamma_0}^3 \\ \mathrm{d}x &= \frac{\mathrm{d}y}{{\gamma_0}^3} \end{aligned}$$

Using the same method we can get the third iterative of the intervals $f_{001}^{3}(x)$, $f_{010}^{3}(x), f_{011}^{3}(x), f_{100}^{3}(x), f_{110}^{3}(x), f_{101}^{3}(x), f_{111}^{3}(x)$. Note that the derivatives of the intervals were given on the previous page.

We split the range of integration into four intervals. Substituting these into the equation (23) we get:

$$\begin{aligned} &= \int_{I_{000}} f_{000}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x \ \ + \int_{I_{001}} f_{001}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x \\ &+ \int_{I_{010}} f_{010}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x + \int_{I_{011}} f_{011}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x \\ &+ \int_{I_{100}} f_{100}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x + \int_{I_{110}} f_{110}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x \\ &+ \int_{I_{101}} f_{101}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x + \int_{I_{111}} f_{111}^{3}(x)\ \ x\ \ \frac{1}{2}\mathrm{d}x \end{aligned}$$

When we replace the x values for y values the integration takes over the whole interval $I = [-1, 1]$.

$$
\begin{aligned}
&= \int_I y \left(\frac{1}{\gamma_0\gamma_0\gamma_0} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_0\gamma_0\gamma_0} + \int_I y \left(\frac{1}{\gamma_0\gamma_0\gamma_1} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_0\gamma_0\gamma_1} \\
&+ \int_I y \left(\frac{1}{\gamma_0\gamma_1\gamma_0} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_0\gamma_1\gamma_0} + \int_I y \left(\frac{1}{\gamma_0\gamma_1\gamma_1} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_0\gamma_1\gamma_1} \\
&+ \int_I y \left(\frac{1}{\gamma_1\gamma_0\gamma_0} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_1\gamma_0\gamma_0} + \int_I y \left(\frac{1}{\gamma_1\gamma_1\gamma_0} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_1\gamma_1\gamma_0} \\
&+ \int_I y \left(\frac{1}{\gamma_1\gamma_0\gamma_1} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_1\gamma_0\gamma_1} + \int_I y \left(\frac{1}{\gamma_1\gamma_1\gamma_1} - b_1\right) \frac{1}{2} \frac{\mathrm{d}y}{\gamma_1\gamma_1\gamma_1}
\end{aligned}
$$

Multiplying out the brackets and collecting the terms together we get:

$$
\begin{aligned}
&= \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_0\gamma_0\gamma_0}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1 \frac{1}{2} \left(\frac{1}{\gamma_0\gamma_0\gamma_0}\right) \mathrm{d}y}_{0} \\
&+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_0\gamma_0\gamma_1}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1 \frac{1}{2} \left(\frac{1}{\gamma_0\gamma_0\gamma_1}\right) \mathrm{d}y}_{0} \\
&+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_0\gamma_1\gamma_0}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1 \frac{1}{2} \left(\frac{1}{\gamma_0\gamma_1\gamma_0}\right) \mathrm{d}y}_{0} \\
&+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_0\gamma_1\gamma_1}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1 \frac{1}{2} \left(\frac{1}{\gamma_0\gamma_1\gamma_1}\right) \mathrm{d}y}_{0} \\
&+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_1\gamma_0\gamma_0}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1 \frac{1}{2} \left(\frac{1}{\gamma_1\gamma_0\gamma_0}\right) \mathrm{d}y}_{0} \\
&+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_1\gamma_1\gamma_0}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1 \frac{1}{2} \left(\frac{1}{\gamma_1\gamma_1\gamma_0}\right) \mathrm{d}y}_{0} \\
&+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_1\gamma_0\gamma_1}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1 \frac{1}{2} \left(\frac{1}{\gamma_1\gamma_0\gamma_1}\right) \mathrm{d}y}_{0}
\end{aligned}
$$

$$+ \int_{-1}^{1} y^2 \left(\frac{1}{\gamma_1\gamma_1\gamma_1}\right)^2 \frac{1}{2}\,\mathrm{d}y - \underbrace{\int_{-1}^{1} y\, b_1\, \frac{1}{2} \left(\frac{1}{\gamma_1\gamma_1\gamma_1}\right) \mathrm{d}y}_{0}$$

Canceling down all the zero terms gives:

$$\begin{aligned} = & \frac{1}{3(\gamma_0\gamma_0\gamma_0)^2} + \frac{1}{3(\gamma_0\gamma_0\gamma_1)^2} + \frac{1}{3(\gamma_0\gamma_1\gamma_0)^2} + \frac{1}{3(\gamma_0\gamma_1\gamma_1)^2} \\ & \frac{1}{3(\gamma_1\gamma_0\gamma_0)^2} + \frac{1}{3(\gamma_1\gamma_1\gamma_0)^2} + \frac{1}{3(\gamma_1\gamma_0\gamma_1)^2} + \frac{1}{3(\gamma_1\gamma_1\gamma_1)^2} \end{aligned}$$

Then we simplify and put the term $\frac{1}{3}$ in evidence:

$$\begin{aligned} = \frac{1}{3}\Bigg[& \frac{1}{(\gamma_0\gamma_0\gamma_0)^2} + \frac{1}{(\gamma_0\gamma_0\gamma_1)^2} + \frac{1}{(\gamma_0\gamma_0\gamma_1)^2} + \frac{1}{(\gamma_0\gamma_0\gamma_1)^2} \\ & + \frac{1}{(\gamma_0\gamma_0\gamma_1)^2} + \frac{1}{(\gamma_0\gamma_0\gamma_1)^2} + \frac{1}{(\gamma_0\gamma_0\gamma_1)^2} + \frac{1}{(\gamma_0\gamma_0\gamma_1)^2} \Bigg] \end{aligned}$$

Factorizing it gives:

$$C(3) = \left\{ \left(\frac{1}{\gamma_0}\right)^2 + \left(\frac{1}{\gamma_1}\right)^2 \right\}^3 \frac{1}{3} \tag{24}$$

This is the Correlation function for $m = 3$.

4.4 Correlation Decay rate

Therefore we demonstrated (on the last subsection) by induction that there exists a General Formula, given by:

$$C(m) = \left\{ \left(\frac{1}{\gamma_0}\right)^2 + \left(\frac{1}{\gamma_1}\right)^2 \right\}^m \frac{1}{3} \tag{25}$$

The General Formula for $C(m)$.

From equation (25), we apply the property of the logarithm, to get:

$$= \frac{1}{3} e^{\ln\left\{\frac{1}{\gamma_0{}^2} + \frac{1}{\gamma_1{}^2}\right\}^m} \tag{26}$$

Next, we bring the power m down:

$$= \frac{1}{3} e^{-m\left(-ln\left(\frac{1}{\gamma_0{}^2}+\frac{1}{\gamma_1{}^2}\right)\right)} \tag{27}$$

Let α be:

$$\alpha = -ln\left(\frac{1}{\gamma_0{}^2} + \frac{1}{\gamma_1{}^2}\right) \tag{28}$$

Finally the correlation function is shown as:

$$C(m) = \frac{1}{3} e^{-m\alpha} \tag{29}$$

Here α is the rate of decay, in other words, how fast the correlation will decay. Note that, for a general one-dimensional map a sequence $x_0, f(x_0), ..., \ f^k(x_0), ...$ can be characterized by:

1. a Lyapunov exponent, which tells us how adjacent points become separeted under the action of f;
2. the Invariant density, which serves as a measure of how the iterates become distributed over the unit interval; and
3. the Correlation function $C(m)$, which measures the correlation between iterates that are m steps apart. For more details the reader may consult reference [4].

5 Relation between Sensitivity and Correlation decay

We want to show that there's an inequality between the quantity α, (that is the Correlation decay), from equation (28); and the Sensitivity (that is the Lyapunov exponent), from equation (12).

$$\alpha \leq \Lambda \tag{30}$$

$$\underbrace{-\ln\left(\frac{1}{\gamma_0{}^2} + \frac{1}{\gamma_1{}^2}\right)}_{\text{rate of decay}} \leq \underbrace{\frac{1}{\gamma_0}\ln\gamma_0 \ \ + \ \ \frac{1}{\gamma_1}\ln\gamma_1}_{\text{Sensitivity}} \tag{31}$$

We are going to prove it using the convexity argument, however we will have to prove the Jensen's Inequality to achieve that.

5.1 Remark on Convex functions

We say that $f(x)$ is Convex if the line segment joining any two points on the graph is never below the graph. A Convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable. This result, is known as Jensen's Inequality. Parts of the presentation are taken from reference[5].

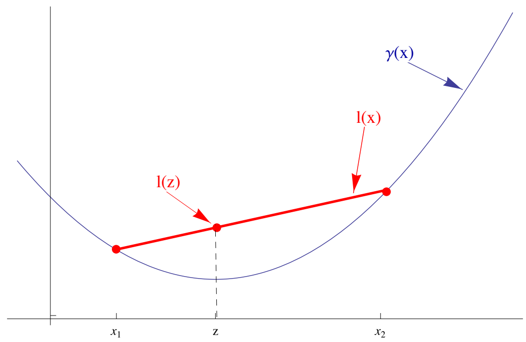

Figure 5: Convex function $f(x)$

More precisely, we can make the following statment:
A function $f : \mathbb{R}^n \to \mathbb{R}$ is called Convex if for any pair of non-equal x_1, x_2 in the domain of f (which is assumed to be a closed convex set) and any pair of real positive numbers $(1-t)$, t, such that $(1-t)+t = 1$, where $t \in [0, 1]$, one has:

$$f\left((1-t)x_1 + tx_2\right) \leq (1-t)f(x_1) + tf(x_2) \tag{32}$$

We call the equation (32) the General convexity equation.

Geometrically, it means that on the graph $(x, y) \in \mathbb{R}^{n+1} : y = f(x)$ of f, for any $l(z)$ lying on the line segment, connecting a pair of chosen points x_1 and x_2 in the domain of f (where z is a point between x_1 and x_2 defined as $z = (1-t)x_1 + tx_2$). The point $(z, f(z))$ lies below the chord,which we will call $l(z)$, connecting the pair of points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, for all the possible choices of the pair x_1, x_2 (the height being measured in terms of the y-coordinate).

Here we are dealing with the one-dimensional case, assuming that f is defined and bounded on some closed interval $[x_1, x_2]$. If in equation (32) one sets $x_2 > x_1$ and $z = (1-t)x_1 + tx_2$, then using $t = 1-(1-t)$, it can be rewritten as:

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2) = l(x) \tag{33}$$

Namely, the right hand side is a linear function $l(x)$ of x (given $x_1, x_2, f(x_1)$, $f(x_2)$), the geometric description above has been made precise.

5.2 Jensen's Inequality

Now we are going to prove the Jensen's Inequality. Let's begin with the General form of the equation of a line, $l(x)$:

$$l(x) = y = mx + c \tag{34}$$

The gradient is an expression that shows how inclined the line is. It can be defined as follows:

$$m = \frac{f(x_2) + f(x_1)}{x_2 - x_1} \tag{35}$$

This now gives the following:

$$y = \frac{f(x_2) + f(x_1)}{x_2 - x_1} x + c \tag{36}$$

Now substituting the point $(x_1, f(x_1))$ into the equation (36) and rearranging to get the constant c :

$$f(x_1) = \frac{f(x_2) + f(x_1)}{x_2 - x_1} x_1 + c, \tag{37}$$

$$c = f(x_1) - \frac{f(x_2) + f(x_1)}{x_2 - x_1} x_1 \tag{38}$$

Putting in the constant c gives:

$$l(x) = y = \frac{f(x_2) + f(x_1)}{x_2 - x_1} x_1 + \underbrace{f(x_1) - \frac{f(x_2) + f(x_1)}{x_2 - x_1} x_1}_{c} \tag{39}$$

Remember that we called z a point between x_1 and x_2, defined as:
$z = (1 - t)x_1 + tx_2$, where $t \in [0, 1]$.

Then when we put the fraction in evidence, we get:

$$l(z) = y = \frac{f(x_2) + f(x_1)}{x_2 - x_1}(z - x_1) + f(x_1) \tag{40}$$

Now we substitute z, and simplify:

$$l(z) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}((1-t)x_1 + tx_2 - x_1) + f(x_1) \tag{41}$$

$$\begin{aligned}
&= \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_1 - x_1 t + tx_2 - x_1) + f(x_1) \\
&= \frac{f(x_2) - f(x_1)}{x_2 - x_1} t(x_2 - x_1) + f(x_1) \\
&= (f(x_2) - f(x_1))\, t + f(x_1) \\
&= tf(x_2) - tf(x_1) + f(x_1)
\end{aligned}$$

$$l(z) = tf(x_2) + f(x_1)(1-t) \tag{42}$$

$$l(z) \quad \geq \quad \underbrace{f(z)}_{Convex function} \tag{43}$$

Therefore, the Jensen's Inequality has been proved.

5.3 Upper bound for the Decay rate

In this subsection we will prove that $\alpha \leq \Lambda$, i.e. there's a relation between sensitivity and correlation decay.

Starting with the General convexity equation (32), which we proved earlier we have:

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$$

Now let:

$$f(x) = \Phi(x) = -\ln(x) \tag{44}$$

Since $f(x) = -\ln(x)$ is a Convex function, see figure (6), we can apply the Jensen's Inequality.

Now we want to prove that $\alpha \leq \Lambda$, equation (30).

Fisrt of all, we substitute equation (44) into equation (32).

$$\Phi\left((1-t)x_1 + tx_2\right) \leq (1-t)\Phi(x_1) + t\Phi(x_2) \tag{45}$$

$$-\ln\left((1-t)x_1 + tx_2\right) \leq -(1-t)\ln(x_1) - t\ln(x_2) \tag{46}$$

Relabling the constansts x_1, x_2 and t:
$x_1 = \frac{1}{\gamma_0}$, $x_2 = \frac{1}{\gamma_1}$, $t = \frac{1}{\gamma_1}$ and $(1-t) = \frac{1}{\gamma_0}$.

Substituting the new values, into equation (46):

$$-\ln\left(\frac{1}{\gamma_0}\frac{1}{\gamma_0} + \frac{1}{\gamma_1}\frac{1}{\gamma_1}\right) \leq -\frac{1}{\gamma_0}\ln\frac{1}{\gamma_0} - \frac{1}{\gamma_1}\ln\frac{1}{\gamma_1} \tag{47}$$

Using the property of the logarithm on the right hand side, we get:

$$-\ln\left(\frac{1}{{\gamma_0}^2} + \frac{1}{{\gamma_1}^2}\right) \leq \frac{1}{\gamma_0}\ln\gamma_0 \quad + \quad \frac{1}{\gamma_1}\ln\gamma_1$$

$$\alpha \leq \Lambda$$

Thus, we have shown that there is an Upper bound for the Correlation decay rate.

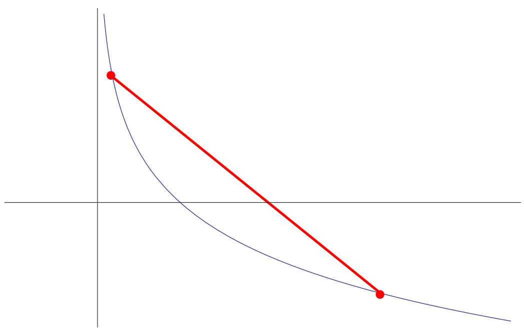

Figure 6: Convex function $\Phi(x) = -\ln(x)$.

6 Summary

The first four Chapters were predominantly used to build a foundation of knowledge to help with the later one. During Chapter 2, we looked at what a Markov partition and a Markov map is, giving detailed statements and examples to gain a firm understanding. Moving on to Chapter 3, we explained explicitly the Frobenius-Perron equation and also derived the Invariant density, giving examples using the Bernoulli Shift maps with same and different slopes. The Lyapunov exponent(Sensitivity) was also studied using the same examples. In Chapter 4, using an inductive argument, we showed the general formula for a Correlation function with m iterates.

Finally, on Chapter 5, we derived a relation between the Correlation decay rate and the Lyapunov exponent, i.e. we have shown that for a particular model, the Bernoulli Shift map with different slopes, where the chaotic dynamics induces a connection between sensitivity and correlations. It was done based on a simple piecewise linear Markov map where all computations can be done explicitly. To show the relation, we explored Jensen's Inequality, proving that if a function is convex and we have a line segment joining any two points on the graph, then the line segment is never below the graph. Then by using the Jensen's Inequality, the connection between sensitivity and correlations was shown.

The result has been derived for a very particular setup. For further study, one can try to check the matrix representations for the Frobenius-Perron equation, or even work on the spectra of Markov maps (on Chapter 3). One could also try to establish a relation for more complicated systems such as two dimensional maps, or even non-Markov maps where the maps are not linear.

7 Bibliography

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[9] Figures 1-6 were created using *Wolfram Mathematica 8.0.*