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BSc Hons Mathematics

The Argument Principle

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Abstract

In this paper, we discuss a non-trivial piece of mathematics which has foundations in complex analysis and applications in engineering. We develop a tool to find the number of roots of nice (i.e. differentiable) complex functions without actually solving them. This tool is called the Argument Principle. First, we will look at some definitions that will give us the background knowledge required to understand the project. Then, there are some examples which have been solved both numerically(using integration) and using analysis. We continue to develop the method of finding the number of zeroes until we are in a position to apply it to the proof of the Fundamental Theorem of Algebra(FTA). The FTA states a polynomial of degree n with complex coefficients, has n roots (zeroes) in the complex plane which are counted with multiplicity. A more general case is presented afterwards: Rouché's Theorem. Finally, we close the project with an application in engineering which uses the Argument Principle to ensure a system is stable. For example, temperature control on a fridge. Note, Rouché's Theorem guarantees stability for small changes in the system.

Acknowledgements

Thank you to Dr Wolfram Just, for supervising this project and providing motivation when needed. He has always been patient and inspired me when I have been stuck. Thank you for encouraging me to learn LaTeX and Mathematica. I would also like to thank Dr Lucas Lacasa and Dr Wajid Mannan for organising various invaluable training sessions and the module as a whole. Finally, thank you to everyone who has read my project and helped me refine it in any way, your feedback is invaluable.

Contents

1 Introduction

This project on the Argument Principle develops a tool which is used to find the number of roots of a complex function $f(z)$ which is nice (i.e. analytic), without actually solving the equation $f(z) = 0$. Solving the equation can sometimes be very difficult. For example, one can easily solve $f(z) = z^2 + \frac{1}{2} = 0$ and see that there are two zeroes, but how about $f(z) = z^2 + \cos(z) + e^{\frac{z}{2}} = 0$? One may now notice the need for the Argument Principle, to simplify and reach a solution more easily and efficiently. We can use integration in the complex plane or a method of proof which is analytical. Meaning, we do not use numerical integration but rather we observe the real part of a function and keep algebra to a minimum while drawing various conclusions. Both methods are shown in the examples section and the complex integral is defined in the background section. Though, in most cases it is best to use the analytical method since solving complicated integrals by hand can be time consuming and one can make errors.

We also look at a very important theorem in complex analysis, the Fundamental Theorem of Algebra and prove it using the Argument Principle. The reason that the Fundamental Theorem of Algebra is so important is because it tells us that a field of complex numbers is algebraically closed. Meaning, a polynomial that has complex coefficients, has complex solutions. We know that $1 + j$ is a complex number and it may not be obvious but so is 5 (i.e. $5 + 0i$). This is an important fact because it tells us that any polynomial, say $p(x)$,with real coefficients, has all of the solutions of $p(x) = 0$ in the set of complex numbers. In other words, the set of real numbers is included in the set of complex numbers. We will see later that this is important in understanding the Fundamental Theorem of Algebra and its proof. All these tools combined will then help us to prove Rouché's Theorem which tells us that we can find the number of zeroes of a function in the complex plane by splitting it into two parts.

At the end of the project we may wish to explore the real life applications of the Argument Principal. The last section on stability problems provides a non-trivial application to practical questions such as, why does a rocket not fall out of the sky once airbourne? Just in case the reader is interested, the project follows in conjunction to a Year 2 module at Queen Mary, UoL titled Complex Variables.

2 Background

In this section, I will list and define some important terms which are useful to understand and keep in mind when reading the project. First, we start by looking at Figure 1 below which serves as a reminder of the notion of the argument of a complex function.

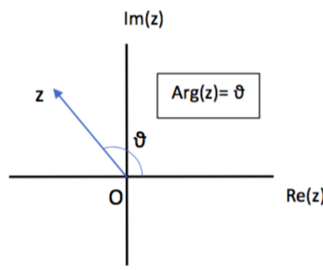


Figure 1: The modulus and argument of a complex function $z = x + iy$. Note, $z \neq 0$, $0 \leq \theta \leq 2\pi$ and θ occurs in multiples of 2π.

Also, we will often see a complex number z written as $z = R(\cos\theta + i\sin\theta)$ which is called the polar form of a complex number $z = x + iy$, where $x = R\cos\theta$ and $y = R\sin\theta$. To make things more compact, we will use the exponential form which is written as $z = Re^{i\theta}$. These are standard identities in the study of complex numbers.

In this project, complex integration is done as usual. For a complex valued function $w(t)$, $(t \in \mathbb{R})$,

$$w(t) = u(t) + iv(t)$$

where u and v are both real valued, the definite integral of $w(t)$ over an interval $a \leq t \leq b$ is defined as

$$\int_a^b w(t) = \int_a^b u(t)\mathrm{d}t + i\int_a^b v(t)\mathrm{d}t$$

provided both integrals on the right exist [2].

The next few definitions are taken verbatim from [1].

Definition 1. A function f which is differentiable at every point of an open set $S \subseteq \mathbb{C}$ is called <u>holomorphic</u> or <u>analytic</u> on S.

Definition 2. We say a curve is simple if there exists a parameterisation γ of C which is injective. We say C is a simple, closed curve if it is parameterised by $\gamma : [a, b] \mapsto \mathbb{C}$ with $\gamma(a) = \gamma(b)$ but $\gamma(t_1) \neq \gamma(t_2)$ for all other $t_1, t_2 \in [a, b]$ with $t_1 \neq t_2$.

Remark 1. In this project we only consider circles which are simple and closed by definition.

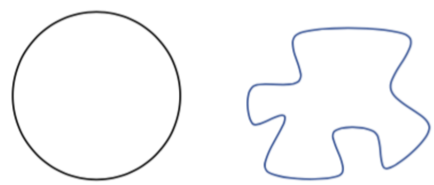


Figure 2: Illustration of simple and closed curves.

Definition 3. A contour is a piecewise-smooth curve, that is a finite union of smooth curves, joined end-to-end.

Definition 4. Let A be an annulus centered at z_o with inner radius R_1 and outer radius R_2, $0 \leq R_1 \leq R_2 \leq \infty$: $A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$.

We can visualise an annulus using Mathematica. In figure 3 we can also see the code which one can use to produce an annulus.

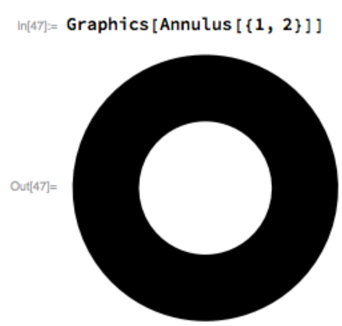


Figure 3: Illustration of an annulus with inner radius 1 and outer radius 2.

Now we have a theorem which is not used directly in the project, but it is helpful to understand the intuition behind it if one wants to understand the general context.

Theorem 1 (Poles)**.** *Suppose that*

(a) two functions p and q are analytic at a point z_0;

(b) $p(z_0) \neq 0$ and q has a zero of order m at z_0.

Then the quotient $\frac{p(z)}{q(z)}$ has a pole of order m at z_0.

Remark 2. We take this verbatim from [2]. Once again, we will not consider poles in this project, but I have included this here as it is mentioned in the application in engineering.

Definition 5. A function f is said to be meromorphic in a domain D if it is analytic throughout D except for poles[2].

Remark 3. In this project, we only consider holomorphic functions - I have only included this definition for completeness of the topic.

Now I define the geometric series and the reverse triangle inequality which are both used later on in the proof of the Fundamental Theorem of Algebra[2, 3].

Lemma 1. *To find the sum of a finite geometric series we have*

$$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r}$$

where $r \neq 1$, n is the number of terms in the series, and r is the common ratio of the series.

In this project, R is the radius of a circle and thus $R > 0$.

Lemma 2. *The reverse triangle inequality states*

$$||z_1| - |z_2|| \leq |z_1 + z_2| \, .$$

In the section on the application of the Argument Principle, I mention a mapping from the s-plane to the w-plane. This is simply a mapping where $s = z$ and $w = f(z)$. Let $z = x+iy$ and $w = u+iv$, then $w = u(x,y)+iv(x,y)$. Now, we move on to the Argument Principle itself.

3 The Argument Principle

This theorem is taken from [2].

Theorem 2 (The Argument Principle for Meromorphic Functions)**.** *Let C denote a positively oriented simple closed contour, and suppose that*

(a) a function $f(z)$ is meromorphic in the domain interior to C;

(b) $f(z)$ is analytic and nonzero on C;

(c) counting multiplicities, Z is the number of zeroes and P is the number of poles of $f(z)$ inside C.

Then,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\mathrm{d}z = Z - P.$$

Remark 4. As stated in the previous section, a function f is meromorphic in a domain if it is analytic throughout the domain, except for poles. When using the Argument Principle, we will suppose f is meromorphic in the domain interior to a positively oriented (anti-clockwise in direction), simple closed contour C and that it is analytic and nonzero on C.

It is not immediately obvious how the integral is related to the argument of a complex function. So where does the idea of $\frac{f'(z)}{f(z)}$ come from? The integral of $\frac{f'(z)}{f(z)}$ over a contour C measures change in $\ln f(z)$ along C and is a multiple of $2\pi i$. The integrand is essentially measuring the change in argument, which is also a multiple of $2\pi i$.

Before we look at some examples, let us take a moment to understand the Argument Principle for Meromorphic functions geometrically. If we have n more zeroes than poles contained in the area of a closed contour, there will be a corresponding plot which is the map of the image of the contour under a function f. The image winds around the origin n times. We can tell the relative difference between the number of poles and zeroes inside a contour by how many times a plot circles the origin, and what direction: clockwise is a zero, anti-clockwise is a pole.

It is worth highlighting, in this project we will only consider functions which have no poles, thus the number of winds around the origin will tell us the number of zeroes of the contour. Note, we only consider circles in this project which are simple and closed by definition. The Argument Principle then becomes as below[1].

Theorem 3 (The Argument Principle for Circles)**.** *If f is holomorphic on and inside a simple closed positively oriented circle C and has a finite number of zeroes inside C, and if $\mathcal{N}_f$ denotes the number of zeroes of f inside C, counted with multiplicity, then*

$$\mathcal{N}_f = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\mathrm{d}z$$

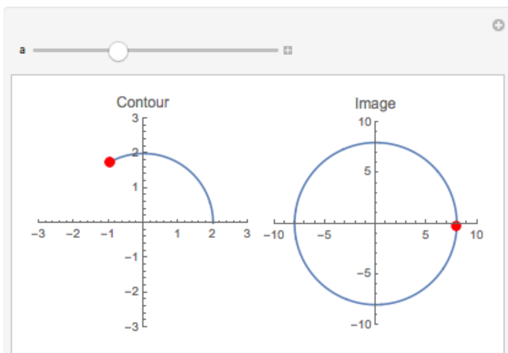
Remark 5. In this special case for holomorphic functions, we can say that $\mathcal{N}_f$ is the same as saying Z which is the winding number. In other words, it measures how many times the image of the circle C under the map f encircles the origin. The images in figure 5 provide an illustration.

It may be difficult to convince yourself of what you see in the following images. Unfortunately, I can only provide static illustrations but I have provided a screenshot of the code, should you wish to replicate this example to see the dynamics.

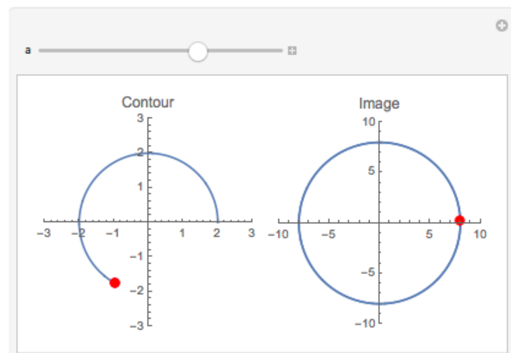
```
Manipulate[
 Grid[
  {{Show[ParametricPlot[{2 Cos[a t], 2 Sin[a t]}, {t, 0, 1},
      PlotLabel → "Contour", PlotRange → {{-3, 3}, {-3, 3}}],
     Graphics[{Red, PointSize[0.05], Point[{2 Cos[a ], 2 Sin[a ]}]}]],
    Show[ParametricPlot[{8 Cos[3 a t], 8 Sin[3 a t]}, {t, 0, 1},
      PlotLabel → "Image", PlotRange → {{-10, 10}, {-10, 10}}],
     Graphics[{Red, PointSize[0.05],
       Point[{8 Cos[3 a ], 8 Sin[3 a ]}]}]]}}], {a, 0, 2 π}]
```

Figure 4: Screenshot of Mathematica code used to produce a dynamic illustration.

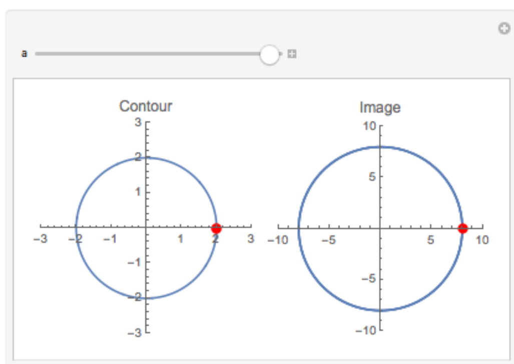
On the left we have a contour which is a circle of radius 2 (i.e. $2\cos[at] + 2i\sin[at]$) and on the right we take the image of the function $f(z) = z^3$ under the contour. The results are as we would expect, the image of $f(z)$ winds around the origin three times. In other words, the function $f(z) = z^3$ has three zeroes. Note, we do not know anything regarding the location of the zeroes.



(a) The image encircles around the origin once.



(b) The image encircles the origin twice.



(c) The image encircles around the origin thrice.

Figure 5: The figures show the change in argument of the function $f(z) = z^3$, the image of $f(z)$ encircles the origin three times under a circle of radius 2, so $f(z) = z^3$ has 3 roots.

4 Examples

In all of these examples, geometrically, we are mapping a distinct function $f(z)$ over a circle of radius R. If one requires a visual aid, feel free to alter the Mathematica code provided above by changing the first parametric plot to a circle of radius R and the second parametric plot to the respective functions in each of the examples in this section.

4.1 Linear Function

In the first example, I will use the Argument Principle to show the total change in argument of $f(z) = z + \frac{1}{2}$. I will consider two separate cases:

1. $R > \frac{1}{2}$;
2. $R < \frac{1}{2}$.

It is also worth noting that this example will be illustrated analytically rather than using integration. This method of reaching our answer analytically will be further developed in subsequent examples.

To find the change in argument of $f(z)$ I will use the exponential polar form $z = Re^{i\theta}$ where $R = |z|$ and $arg(z) = \theta$. Using this definition of z, we can re-write $f(z)$:

$$f(z) = f(Re^{i\theta}) = Re^{i\theta} + \frac{1}{2} = \varrho e^{i\psi}$$

where $\varrho = \left|f(Re^{i\theta})\right|$.

Note, $Re^{i\theta} + \frac{1}{2} \approx Re^{i\theta}$ for very large R. Intuitively, as R becomes very large, the $\frac{1}{2}$ does not really affect $f(Re^{i\theta})$, so we can neglect it. Now, we must find ψ by equating coefficients of $\varrho e^{i\psi}$ and $Re^{i\theta}$ and we can say $\psi = \theta$. This means $\psi(2\pi) - \psi(0) = 2\pi$ which we can write as $2\pi(\mathcal{N}_f)$ where $\mathcal{N}_f = 1$.

We are now in a position to consider case 1.

By factoring out the dominant term $Re^{i\theta}$, we write

$$\varrho e^{i\psi} = Re^{i\theta}(1 + \frac{1}{2R}e^{-i\theta}).$$

Let $(1 + \frac{1}{2R}e^{-i\theta}) = re^{i\alpha} = \zeta$ where α is a function of θ. By rearranging our condition $R > \frac{1}{2}$ we get $0 < \frac{1}{2R} < 1$.

Now, we are interested in the real part of ζ for further analysis. Here, and in further arguments, we observe $Re(\zeta) > 0$ which implies $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$. This can be seen in figure 6.

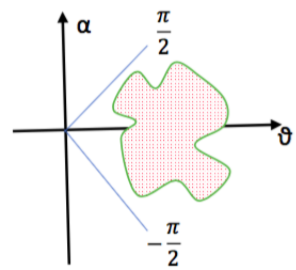


Figure 6: The image of the circle with radius R under the function $f(z) = z + \frac{1}{2}$.

We have shown above that $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$ which proves $\alpha(2\pi) - \alpha(0) = 0$ and thus $\alpha(2\pi) = \alpha(0)$. In other words, the change in argument comes from the term θ in the equation $\psi(\theta) = \theta + \alpha(\theta)$. Now, we must find a lower bound for $Re(\zeta)$ (by computation) because $Re(\zeta) > 0$ implies that $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$.

Re-writing $f(Re^{i\theta})$,

$$Re(\zeta) = 1 + \frac{1}{2R} \cos\theta.$$

We consider the minimum values of cosine and the term $\frac{1}{2R}$ in order to find the lower bound for the expression $\frac{1}{2R}\cos(\theta)$. Using $-1 \leq \cos\theta \leq 1$ and $\frac{1}{2R} > 0$ we have, $Re(\zeta) > 1 + 0(-1) = 1$. Finally, we can conclude the total change in argument of $f(z)$ along the curve is 2π, so we only have one zero for the case $R > \frac{1}{2}$, by the Argument Principle. This is because the number of zeroes is given by $2\pi n$, here $n = 1$.

Now, we will consider case 2:

Firstly, we will re-write $\varrho e^{i\psi}$ in a different form, $\varrho e^{i\psi} = \frac{1}{2}(1 + 2Re^{i\theta})$. Let $(1 + 2Re^{i\theta}) = re^{i\beta} = \zeta$ where β is a function of θ. Since we already know $R < \frac{1}{2}$ we can rearrange this to get $0 \leq 2R < 1$. Once again, we are interested in considering $Re(\zeta)$ at the minimum values of R and θ. Using $0 \leq 2R < 1$ and $cos\theta \geq -1$ we have $2R\cos\theta > -1$ which give us $Re(\zeta) > 1 + 2R\cos\theta > 1 + (1)(-1) = 0$.

Thus, for the case $R < \frac{1}{2}$ we can conclude the total change of argument of $f(z)$ is $\psi(2\pi) - \psi(0) = 0(2\pi) = 0$. In other words, using the Argument Principle and considering the change in argument, we can see there are no zeroes for this case. Geometrically, this is when the zero(s) of $f(z)$ lie outside of the circle.

4.2 Quadratic Equations

Example 1

We will use the statement of the Argument Principle to show the number of zeroes of the function $f(z) = z^2$. We expect to obtain a result showing there are two zeroes; this will be illustrated using integration. Recall, in school we were always taught: "...the number of roots of a function is identical to its' highest power."

In order to use the Argument Principle, we must know or calculate the first derivative, which in this example is $f'(z) = 2z$. Now, we are in a position to use the statement of the principle. In the integral below, C is a circle with radius R.

$$\begin{aligned} \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\mathrm{d}z &= \frac{1}{2\pi i}\int_C \frac{2z}{z^2}\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{2e^{i\theta}}{e^{2i\theta}}(ie^{i\theta})\mathrm{d}\theta \qquad (1) \end{aligned}$$

To obtain (1), we have used $z = e^{i\theta}$ and $\mathrm{d}z = ie^{i\theta}$ to perform a change of variables from z to θ.

$$\begin{aligned} \frac{1}{2\pi i}\int_0^{2\pi} \frac{2ie^{2i\theta}}{e^{2i\theta}}\mathrm{d}\theta &= \frac{1}{2\pi i}\int_0^{2\pi} 2i\mathrm{d}\theta \\ &= \frac{1}{2\pi i}[2i\theta]\Big|_0^{2\pi} \\ &= \frac{1}{2\pi i}(4\pi i - 0) \\ &= \frac{4\pi i}{2\pi i} \\ &= 2. \end{aligned}$$

The total change of argument is 4π. The result confirms that the function has two zeroes (i.e. $2\pi n$ where $n = 2$). This is exactly what we expected.

Example 2

Now, we will use the method of analysis developed in section 4.1 on linear functions, proving analytically that the function $f(z) = z^2 + z + \frac{1}{2}$ has two zeroes. Let us consider a circle of radius R, then using the exponential and polar form of a complex number,we have $z = Re^{i\theta}$ where $R = |z|$ and $arg(z) = \theta$. We first find the radius of the outer circle of an annulus, and then the radius of the inner circle. Recall, annulus was defined in definition 4.

We are able to re-write $f(z)$ as $f(Re^{i\theta}) = R^2e^{2i\theta} + Re^{i\theta} + \frac{1}{2} = \varrho e^{i\psi}$ where $\varrho = \left|f(Re^{i\theta})\right|$. For very large R ($R \to \infty$) we say,

$$R^2e^{2i\theta} + Re^{i\theta} + \frac{1}{2} \approx R^2e^{2i\theta}.$$

Equating the indicies of $\varrho e^{i\psi}$ and $R^2e^{2i\theta}$ we get $\psi = 2\theta$ which means $\psi(2\pi) - \psi(0) = \mathcal{N}_f 2\theta$.

Let us re-write $f(Re^{i\theta})$ in a different form by factoring out the dominant term(term with the highest power):

$$\varrho e^{i\psi} = R^2e^{2i\theta}\left(1 + \frac{1}{R}e^{-i\theta} + \frac{1}{2R^2}e^{-2i\theta}\right).$$

Let $\left(1 + \frac{1}{R}e^{-i\theta} + \frac{1}{2R^2}e^{-2i\theta}\right) = re^{i\alpha} = \zeta$, this allows for simpler notation. In order to make any further conclusions, we analyse the real part of ζ, $Re(\zeta)$:

$$Re(\zeta) = 1 + \frac{1}{R}\cos(\theta) + \frac{1}{2R^2}\cos(2\theta).$$

For $Re(\zeta) > 0$, it is implied $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$ which can be seen in figure 7.

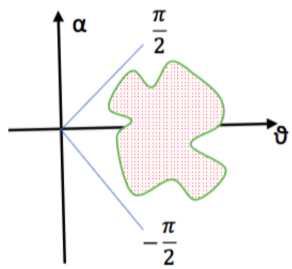


Figure 7: The image of the circle C with radius R under the function $f(z) = z^2 + z + \frac{1}{2}$.

Also, $\psi(\theta) = 2\theta + \alpha(\theta)$ where the change in argument comes from the term 2θ because $\alpha(2\pi) - \alpha(0) = 0$ when computing $\psi(2\pi) - \psi(0) + (\alpha(2\pi) - \alpha(0))$ and thus $\alpha(\theta)$ does not contribute to the change in argument.

Using the properties of cosine: $-1 \leq \cos(\theta) \leq 1$, we know $\cos(\theta) \geq -1$ and $\cos(2\theta) \geq -1$. When $\cos(\theta) \geq -1$ and $\cos(2\theta) \geq -1$, we say

$$Re(\zeta) \geq 1 - \frac{1}{2R} - \frac{1}{2R^2} > 0. \tag{2}$$

As stated previously, $Re(\zeta) > 0$ implies $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$ so we need to compute a lower bound for $Re(\zeta)$. Below, we try to find an appropriate R which gives us $Re(\zeta) > 0$.

Table 1: This table shows the value of $Re(\zeta)$ at different values of R which are substituted into equation 2.

$R > n$	$Re(\zeta) > m$
$n = 5$	$m = \frac{22}{25}$
$n = 4$	$m = \frac{27}{32}$
$n = 3$	$m = \frac{7}{9}$
$n = 2$	$m = \frac{5}{8}$
$n = 1$	$m = 0$

From the table 1, we see for $R > 2$, $\psi = 2\theta + \alpha(\theta)$ because $m > 0$ (i.e.

$Re(\zeta) > 0)$. By proving $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$ in figure 7, we have shown that $\alpha(2\pi) = \alpha(0)$ and thus $\alpha(2\pi) - \alpha(0) = 0$.

Thus for $R > 2$ we conclude the total change in argument of $f(z)$ along the curve is 4π, so we have two zeroes of the function.

Now, we can also look at a different case,

$$\varrho e^{i\psi} = \frac{1}{2}(1 + 2Re^{i\theta} + 2R^2 e^{2i\theta}).$$

In this part, we are finding the radius of the inner circle of an annulus. Here, we take $\zeta = 1 + 2Re^{i\theta} + 2R^2 e^{2i\theta}$ and are interested in $Re(\zeta)$ to make further conclusions and find a value for R.

$$Re(\zeta) = 1 + 2Rcos(\theta) + 2R^2 cos(2\theta)$$

When $Re(\zeta) > 0$, we see that $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$. This fact is illustrated in figure 7. As in the case for $R > 2$, we have here that $\psi = 2\theta + \alpha(\theta)$. Using the property of cosine being bound by $(-1, 1)$, we have $Re(\zeta) \geq 1 - 2R - 2R^2 > 0$. Now, there are different ways to find an appropriate value of R, but I choose to solve to the quadratic $1 - 2R - 2R^2 = 0$ by using the quadratic formula.

Remark 6. The quadratic formula states that a quadratic function of the form $ax^2 + bx + c = 0$ has solutions $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Using the formula as defined, we have

$$\begin{aligned} R_1, R_2 &\approx \frac{2 \pm \sqrt{4 - (1)(4)(-2)}}{(-2)(2)} \\ &= \frac{2 \pm 2\sqrt{3}}{-4}. \end{aligned}$$

Simplifying $\frac{2 \pm 2\sqrt{3}}{-4}$ gives us $R_1 \approx -1.366$ and $R_2 \approx 0.366$. In this context, R must be a positive number since it represents the radius of a circle, thus, $R \approx 0.36$. Note, we want $Re(\zeta) > 0$ so we take $R < 0.366$ because $R > 0.366$ gives us $Re(\zeta) < 0$.

In the previous part, we found $R > 2$, this is the radius of the outer circle($R = 2$). The inner circle has radius $R = 0.366$ To illustrate this, I have provided an illustration of the annulus, made in Mathematica.

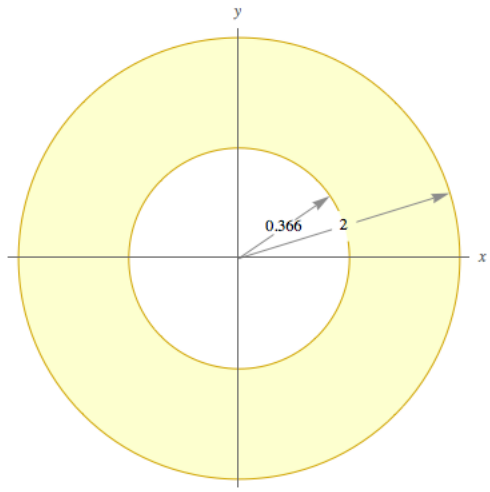


Figure 8: An annulus with inner radius 0.366 and outer radius 2.

4.3 Monomial of Order N

In this example, we find the number of zeroes of the function $f(z) = z^n$. From previous mathematical knowledge, we expect the function $f(z) = z^n$ to have n zeroes. Here, the first derivative is $f'(z) = nz^{n-1}$ where n is any natural number. Also, let $z = e^{i\theta}$ and $\mathrm{d}z = ie^{i\theta}$ which we will use to perform a change of variables from z to θ.

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\mathrm{d}z &= \frac{1}{2\pi i}\int_C \frac{nz^{n-1}}{z^n}\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{ne^{i\theta(n-1)}}{e^{ni\theta}}(ie^{i\theta})\mathrm{d}\theta \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{nie^{ni\theta}e^{-i\theta}e^{i\theta}}{e^{ni\theta}}\mathrm{d}\theta \\
&= \frac{1}{2\pi i}\int_0^{2\pi} ni\mathrm{d}\theta \\
&= n.
\end{aligned}$$

Thus, we can conclude the change in argument of $f(z)$ is $2\pi n$ and by the Argument Principle, $f(z)$ has n roots.

Remark 7. After looking at these examples, one should deduce that the

Argument Principle allows us to find the change in argument in the form $2\pi n$ which then tells us how many zeroes the function has (n).

5 The Fundamental Theorem of Algebra

Theorem 4 (The Fundamental Theorem of Algebra[1]). *Every polynomial, $p(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ... + a_2 z^2 + a_1 z^1 + a_0$, of degree n with real or complex coefficients has exactly n zeroes (counting multiplicities) in $\mathbb{C}$, $(n \geq 1, a_j \in \mathbb{C}, j = 0, ..., n, a_n \neq 0)$.*

The method of proof which we have developed in previous examples, will be used here to prove that there are n zeroes of the Fundamental Theorem of Algebra,

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ... + a_1 z^1 + a_0$$

where $a_n \neq 0$.

First, we consider a non-rigorous reasoning which gives a taste of the proof to follow. As usual, we write $p(z)$ using $z = Re^{i\theta}$. Let us consider R to be so large that we can ignore all of the terms except the one with highest power.

$$p(Re^{i\theta}) = a_n R^n e^{in\theta} + a_{n-1}R^{n-1}e^{(n-1)i\theta} + ... + aRe^{in\theta} + a_0.$$

We see the complex argument of $p(Re^{i\theta})$ is $n\theta$, which increases by $2\pi n$ when θ goes from 0 to 2π. Hence, $N_f = \frac{1}{2\pi}2\pi n = n$. Now, let us make this a rigorous argument.

Proof. As previously, we will use $z = Re^{i\theta}$ where $R = |z|$ and $arg(z) = \theta$. Then, substituting z with $Re^{i\theta}$ gives

$$p(Re^{i\theta}) = a_n R^n e^{in\theta} + a_{n-1}R^{n-1}e^{(n-1)i\theta} + ... + aRe^{i\theta} + a_0.$$

To simplify the problem, we say $p(Re^{i\theta}) = \varrho e^{i\psi}$ where $\varrho = \left|p(Re^{i\theta})\right|$. Note, for sufficiently large R,

$$a_n R^n e^{in\theta} + a_{n-1}R^{n-1}e^{(n-1)i\theta} + ... + aRe^{i\theta} + a_0 \approx a_n R^n e^{in\theta}.$$

Now, we can re-write $\varrho e^{i\psi}$ by factoring out the dominant term $R^n e^{ni\theta}$:

$$\varrho e^{i\psi} = R^n e^{ni\theta}(a_n + \frac{a_{n-1}e^{-i\theta}}{R} + ... + \frac{a_0 e^{-ni\theta}}{R^n}).$$

Let $a_n + \frac{a_{n-1}e^{-i\theta}}{R} + ... + \frac{a_0 e^{-ni\theta}}{R^n} = re^{i\alpha} = \zeta$. It is important to note that $\psi(\theta) = n\theta + \alpha(\theta)$ where α is a function of θ. We also conclude that $\psi = n\theta$ by equating the indices of $e^{i\psi}$ and $e^{ni\theta}$.

When $Re(\zeta) > 0$, the implication $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$ means $\alpha(2\pi) - \alpha(0) = 0$. We can write $Re(\zeta)$ as:

$$Re(\zeta) = a_n + \frac{a_{n-1}\cos\theta}{R} + \frac{a_{n-2}\cos 2\theta}{R^2} + ... + \frac{a_1\cos(n-1)\theta}{R^{n-1}} + \frac{a_0\cos n\theta}{R^n}. \quad (3)$$

From the statement of the Fundamental Theorem of Algebra, we know $a_n \neq 0$, but we do not know if a_n is positive or negative. Thus, we must consider two separate cases:

1. a_n is positive ($a_n > 0$)
2. a_n is negative ($a_n < 0$)

We will consider case 1 first and start by using equation 3 from above.

$$\begin{aligned} Re(\zeta) &= a_n + \frac{a_{n-1}\cos\theta}{R} + \frac{a_{n-2}\cos 2\theta}{R^2} + ... + \frac{a_1\cos(n-1)\theta}{R^{n-1}} + \\ & \frac{a_0\cos n\theta}{R^n} && (4) \\ &\geq a_n - \left|\frac{a_{n-1}\cos\theta}{R}\right| - \left|\frac{a_{n-2}\cos 2\theta}{R^2}\right| - ... - \left|\frac{a_1\cos(n-1)\theta}{R^{n-1}}\right| - \\ & \left|\frac{a_0\cos n\theta}{R^n}\right| && (5) \end{aligned}$$

This inequality holds due to the reverse triangle inequality (Lemma 2 from the background section), which allows us to find a lower bound estimate for $Re(\zeta)$. In the next step, we use a property of the cosine function: $\cos \geq -1$.

$$\begin{aligned} Re(\zeta) &\geq a_n - \left|\frac{a_{n-1}\cos\theta}{R}\right| - \left|\frac{a_{n-2}\cos 2\theta}{R^2}\right| - ... - \left|\frac{a_1\cos(n-1)\theta}{R^{n-1}}\right| - \left|\frac{a_0\cos n\theta}{R^n}\right| \\ &\geq a_n - |-1|\left|\frac{a_{n-1}}{R}\right| - |-1|\left|\frac{a_{n-2}}{R^2}\right| - ... - |-1|\left|\frac{a_1}{R^{n-1}}\right| - |-1|\left|\frac{a_0}{R^n}\right| \\ &\geq a_n - \left|\frac{a_{n-1}}{R}\right| - \left|\frac{a_{n-2}}{R^2}\right| - ... - \left|\frac{a_1}{R^{n-1}}\right| - \left|\frac{a_0}{R^n}\right| \\ &= a_n - \frac{|a_{n-1}|}{R} - \frac{|a_{n-2}|}{R^2} - ... - \frac{|a_1|}{R^{n-1}} - \frac{|a_0|}{R^n}. \end{aligned}$$

The final equality holds because we know that $R > 0$ due to the context of the problem, R is the radius so it must be larger than 0.

We now introduce a constant $\alpha = max\{|a_k|\}$ where $0 \leq k \leq n-1$. In other words, α is the largest coefficient. This means $|a_{n-1}| \leq \alpha$, $|a_{n-2}| \leq$

α,..., $|a_0| \leq \alpha$. Thus, we can use geometric series to find the desired value of R; keeping in mind that the aim of this part of the proof is to show $a_n + ... + a_0 > 0$. So,

$$\begin{aligned} a_n - \left(\frac{|a_{n-1}|}{R} - \frac{|a_{n-2}|}{R^2} - ... - \frac{|a_1|}{R^{n-1}} - \frac{|a_0|}{R^n}\right) &\geq a_n - \left(\frac{\alpha}{R} - \frac{\alpha}{R^2} - ... - \frac{\alpha}{R^{n-1}} - \frac{\alpha}{R^n}\right) \\ &= a_n - \alpha\left(\frac{1}{R} + \frac{1}{R^2} + ... + \frac{1}{R^{n-1}} + \frac{1}{R^n}\right) \\ &= a_n - \frac{\alpha}{R}\left(1 + \frac{1}{R} + \frac{1}{R^2} + ... + \frac{1}{R^{n-1}}\right). \end{aligned}$$

In the steps above, we have first factored out α and then factored out $\frac{\alpha}{R}$. From the last equality, we are now in a position to make use of the geometric series which was defined earlier.

Using the definition of geometric series,

$$a_n - \frac{\alpha}{R}\left(1 + \frac{1}{R} + \frac{1}{R^2} + ... + \frac{1}{R^{n-1}}\right) = a_n - \frac{\alpha}{R}\left(\frac{1 - \frac{1}{R}^n}{1 - \frac{1}{R}}\right) \tag{6}$$

We can expand 6 to give

$$a_n - \frac{\alpha}{R}\left(\frac{1}{1-\frac{1}{R}}\right) + \frac{\alpha}{R}\left(\frac{\frac{1}{R^n}}{1-\frac{1}{R}}\right).$$

We know that

$$\frac{\alpha}{R}\left(\frac{\frac{1}{R^n}}{1-\frac{1}{R}}\right) > 0$$

since we assume that $R > 1$, by the definition of geometric series.

Then, we can say

$$Re(\zeta) \geq a_n - \frac{\alpha}{R}\left(\frac{1}{1-\frac{1}{R}}\right) = a_n - \frac{\alpha}{R-1} > 0$$

where $a_n > 0$ and $R > 1$. Now, we find a range for R in which we can guarantee $Re(\zeta) > 0$:

$$\begin{aligned}
a_n - \frac{\alpha}{R-1} &> 0 \\
a_n &> \frac{\alpha}{R-1} \\
a_n(R-1) &> \alpha \\
R - 1 &> \frac{\alpha}{a_n} \\
R &> \frac{\alpha}{a_n} - 1.
\end{aligned}$$

Thus,

$$R > max\left(\frac{\alpha}{-a_n} - 1, 1\right).$$

Finally, for the case where $a_n > 0$, we can conclude that $\psi(2\pi) - \psi(0) = 2\pi n$. The polynomial $p(z)$ has a change in argument of $2\pi n$ and thus we have n zeroes. We have now proved case 1 for positive a_n of the Fundamental Theorem of Algebra, using the Argument Principle.

We will consider case 2 now and start from re-writing equation 3 by factoring out -1.

$$Re\zeta = -\left(-a_n - \frac{a_{n-1}\cos\theta}{R} - \frac{a_{n-2}\cos 2\theta}{R^2} - ... - \frac{a_1\cos(n-1)\theta}{R^{n-1}} - \frac{a_0\cos n\theta}{R^n}\right)$$

By the expression

$$-a_n - \frac{a_{n-1}\cos\theta}{R} - \frac{a_{n-2}\cos 2\theta}{R^2} - ... - \frac{a_1\cos(n-1)\theta}{R^{n-1}} - \frac{a_0\cos n\theta}{R^n}$$

we can say that $-a_n > 0$ which is the same as saying $a_n < 0$. Thus we can use case 1 to show case 2.

Note, the previous case $a_n > 0$ tells us

$$Re(\zeta) = -a_n - \frac{a_{n-1}\cos\theta}{R} - \frac{a_{n-2}\cos 2\theta}{R^2} - ... - \frac{a_1\cos(n-1)\theta}{R^{n-1}} - \frac{a_0\cos n\theta}{R^n} > 0$$

if $R > max\left(\frac{\alpha}{a_n} - 1, 1\right)$ where $\alpha = max\{|-a_k|\}$ for $0 \le k \le n-1$.

Thus we can deduce

$$Re\zeta = -\left(-a_n - \frac{a_{n-1}\cos\theta}{R} - \frac{a_{n-2}\cos 2\theta}{R^2} - ... - \frac{a_1\cos(n-1)\theta}{R^{n-1}} - \frac{a_0\cos n\theta}{R^n}\right) < 0.$$

For the case where $a_n < 0$, we can say the term $\alpha(2\pi) - \alpha(0) = 0$ and thus $\psi(2\pi) - \psi(0) = 2\pi n$. The change in argument of $p(z)$ is $2\pi n$ so by the Argument Principle, we have n zeroes.

Using the Argument Principle, we have proved the Fundamental Theorem of Algebra has n zeroes by considering two cases: $a_n < 0$ and $a_n > 0$. $\square$

6 Special Case of Rouché's Theorem

Theorem 5 (Rouché's Theorem for Circles)**.** *If f and g are holomorphic on and inside a circle C with radius R and $|f(z)| > |g(z)|$ for all z on C, then*

$$\mathcal{N}_{f+g} = \mathcal{N}_f$$

where $\mathcal{N}_f$ denotes the number of zeroes of f inside C, counted with multiplicity.

A more general Rouché's Theorem can be found in [1] and [2].

Remark 8. The theorem above is essentially telling us that the change in argument of $|f(z)|$ is $2\pi\mathcal{N}_f$, which we get by evaluating $\psi(2\pi) - \psi(0)$ with $\psi(\theta)$ being the argument of the function $f(Re^{i\theta})$ in polar representation. I will reiterate this in the proof below - for context.

Before we look at the proof, I have given a lemma below which will be essential to the proof of Rouché's Theorem.

Lemma 3. *If $z \in \mathbb{C}$ and $|z| < 1$ then $Re(1 + z) > 0$.*

I omit the proof of the lemma as it is trivial.
We can now prove Rouché's Theorem.

Proof. We are given $|f(z)| > |g(z)|$ which implies $|f(z)| > 0$ and $|g(z)| \geq 0$. Using polar representation, we know $f(Re^{i\theta}) = r(\theta)e^{i\psi(\theta)}$. We also know $\psi(2\pi) - \psi(0) = 2\pi\mathcal{N}_f$ from the statement of the theorem. Let us start by looking at $f(Re^{i\theta}) + g(Re^{i\theta})$:

$$\begin{aligned} f(Re^{i\theta}) + g(Re^{i\theta}) &= \varrho(\theta)e^{i\gamma(\theta)} &(7)\\ &= f(Re^{i\theta})\{1 + \frac{g(Re^{i\theta})}{f(Re^{i\theta})}\}. &(8) \end{aligned}$$

Above, we wrote equation 7 using polar form and factored out $f(Re^{i\theta})$ in equation 8 because we know that $\left|f(Re^{i\theta})\right| > 0$.

Let $1 + \frac{g(Re^{i\theta})}{f(Re^{i\theta})} = \zeta$, then we can re-write $f(Re^{i\theta}) + g(Re^{i\theta})$ as

$$\begin{aligned} f(Re^{i\theta}) + g(Re^{i\theta}) &= f(Re^{i\theta})\zeta &(9)\\ &= r(\theta)e^{i\psi(\theta)}\left|\zeta\right|e^{i\alpha(\theta)}. &(10) \end{aligned}$$

Here, $Re(\zeta) > 0$, and as seen in previous sections this implies $-\frac{\pi}{2} \leq \alpha(\theta) \leq \frac{\pi}{2}$ which can be seen in figure 9.

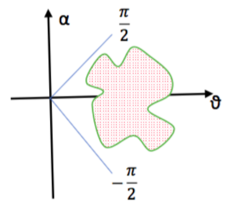


Figure 9: The image of a circle C with radius R under the function $f(z)$.

It follows that $\alpha(2\pi) = \alpha(0)$, meaning $|\zeta| e^{0i} = |\zeta| e^{2\pi i}$ is the same complex number. By equating the indices of $e^{i\gamma(\theta)}$ and $e^{i\psi(\theta)}e^{i\alpha(\theta)}$ we get that $\gamma(\theta) = \psi(\theta) + \alpha(\theta)$.

Finally, we conclude that

$$\begin{aligned} 2\pi\mathcal{N}_{f+g} &= \gamma(2\pi) - \gamma(0) \\ &= \psi(2\pi) - \psi(0) + (\alpha(2\pi) - \alpha(0)) \\ &= \psi(2\pi) - \psi(0) \\ &= 2\pi\mathcal{N}_f. \end{aligned}$$

Note, $\alpha(\theta)$ vanishes when we compute $\psi(2\pi)-\psi(0)$ because $\alpha(2\pi)-\alpha(0) = 0$.

From this result we see that ζ does not contribute to the change in argument.

In other words, the number of roots of $f(z) + g(z)$ is identical to the number of roots of $f(z)$. Furthermore,the change in argument is $2\pi\mathcal{N}_f$ and thus $f(z)$ has $\mathcal{N}_f$ roots. We have now proved this special case of Rouché's Theorem for circles using the Argument Principle. $\square$

7 Applications of the Argument Principle in Stability Problems

In engineering, certain feedback control systems need to be stable. Firstly, what is a feedback control system? Essentially, performance information is measured and then fed back into the system to ensure it performs correctly.

For example, What happens when a person drinks too much in a given time period? The kidneys will push urine through to the bladder and after it has become full to a certain extent, various receptors will trigger a message to be sent to via nerve strands to the spine. The spine will then send a message to their brain, letting them know that they should probably pass urine. Once this control system starts to become unstable (i.e. starts to fail), a person experiences involuntary urination. This is a biological control system, but there are also many which are created by engineers.

Let's take a formula 1 race car, according to regulations set by the Fédération Internationale de l'Automobile (FIA), they must have a range of control systems. A specific one we can look at is the medical warning light control system, it must be fitted to each race car in order to indicate the severity of an accident. Without this light control system, the rescue team would not be able to efficiently bring the driver to safety and ensure there are no life threatening injuries (regulation 8.10 from [5]).

The Argument Principle forms the basis for a stability criterion called the Nyquist Stability Criterion in the design and analysis of control systems. This application is very powerful and important in everyday life, also having the ability to provide relative stability and margins. In this section, we will not delve too deep in the theory behind the Nyquist Stability Criterion, but rather show what it is and where the Argument Principle comes into use.

In figures 10 and 11 we can see two different types of systems - open loop and closed loop. In the open loop system, we have two processes G and H which receive an input and produce an output, respectively. In everyday life, we use washing machines which are a great example of an open loop system.

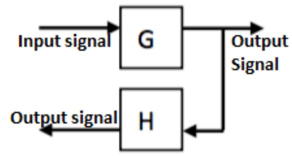


Figure 10: An open loop system.

In the closed loop system, we have the additional feature of an adjusting device, which interprets the feedback received from H and adjusts the input signal for process G. This can be an advantage in everyday life because the system adjusts its own output. For example, an iron which stops producing heat after a certain temperature setting is reached and has a sensor which cuts off the power if it has been left on for too long. We wouldn't want a fire, it's important to ensure the temperature remains stable.

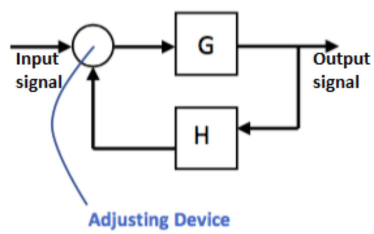


Figure 11: A closed loop feedback control system.

Note, it is not uncommon to say H is identical to 1, then we have a system which is illustrated in figure 12, called the unity feedback system [4].

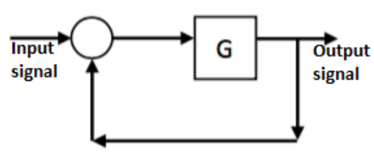


Figure 12: A unity feedback control system.

In the interest of building a basic understanding, the signals relevant to each arrow have been omitted. Usually, we would have to find the Laplace Transform of each of the signals which are originally time dependent in order to find $T(s)$, the transfer function. This step is omitted. See table 2 for the results. When we say transfer function, we mean a function which maps points from one plane to another, we will use the s-plane to w-plane. For a reminder of what this is, you may wish to refer to the background section.

Table 2: Transfer functions for three different types of systems.

$Open - Loop$	$Closed - Loop$	$Unity - Feedback$
$T(s) = G(s)H(s)$	$T(s) = \frac{G(s)}{1+G(s)H(s)}$	$T(s) = \frac{G(s)}{1+G(s)}$

For the open loop system, we need to find the poles of the transfer function. That is, the values of s which cause $T(s) \to \infty$. If there is a pole in $Re(s) \geq 0$ (i.e. the right half plane), then the open loop system is unstable.

Remark 9. We can refer back to the background section to remind ourselves what a pole is.

For the closed loop system, we need to find the zeroes of the denominator of the transfer function which cause $T(s) \to \infty$. If there is a zero of $1 + G(s)H(s)$ in $Re(s) \geq 0$, then the closed loop system is unstable.

Finally, in the unity feedback system, we need to ensure $T(s)$ does not have any poles in $Re(s) \geq 0$. In other words, we must find the zeroes of $1 + G(s)$. From here on, we only look at the unity feedback system.

Remark 10. $G(s)$ and $T(s)$ are both rational functions. We assume the denominator of the rational function $G(s)$ has higher order than the numerator. Also, we assume that there are no poles of $G(s)$ on the imaginary axis [4]. Things would start to get complicated if there were, but in the real world there are such scenarios. For the purpose of this project, we are not interested in them.

Now, the question that interests us is: are there any zeroes in $Re(s) \geq 0$ of $1 + G(s)$? To find out, we can sketch a Nyquist contour, generally we could use software such as Matlab to produce this. An example sketch can be seen in [4] on page 10. The Nyquist contour is a semi-circle contained in the right half plane and runs from $[-iR, iR]$ (the base of the semi-circle) where R is the radius of the semi-circle. Taking the semi-circle in the right half plane with very large R ensures that we have contained any zeroes and poles in the contour. This is now mapped onto the w-plane as $R \to \infty$ and the image produced is a Nyquist plot. Here, we use the Argument Principle.

Just as we did previously, we are counting the number of encirclements of the plot around the origin. As a reminder of what is happening here, we can refer back to the Mathematica screenshots provided earlier, if needed. As stated previously, the number of encirclements only tells us how many more zeroes we have than poles (if the image is clockwise in direction) or how many more poles than zeroes (if it is anti-clockwise in direction).

The number of encirclements around the origin is called the winding number, $W(f(C), 0) = P_G - P_T$ where P is a pole, because the zeroes of $1 + G(s)$ are the same as the poles of $T(s)$ [4]. The Nyquist Criterion below is taken verbatim from source [4].

Theorem 6 (Nyquist Criterion)**.** *The unity feedback system is asymptotically stable if and only if*

(a) the Nyquist plot does not pass through zero;

(b) the winding number of the Nyquist plot around zero equals the number of poles of the open loop system.

In the last section we looked at a handy theorem, Rouché's Theorem. It is precisely this which guarantees stability for small deviations of G [4].

8 Conclusion

Throughout the project we have seen examples of the Argument Principle in action, in helping us find the number of zeroes of a function using analysis. We also used numerical integration but pointed out that analytical methods are generally easier. This method is non-trivial and forms the basis of a very important application in engineering: the Nyquist Stability Criterion. Without the Argument Principle, engineers would have much difficulty on determining the stability of a system.

We also looked at the Fundamental Theorem of Algebra which tells us that a polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z^1 + a_0$ where $a_n \neq 0$ and the coefficients are complex, has n roots in the complex plane, counted with multiplicity. There are various ways of proving this theorem but in this project we looked specifically at the proof using the Argument

Principle. We first used polar representation to re-write $p(z)$ and then approximated $p(Re^{i\theta})$ for very large values of R. The punch line was, as z traverses a unit circle in the anti-clockwise direction, $p(z)$ winds around the origin n times in the anti-clockwise direction. This positive integer is the number of zeroes of the function.

The proof of Rouché's Theorem saw the next major application of the Argument Principle. Rouché's Theorem says if f and g are holomorphic on and inside a circle with radius R and $|f(z)| > |g(z)|$ for all z on C, then $\mathcal{N}_{f+g} = \mathcal{N}_f$ where $\mathcal{N}_f$ denotes the number of zeroes of f inside C, counted with multiplicity. Rouché's Theorem is important because it guarantees the stability of a unity feedback system.

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