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The Kuramoto Model

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Declaration of original work

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This work is dedicated to my family and my friends.

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Abstract

The Kuramoto model is an illustrative model that analyses the synchronisation of coupled limit-cycle oscillators. In the model, the number of oscillators N is large, and the natural frequencies are distributed according to a unimodal and symmetric probability density. The synchrony state is governed by the strength of the coupling between the oscillators. When the coupling strength surpasses a particular value, called threshold, the system displays a phase transition. Below the threshold, the system is fully incoherent and the oscillators are running on their own frequencies. If the coupling strength exceeds the threshold, the system is partially synchronised. Some of the oscillators synchronise while others stay asynchronous.

1 Introduction

The story of the Kuramoto model began in 1975 [1–6]. The story has been through many turns and it ends up with more questions than answers. In the 1990s, Crawford tackled the problem and wrote some papers regarding this topic [7–9].Though some of his papers look fearsome, he elucidated some questionable problems. Lately, Strogatz put Crawford's work in context in 2000 [10].

Along the way, the current paper follows Strogatz work. The background identifies the history of the Kuramoto model. The section that follows the background gives a complete description of the Kuramoto model and how to measure synchronisation. Later, section 4 illustrates Kuramoto Self-consistency Analysis, which gives the exact formula for critical coupling. Finally , some issues regarding stability are pointed out.

2 Background

Synchronisation is a cooperative phenomenon that takes place when a large set of individuals entities coordinate to work simultaneously. Examples of synchronisation can be found in biology, physics, chemistry, engineering, and social systems. A conventional example of synchronisation is the synchronous flashing of fireflies. The fireflies start to flash incoherently then start to flash at the same time after a period of time. The fireflies synchronisation exist through the relation (coupling) between males and females. One of the successful model of synchronisation is the Kuramoto model.

The Kuramoto model is mainly stimulated by collective synchronisation in which a huge number of oscillators locks to a common frequency though these individual oscillators have different frequencies [11, 12]. The first mathematical study of collective synchronisation was done by Wiener [13,14]. Wiener's study of collective synchronisation, which was based on Fourier integrals [13], came to a dead end.

Later, Winfree produced more predictive strategy [11]. He put the problem in terms of an enormous set of interacting limit-cycle oscillators. The problem was unmanageable, but Winfree simplified it by assuming that the coupling is weak and the oscillators are identical. The oscillators showed different behaviour over a fast timescale and over a long timescale. Another simplification, made by Winfree, occurred when each oscillator was coupled to the unified rhythm. The Winfree model is given by:

$$\dot{\theta} = \omega_i + \sum_{j=1}^{N} \Big(X(\theta_j) \Big) Z(\theta_i) \qquad i = 1, ..., N$$

where θ_i is the phase of oscillators, ω_i is the natural frequency and $Z(\theta_i)$ is the sensitivity function. Using a numerical and analytic approach, Winfree discovered the relationship between the oscillators and phase transition. When the scatter of the frequencies is greater than the coupling, the system is incoherent and each oscillator moves at their frequencies. As the scatter becomes smaller, the incoherence exists until a particular value called the threshold is reached. After that, the system behaves coherently. This phenomenon motivated Kuramoto to write a paper with his student Nishikawa [5]. However, Kuramoto himself wrote his first paper on the topic in 1975, and it has been known as the Kuramoto model ever since.

3 The Kuramoto Model

3.1 - Model Description

The Kuramoto model is a simplification of Winfree's model to study the evolution of a huge population of coupled limit-cycle oscillators [2]. The model displays that if the coupling is weak and the oscillators are nearly identical, the long-term dynamics for any system are given by a phase equation of the following form:

$$\dot{\theta} = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), \qquad i = 1, ...N \tag{1}$$

where θ_i, θ_j are the phases and Γ_{ij} is the interaction function. The oscillators could be connected in a different random graph due to the arbitrariness of the interaction functions. To simplify the analysis of the model, Kuramoto used a sine function to couple the oscillators, and used the mean-field case to trace the model as shown below:

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N} sin(\theta_j - \theta_i) \tag{2}$$

combining (1) and (2) gives the governing equation:

$$\dot{\theta} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} sin(\theta_j - \theta_i), \qquad i = 1, ...N \tag{3}$$

where $k \geqslant 0$ is the coupling strength, a key parameter in the problem, and ω_i is the natural frequencies that distribute to a probability density g(w). While g(w) is unimodal and symmetric $g(\Omega + \omega) = g(\Omega - \omega)$ around the origin, one can set the mean to $\Omega = 0$ because of the rotational symmetry of the model.This transformation will not affect the model, since the Kuramoto governing equation will be invariant to such a change . This gives $g(\omega) = g(-\omega)$ for all ω [10].

For instance, If $\theta_i(t) = \theta(j)$, the angle variables synchronise. As opposed to that, they are not synchronised if $\theta_i(t) \neq \theta(j)$. While each oscillator tends to run independently at its own frequency, the coupling tries to synchronise them together. To see how that depends on the coupling strength K, consider the case if there is no coupling ($K = 0$), The governing equation becomes:

$$\theta_i(t) = \omega_i + \theta_i(0)$$

For the identical initial condition $\theta_i(0) = 0$ for all i, it is clear the angle variables depend only on the natural frequencies ω_i. If all frequencies ω_i are the same, the angles synchronise, but they do not synchronise if ω_i differ (Figure 1 and 2).

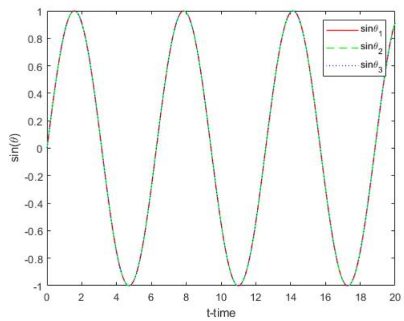

Figure 1: Three oscillators start from the same initial condition with same frequencies $_i$ (synchronised oscillators)

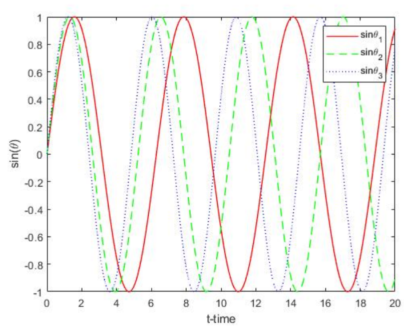

Figure 2: Three oscillators start from the same initial condition but with different frequencies $_i$(asynchronous oscillators)

To figure this out, the next section introduces the Order Parameter :

3.2 - Order Parameter

The state variable θ is called the phase of the oscillator and the phases of any system can be viewed by points going around the simplest loop, a unit circle [2]. The phases live in the interval [0, 2pi] as the unit circle is periodic and we can represent the state of an oscillator by its phase $\theta(t)$.

To describe the unit circle, it is appropriate to set it in the complex plane, where a complex number Z is on the circle if it has length one, i.e $|Z| = 1$ and can be written as

$$Z = e^{i\theta} = cos\theta + isin\theta, \tag{4}$$

where $i = \sqrt{-1}$ and θ is the counterclockwise angle between the positive real axis and the vector from the centre to the complex number that represents the phase at the circle.

One way to measure the collective behaviour of a collection of phase oscillators is their degree of synchrony. The oscillators are completely synchronised if their phases are moving together around the unit circle. On the contrary, if the phases are spread around the circle, the oscillators are asynchronous. The Kuramoto order parameter r is used to measure the level of synchrony and it can be defined by the average of all the complex numbers representing the phases of the oscillators.

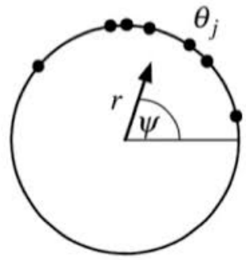

Figure 3: order parameter r has magnitude corresponding to the average of θ_j phases plotted around the unit circle [10]

For N oscillators with θ_j phases, the order parameter is given by:

$$Z = re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \qquad j = 1, 2, ...N \tag{5}$$

where ψ is the average phase or the argument of Z, i.e.the counterclockwise angle between Z and the positive real axis, and r measures the phase coherence (Figure 3).

To explain the idea, consider two oscillators with their phases represented by a vector between the origin and the corresponding points on the circle. The sum of these two vectors, using the geometric vector definition, is a vector that points to the average direction of the two oscillators. So, if the two phases are equal, the average vector will be the unit vector. In contrast, if the phases are opposite of each other, their average vector will be zero. Kuramoto used the magnitude of Z as a measure of synchronisation i.e $r = |Z|$.

Basically, r could capture the degree of phase coherence in the system. It vanishes ($r \approx 0$) when the phases are distributed around the circle (incoherent system) and approaches one ($r \approx 1$) when the phases of all oscillators are acting together like a giant oscillator (coherent system). Kuramoto rewrote the governing equation (3) in terms of the order parameter by multiplying both sides of equation (5) by $e^{-\theta_i}$ to obtain:

$$re^{i\psi-\theta_i} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j-\theta_i} \tag{6}$$

The L.H.S can be expressed in the form :

$$re^{i\psi-\theta_i} = r(cos(\psi-\theta_i) + isin(\psi-\theta_i))$$

and the R.H.S can be expressed as:

$$\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j-\theta_i} = \frac{1}{N}\sum_{j=1}^{N}(cos(\theta_j-\theta_i) - isin(\theta_j-\theta_i))$$

consider only the imaginary part from each side as it is the exact expression that appears in the Kuramoto governing equation which yields:

$$rsin(\psi-\theta_i) = \frac{1}{N}\sum_{j=1}^{N} sin(\theta_j-\theta_i)$$

The governing equation (3) can be rewritten in terms of the order parameter as follows:

$$\sum_{j=1}^{N} sin(\theta_j-\theta_i) = Nrsin(\psi-\theta_i)$$

by substitution, equation (3) becomes:

$$\dot{\theta} = \omega_i + Krsin(\psi-\theta_i), \qquad i = 1, ..., N \tag{7}$$

Equation (7) appears to be an equation for one oscillator, but by looking at the definition of the order parameter, these two quantities r and ψ involve all the other oscillators. In other words, the oscillators are coupled only through the mean-field quantities r and ψ.In Particular, each oscillator is coupled to the mean phase ψ with a coupling strength Kr, rather than individual phase [10]. Moreover, the coherence r is proportional to the coupling strength K in a positive way. If the system becomes more coherent, then the effective coupling Kr increases and more oscillators will pull toward the synchronise state. In contrast, if r decreases, it becomes self-limiting (Winfree) [11].

3.3 - Simulations and Numerical Results

The simulation and numerical methods are used to clarify how r evolves with time and what role coupling strength plays. For solidity, the distribution g(ω) is chosen from any infinite tails distribution and K is varied. Kuramoto showed that for N number of oscillators there is a critical coupling k_c delimiting the synchronisation state. Also, For all K below the certain threshold $K < K_c$, all the oscillators are incoherent and act like if they were not coupled (fully asynchronous). In this case, the phases distribute around the circle and r decays of size $O(N^{-\frac{1}{2}})$ to fluctuate around zero (Figure 4). However, for all K above that critical value $K > K_c$, the oscillators become coherent and act like one oscillator (fully synchronszed). Now all the phases gather towards the average phase and $r(t)$ grows exponentially until it reaches some level $r_\infty < 1$ and fluctuates around with size $O(N^{-\frac{1}{2}})$ (figure 4). By looking at that, the oscillators become synchronised for a certain value of the coupling strength K_c and the connection between the Kuramoto model and phase transition becomes obvious.

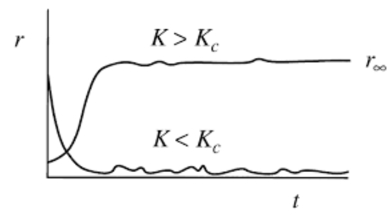

Figure 4: evolution of r(t) [10]

In some cases of individual oscillators, the population splits into two different groups: part of the oscillators are in the synchronised state and the others are in the asynchrony state. The part of the oscillators around the centre of the distribution lock together and synchronise, while the other oscillators in the tails drift and run around their natural frequencies. This mixed state is called 'partially synchronised' where more oscillators are recruited into the synchronised state by increasing the coupling strength K, and r grows accordingly (Figure 5). As a numerical result, r_∞ depends only on the value of K_c, and the fluctuation of r becomes smaller as the number of oscillators increases. Kuramoto affords formulas to calculate the critical coupling K_c and coherence measure $r - \infty$, and derives definite results [10].

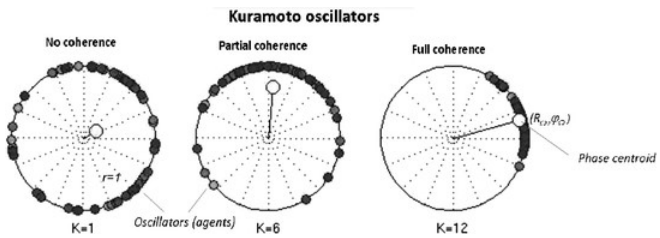

Figure 5: More oscillators are recruited into the synchronised state as K increases and r grows accordingly [15]

For $K < K_c$, the oscillators are running individually in the incoherent state and r fluctuates around zero. As K increases, more oscillators get closer to synchronize until K saturates at the theoretical value K_c in the limit of large N. The oscillators reach the critical value to transform from incoherent state to coherent one. Then, at that level when $K > K_c$, the oscillators are moving together coherently and r tends to 1 (Figure 6).

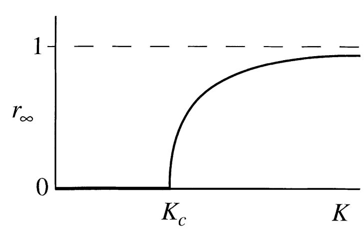

Figure 6: r evolution with increase of K (bifurcation diagram) [10]

4 Kuramoto's analysis and self-consistency

Kuramoto found out the long-term behaviour of the solutions in the limit $N \to \infty$ by what is called self-consistency. He tried to obtain a steady solution, where $r(t)$ is constant and ψ rotates uniformly by going into a suitable rotating frame with frequencies Ω. The average of phases could be set to $\psi(t) \equiv 0$ without effecting the model, since the Kuramoto equation is invariant to such a change. Then the governing equation (7) gives:

$$\dot{\theta} = \omega_i + Krsin\theta_i, \qquad i = 1, ..., N \tag{8}$$

As a result of the assumption that r is constant, all the oscillators are independent and Kuramoto described the motions of all the oscillators. These resulting motions depend on the parameter r and denote a consistent value for r and ψ. By finding solutions of (8), two types of behaviour will result. These types of long-term behaviour depend on the size of frequencies relative to Kr. The oscillators with $|\omega_i| \leq Kr$ reach a stable fixed point which can be obtained by setting $\dot{\theta} = 0$ in (8). The stable fixed point is defined by

$$\omega_i = Krsin\theta_i \tag{9}$$

Oscillators in this case will be called locked oscillators where $|\theta_i| \leq \frac{1}{2}\pi$. On the other hand, the oscillators with $|\omega_i| > Kr$ do not approach the fixed point. They move around the circle non-uniformly, so they are called drifting oscillators. In particular when the coupling K is strong and r is high, they could pull the oscillators away from their natural frequencies. Conversely,

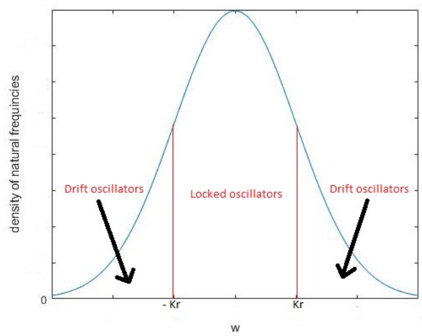

Figure 7: Locked and Drifting Oscillators

weak coupling with small coherence could not pull the oscillators away from their natural frequencies. All the oscillators that locked to a common frequency are consistent with the centre of $g(\omega)$ while the drifting corresponds to the tails [10] (Figure 7).

Kuramoto clarified how the population can be divided into locked and drifting oscillators. In order for the drifting oscillators to keep the value of r constant as assumed, Kuramoto required that the drifting oscillators $\rho(\theta\omega)$ form a stationary probability distribution. Then, for r to be constant we must have

$$\rho(\theta\omega)\dot{\theta} = C$$

As stationary demands inverse proportion between $\rho(\theta, \omega)$ and the speed of θ

$$\rho(\theta\omega) = \frac{C}{\dot{\theta}} = \frac{C}{|\omega - Krsin\theta|} \tag{10}$$

where C can be determined by the normalization of the probability distribution ρ to give:

$$C = \frac{\sqrt{\omega^2 - (Kr)^2}}{2\pi}$$

The order parameter (5) can be re-expressed as an integral with respect to the distribution, as follows:

$$\langle e^{i\theta} \rangle = \langle e^{i\theta} \rangle_{lock} + \langle e^{i\theta} \rangle_{drift}$$

the angular brackets used here to indicate the average of population and $\langle e^{i\theta} \rangle = re^{i\psi} = r$ since $\psi = 0$, then

$$r = \langle e^{i\theta} \rangle_{lock} + \langle e^{i\theta} \rangle_{drift}$$

Starting with locked oscillators where $|\omega_i| \leq Kr$

$$\langle e^{i\theta} \rangle_{lock} = \langle cos\theta \rangle_{lock} + i\langle sin\theta \rangle_{locked}$$
$$= \int_{|\omega_i| \leq Kr} cos\theta(\omega)g(\omega)d\omega + i\int_{|\omega_i| \leq Kr} sin\theta(\omega)g(\omega)d\omega$$

The symmetry of the locked oscillators distribution will give 0 for the imaginary part and the following for the real part:

$$\langle e^{i\theta} \rangle_{lock} = \langle cos\theta \rangle_{lock} = \int_{-Kr}^{Kr} cos\theta(\omega)g(\omega)d\omega$$

By using (9)

$$\langle e^{i\theta} \rangle_{lock} = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos\theta g(Krsin\theta)Krcos\theta d\theta = Kr\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos^2\theta g(Krsin\theta)d\theta$$

Now evaluate the drifting oscillators where $|\omega_i| > Kr$

$$\langle e^{i\theta} \rangle_{drift} = \int_{-\pi}^{\pi}\int_{|\omega| < Kr} e^{i\theta}\rho(\theta,\omega)g(\omega)d\omega d\theta$$

From equation (10), it can be seen that $\rho(\theta + \pi, -\omega) = \rho(\theta, \omega)$ and since $g(\omega) = g(-\omega)$, this integral turns to zero.

Collecting all the results from above will give the self-consistency condition :

$$r = Kr \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos^2\theta g(Krsin\theta)d\theta \tag{11}$$

For which $r = 0$ is always a trivial solution corresponding to a completely incoherent state with $\rho(\theta, \omega) = \frac{1}{2\pi}$.

A non trivial branch of the solution corresponds to a partially synchronised state meets:

$$1 = Kr \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos^2\theta g(Krsin\theta)d\theta \tag{12}$$

This branch of the solution bifurcates at the critical point $K = K_c$ where the order parameter $r = 0$ starts to grow to r_∞. The critical point K_c can be obtained by putting $r \to 0^+$ in (12) [10]

$$K_c = \frac{2}{\pi g(0)} \tag{13}$$

The whole idea is as follows:

Below the critical point K_c, r = 0 is the stable solution corresponding to a completely incoherent state. As the coupling reaches K_c the incoherent state becomes unstable and a nonzero solution appears. As K increases, r tends to 1 and the state becomes completely coherent where the synchronisation sets in.

By expanding the integral in (12), the bifurcation is subcritical if $g''(0) > 0$ and supercritical if $g''(0) < 0$, The size of the bifurcation complies with the square root scaling law [10] (Figure 6):

$$r \approx \sqrt{\frac{16}{\pi K_c^3}} \sqrt{\frac{K - K_c}{K_c} \frac{1}{-g''(0)}} \tag{14}$$

5 Kuramoto model for 100 oscillators

Through the simulation, the Kuramoto model is used with 100 oscillators and the frequencies ω_i are drawn from the normal distribution. By varying K, Different states of synchronisation appear. In the first case, the coupling strength has been chosen to be $K = 1$, so the resultant state is fully incoherent and the oscillators are distributed around the circle (Figure 8).

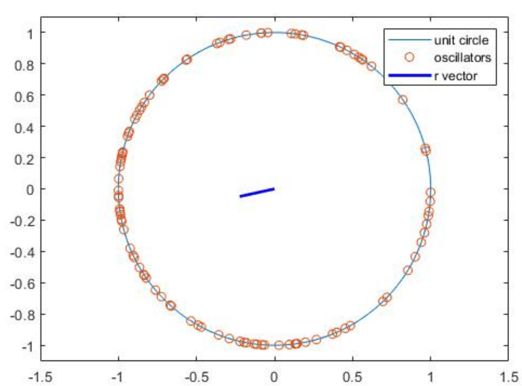

Figure 8: asynchronous oscillators due to a weak coupling

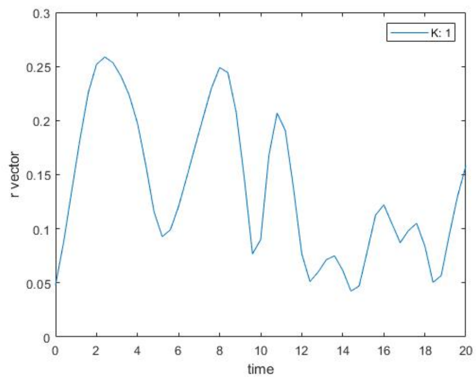

Figure 9: How r evolves with time at k = 1

The vector r fluctuates around zero, as the oscillators act as if they were uncoupled (Figure 9).

When K =1.5 the state is still incoherent but the magnitude of vector r is increased. As K increases, the oscillators split into two groups, part of the oscillators synchronised while the others stay incoherent. This case is called partially synchronised and r starts to grow (Figure 10)

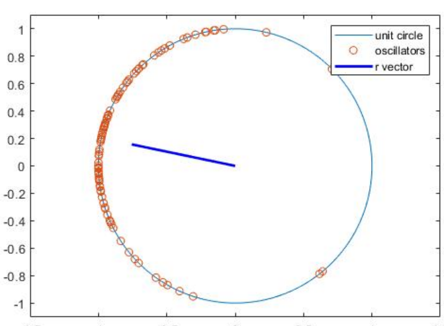

Figure 10: Partially Synchronized Oscillators

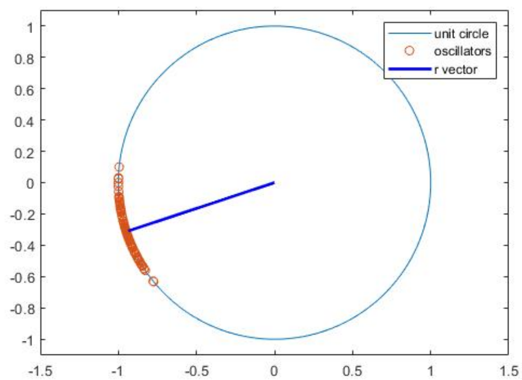

Figure 11: fully Synchronised State

With an increase in K, more oscillators begin to synchronise. When K=6 the oscillators move together coherently and the synchronisation sets in (Figure 11).

The measure of synchrony r tends to 1 because of the prominent strength of the coupling between oscillators(Figure 12)

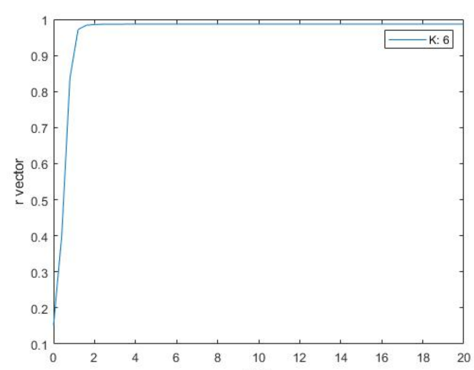

Figure 12: r evolution with time at K=6

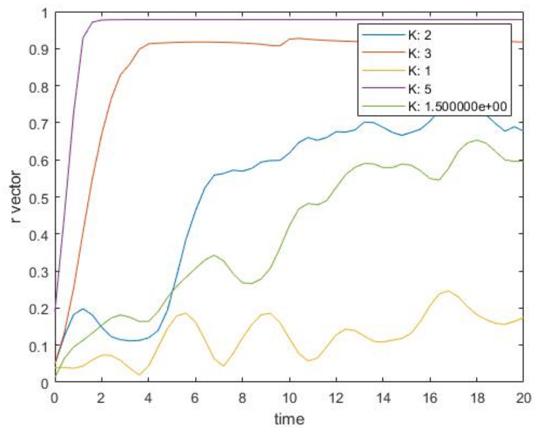

Figure 13: r evolution with different values of K

The vector r is changing with different values of K, For each value of K, r shows a different attitude. (Figure 13).

From the previous results, it is clear that any value $K > 2$, r tends to 1, and for any value $K < 2$, r decays to 0. That means the threshold seems to be in some value around 2. Simulation is used here to find the approximate critical point where $K = K_c$ (Figure 14).

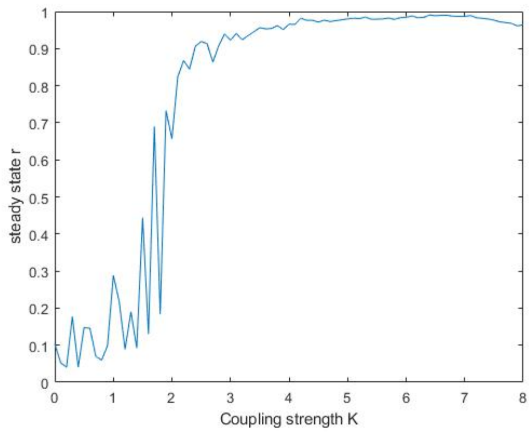

Figure 14: Bifurcation Diagram

From the plot of bifurcation, it is clear that $K_c \simeq 2$. By using the Kuramoto theoretical formula for K_C (13), the critical value for the normal distribution when $\sigma = 1$ is:

$$K_c = \sqrt{\frac{2^3}{\pi}}\sigma \simeq 0.2108 \tag{15}$$

As the number of oscillators N increases, the transition value starts to approach the numerical value for K_c (15). With further increases in N, one could get a sequence of numbers which do not converge monotonically. Those sequences converge very slow by $\frac{1}{N}$.

6 Conclusion

In this paper the principle properties of the Kuramoto model are explored. The model is given by a differential equations studied in the mean field with large numbers of oscillators. The phases of oscillators are represented by points on the unit circle in the complex plane. The model derives several results analytically. The measure of synchrony is a vector r which decays to zero if the system is incoherent and tends to 1 if the system is coherent. The synchronisation depends on the strength of the coupling K between oscillators. There is a transition value for K: when K is below that value, the state is fully asynchronous and when K is above that value, the state is partially synchronised. The Kuramoto model extracts the formula for the critical value of a large number of oscillators. In the self-consistency approach, where r is constant and $\psi = 0$, the behaviour of the system depends on the size of the frequencies. If $|\omega_i| \leq Kr$ the oscillators are locked, while they are drifting if $|\omega_i| > Kr$. The model is then derived with 100 oscillators. Stability is a difficult issue and the stability of the steady solution is left unsolved by Kuramoto. He was conscious of that problem and he explained some points in [2]. Kuramoto at first time tried to address the stability problem with Nishikawa. They suggest two different theories, but neither of them are correct [5, 6]. Strogatz and Mirollo proved that the incoherent state is stable for $K < K_c$ then becomes unstable when $K > K_c$ [16]. Crawford confirms that result later [7]. The stability problem for the non-zero branch remains untouched by anyone to this day.

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