Stability Analysis of Linear Delay Differential Equations

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To may late mother Hawa Adam.

Abstract

The concept of delay differential equations is discussed briefly along with where it was encountered first. We will investigate linear delay differential equations by studying the solutions of the characteristic equations. This will be done by writing the solutions in terms of exponential functions and hence obtaining a transcendental characteristic equation. Stability of the characteristic equation is established by evaluating eigenvalues, where those with positive real parts indicate unstable solutions. A numerical approach based on complex analysis is taken to judge the stability. We will then extend our investigation to second order delay differential equations where the stability boundaries undergo bifurcations.

Acknowledgements

I would like to thank Allah first and foremost for giving me the strength and ability to complete this dissertation. I would like to thank my supervisor Dr Wolfram Just for his continuous support. I am sincerely grateful for all his outstanding efforts which have made this dissertation possible.

I would also like to thank Dr Rainer Klages for his advice, guidance and support through out all my years of studying at this university, and Prof Leonard H. Soicher my MSc adviser for his help and support throughout a very difficult time in my life.

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Contents

Chapter 1

Introduction

Delay differential equations (DDEs) are a type of differential equation where the derivative of the unknown function at a certain time is given in terms of the values of the function at previous time. The general form is defined for $x \in \Re$

$$\dot{x} = f(t, x(t), x_t)$$

where $x_t = x(\tau) : \tau \leq t$ represents the trajectory of the solution in the past. Delay differential equations, also commonly known as 'difference differential equations', are a special class of differential equations called 'functional differential equations'. Delay differential equations were first encountered in the late eighteen century by Bernoulli, Laplace and Condorcet. Unfortunately, very little was accomplished during the nineteen century and the early part of all twentieth century. Instead, great development of the theory and application began after the Second World War [1], and from then on they continued to be investigated at a very rapid pace. Delay differential equations are now used to describe many phenomena in science - economics, engineering, life sciences, etc. DDE's can be solved either numerically or analytically. An analytical approach can be very difficult; DDEs have infinite dimensional phase-space which essentially means that the characteristic equations corresponding to the linear system will have an infinite number of solutions [2]. Although it would be worth mentioning that there exists an analytical approach for obtaining the complete solutions for a systems of delay differential equations based on the concept of Lambert function. Some of the numerical methods most often used to solve DDE's are Runge-Kutta and asymptotic solution.

A brief description of the organisation of the project follows. In the second chapter we introduce a linear first order of delay differential equation. We study their corresponding characteristic equation, this will be achieved by assuming $\exp(\lambda t)$ is a solution of delay differential equation. We will then examine numerically stability boundaries using a theorem borrowed from complex analysis. Examples will be given to illustrate the results. In the third chapter we extend to a certain two dimensional system. After investigating the stability of first order linear delay differential equation it becomes interesting to consider a specific two dimensional system. We use the same method proposed in chapter two. Fourth chapter will be devoted to second order delay differential equations. Complex analysis will not be used in this section, instead we will take a simpler approach. At first will try to see the effect of parameter change on the stability region. We will then construct bifurcation plots showing the location of bifurcation as the parameter is varied.

Chapter 2

Stability of linear delay differential equations and the argument principle

This chapter serves as an introduction to the more general types of equations that will be encountered in later chapters. We discuss first order delay differential equations

$$\dot{x}(t) = bx(t-\tau)\,. \tag{2.1}$$

Since Equation (2.1) is a linear equation, the solution may be written in terms of exponential functions. Thus if we use $x(t) = \exp(\lambda t)$ Equation (2.1) results in an algebraic equation for λ, namely

$$\lambda = b\exp(-\lambda\tau). \tag{2.2}$$

It is now quite useful to introduce the non-dimensional quantities $z = \lambda\tau$ and $\hat{b} = b\tau$ as Equation (2.2) then attains the slightly simpler form

$$0 = f(z) = z - \bar{b}\exp(-z). \tag{2.3}$$

This so-called characteristic equation determines in particular the stability of the trivial solution $x(t) \equiv 0$ of Eq.(2.1). One of the difficulties in the analysis of DDEs lies in finding an analytic method to solve this transcendental equation. DDEs are often solved using numerical methods, asymptotic solutions, and graphical tools. The approach we will be taking to tackle the problem will be using complex analysis.
The problem to determine the stability of the trivial solution reduces to the question of whether the characteristic equation admits eigenvalues λ with positive real parts. Thus to judge stability we have to count the number of solutions of Eq.(2.3) in the right half of the complex plane. Phrased differently, namely in terms of the inverse multipliers

$\mu = \exp(-\lambda\tau) = \exp(-z)$, we have to count the number of multipliers within the complex unit circle. To solve such an issue, i.e., to count the number of zeros (of the characteristic equation) inside a domain C, here the unit circle, we will employ a theorem borrowed from the theory of complex functions. Suppose f denotes a 'meromorphic function'(i.e. a function which has at most poles as singularities) and $\mathcal{C}$ denotes a closed Jordan curve (i.e. a curve which does not intersect itself) with positive orientation, in our case for instance the unit circle. Then the following statement holds [3] =)

$$\oint_{\mathcal{C}} \frac{f(z)'}{f(z)}\, dz = 2\pi i(N-P) \tag{2.4}$$

where N and P respectively denote the number of zeros and poles of $f(z)$ inside the contour $\mathcal{C}$, with each zero counted as many times as its multiplicity, and each pole counted as many times as its order. The common reason to consider the notion of multiplicity is to count correctly, without specifying exceptions (for example, double roots counted twice). It is assumed that $f(z)$ is analytic at every point on the contour. The integral on the left hand side of Eq.(2.4) can now conveniently be expressed in terms of the complex argument of the function $f(z)$. For that purpose recall that the complex argument $\Phi(z)$ is defined by

$$f(z) = |f(z)| \exp(i\Phi(z)). \tag{2.5}$$

Using the complex logarithm or otherwise we then obtain for the kernel of the integral

$$\frac{f'(z)}{f(z)} = (\ln f(z))' = (\ln |f(z)|)' + i\Phi'(z). \tag{2.6}$$

Using Eq.(2.6) the integral in eq.(2.4) becomes

$$\begin{aligned} \oint_{\mathcal{C}} \frac{f(z)'}{f(z)}\, dz &= \oint_{\mathcal{C}} (\ln |f(z)|)' dz + \oint_{\mathcal{C}} i\Phi'(z) dz \\ &= 0 + i\Delta\Phi \end{aligned} \tag{2.7}$$

where $\Delta\Phi = \Delta\arg(f)$ denotes the change of the complex argument along the contour $\mathcal{C}$. Thus Eq.(2.4) yields

$$\Delta\arg(f) = 2\pi(N-P). \tag{2.8}$$

Finally, we are able to work out the change of argument $\Delta\arg(f)$ by a simple geometric construction. For the purpose of simplicity, and since this case will be relevant for our applications, consider the unit circle. Then the contour $\mathcal{C}$ is given by the parametric representation $z = \exp(it)$ with $t \in [0, 2\pi]$. Then (as f is continuous on $\mathcal{C}$) $f(\exp(it))$

determines a closed curve, say $\mathcal{D}$, in the complex plane and the argument of $f(z)$ is given by the angle between the positive real axis and the ray from 0 to $f(z)$. Thus, if the closed curve $\mathcal{D}$ encircles the origin ℓ times then the argument of $f(z)$ changes by $\ell \times 2\pi$. Plotting the curve $\mathcal{D}$ it is easy to determine the change in argument, and thus to determine the number of zeros and poles inside the unit circle. By the argument principle, Eq.(2.8) the factor of 2π cancel and so we are left with $N - P = I(\mathcal{C}, 0)$, where $I(\mathcal{C}, 0) = \ell$ denotes the 'winding' number of f over $\mathcal{C}$ about 0.

Let us illustrate this statement using a very simple example, the function

$$F(z) = z^2 \tag{2.9}$$

and take the domain to be the unit circle. The function $F(z)$ has a doubly degenerated zero at $z = 0$ which is inside the unit circle. Now consider the boundary of the unit circle, i.e. the z-values $z = \exp(it)$ where t changes from 0 to 2 π. The variable z moves around the unit circle once shown in Figure 2.1(a). Consider the change of $F(z)$. We have

$$F(z) = F(\exp(it)) = (\exp(it))^2 = \exp(2it). \tag{2.10}$$

When t changes from 0 to 2π the argument of $F(z)$ changes from 0 to $4\pi = 2 \times 2\pi = N \times 2\pi$, where N is the number of zeros inside the unit circle (counting the multiplicity). In Figure 2.1(b) we can observe the circle appears two times thicker compared to 2.1(a), which means it surrounds the unit circle twice. That concludes the illustration of the theorem.

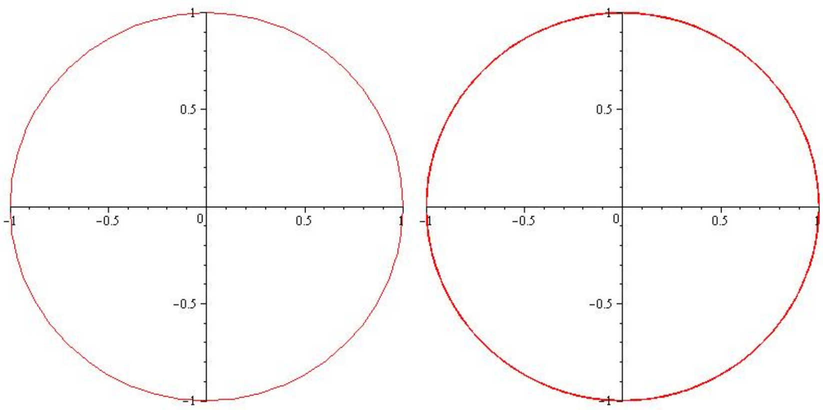

Figure 2.1: Contour plots (a) unit circle (left), (b) expression for eq.(2.10)

Now we are dealing with the function (cf. eq.(2.3))

$$F(z) = z - b\exp(-z) \tag{2.11}$$

and we want to count the solutions in the right half plane, i.e. the “unstable eigenvalues”. We could use the theorem [3] to count the number of eigenvalues. The right half plane is unbounded and it is not completely straightforward to deal with unbounded sets. For that purpose we introduce the new complex variable

$$\mu = \exp(-z). \tag{2.12}$$

The unstable solutions, i.e. z-values with real part larger then zero, are transformed to μ-values which are inside the unit circle since $|\mu| = \exp(-\mathrm{Re}(z))$. Hence we need to count solutions inside the unit circle. Rewriting Eq.(2.11) in terms of μ by taking the exponential on both sides we obtain

$$1 = \exp(z)\exp(-b\exp(-z)) \tag{2.13}$$

so that

$$\mu = \exp(-z) \Longrightarrow 1 = \frac{1}{\mu\exp(-b\mu)} \tag{2.14}$$

and finally

$$\mu = \exp(-b\mu)\,. \tag{2.15}$$

To determine the zeros of eq.(2.15), i.e. the zeros of the function

$$G(\mu) = \mu - \exp(-b\mu) \tag{2.16}$$

we may apply Theorem (2.4) with $\mu = \exp(it)$ where t changes from 0 to 2π

$$G(\exp(it)) = \exp(it) - \exp(-b\exp(it). \tag{2.17}$$

We could plot Equation (2.17) on the complex plane to assess whether its argument does indeed change by a multiple of 2π. Thus as we have seen, if the curve encircles the origin n times then there are n unstable eigenvalues, and if the curve does not encircle the origin then there are none.

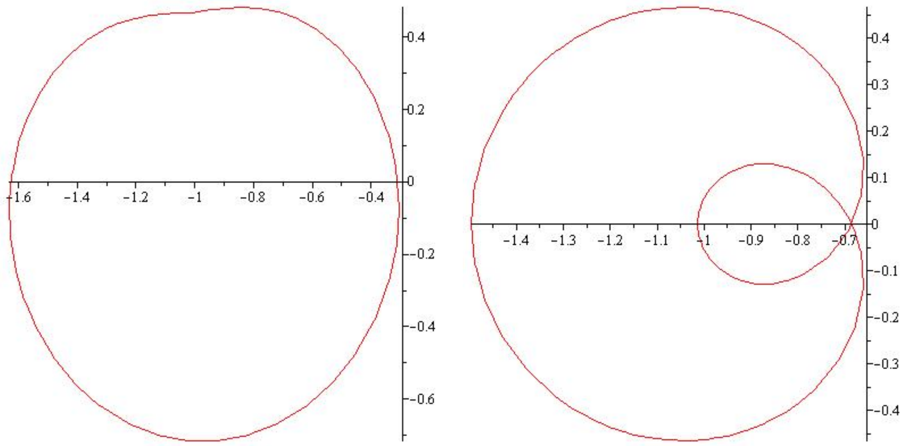

Figure 2.2: Contour plots of the expression (2.17) for $b = -0.4$ (left), $b = -0.7$ (right).

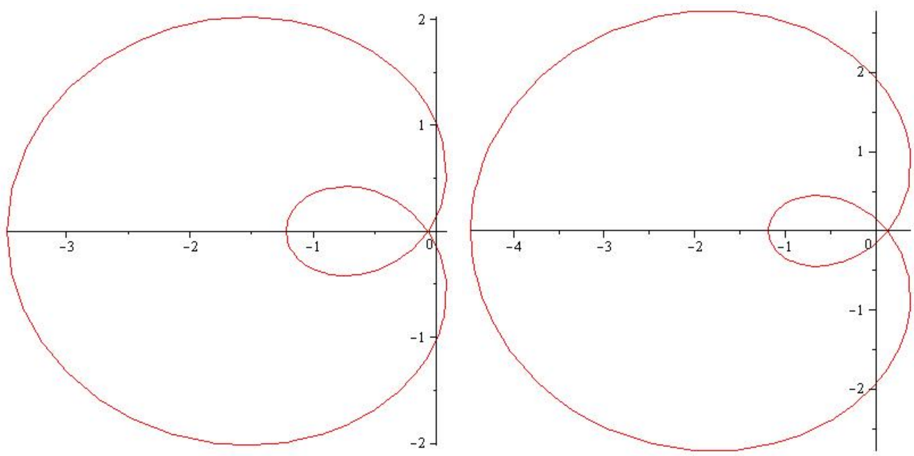

Figure 2.3: Contour plots of the expression (2.17) for $b = -1.5$ (left), $b = -1.8$ (right).

We have used Maple to produce the figures 2.2 and 2.3. We can see in Figure 2.2 (a),(b) and 2.3 (a) that the curve does not encircle the origin, therefore there are no unstable eigenvalues. In Figure 2.3 (b), we can see that the curve surrounds the origin twice therefore there are two unstable eigenvalues. In fact, we can see that at some value between $b = -1.5$ and -1.8 some eigenvalues with positive real parts occur, i.e., the original equation of motion (2.1) becomes unstable. Thus, we would like to determine the stability region for b. Of course, there can be an infinite number of stability intervals , but varying b we were able to numerically approximate one stability interval $b \in (-\pi/2, 0)$. The difficulty in the analysis of the DDEs lies in finding an analytic method to solve for the zeros of Eq.(2.11). We can rewrite such a condition as follows

$$z = b\exp(-z) \quad \Rightarrow \quad g(z) := z\exp(z) = b \tag{2.18}$$

$$z = g^{-1}(b)\,. \tag{2.19}$$

The inverse function which has been formally introduced in Eq.(2.19) is called the Lambert function. The analytic approach to obtain the complete solution for systems of delay differential equations is based on such a function. It is often called "Product log", which is the function $W(z)$ such that $z = W(z)\exp(W(z))$ solves $z = W(z)\exp(W(z))$ for a given value z. The Lambert function is defined as the solution of a transcendental equation $W\exp(W) = z(z \in \mathbb{C})$. It has infinite branches , denoted by $W_k(z)$, $k = 0, \pm1, \pm2, ..$ respectively. $W_0(z)$ is the unique branch that is analytic at the origin $z = 0$, and is called the principal branch [4]. This concept is applied to solve transcendental characteristic equation of DDE. The Lambert function also gives us the full stability region - i.e. the complete set of stability intervals - for the function in Equation (2.19). A full derivation of this result lies outside the scope of this thesis, for full details please refer to [5].

Chapter 3

Extension to a certain two dimensional system

In this section, the method proposed in chapter 2 will be applied to find the stability region of a two dimensional system of DDEs. In order to do that we quote the main result of Hara and Sugie [6]. Consider the delay differential equation system

$$\dot{x}(t) = Ax(t-\tau) \tag{3.1}$$

where A is a 2×2 matrix. Stability criteria are expressed explicitly in terms of the trace, the determinant and the eigenvalues of the matrix in the case of

$$A = -\rho \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.2}$$

where $\rho > 0$, and -$\pi \leq \theta \leq \pi$ denote the parameters of the system.
The zero solution of (3.1) is asymptotically stable if and only if

$$0 < \rho\tau < \frac{\pi}{2} - |\theta| \tag{3.3}$$

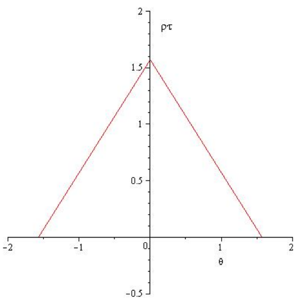

Figure 3.1: The stability region for system (3.1) with matrix (3.2), according to Eq.(3.3)

Eq.(3.3) determines a stability region in the ρ-θ parameter plane. Figure 3.1 shows a sketch of this region. We clearly see that the stability region has a triangular shape, and that no stability is possible if $\rho\tau$ exceeds $\pi/2$.

Our aim is to verify that this necessary and sufficient condition is satisfied for this specific matrix A using the method proposed in chapter 2. In fact, we will show that the original equation of motion (3.1) can be written as a single complex equation, namely

$$\dot{\xi} = -\rho \exp(i\theta)\xi(t-\tau) \tag{3.4}$$

For that purpose define

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \xi(t) = x_1(t) + ix_2(t)\,. \tag{3.5}$$

We want to convert the two dimensional (3.1) system to a one dimensional one, Eq.(3.4). We could do this by observing that

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\rho \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} = -\rho \begin{pmatrix} \cos\theta x_1(t-\tau) - \sin\theta x_2(t-\tau) \\ \sin\theta x_1(t-\tau) + \cos\theta x_2(t-\tau) \end{pmatrix} \tag{3.6}$$

Therefore, using the definition (3.5) we have

$$\begin{aligned} \dot{\xi}(t) &= \dot{x}_1 + i x_2(t) \\ &= \rho(x_1(t-\tau)\cos\theta - x_2(t-\tau)\sin\theta + i(x_1(t-\tau)\sin\theta + x_2(t-\tau)\cos\theta)) \\ &= \rho(x_1(t-\tau)(\cos\theta + i\sin\theta) + x_2(t-\tau)(-\sin\theta + i\cos\theta) \\ &= \rho(x_1(t-\tau) + ix_2(t-\tau))\exp(i\theta) \\ &= -\rho\xi(t-\tau)\exp(i\theta) \end{aligned} \tag{3.7}$$

Thus

$$\dot{\xi} = -\rho \exp(i\theta)\xi(t-\tau). \tag{3.8}$$

The characteristic equation is obtained by assuming a solution in the form of $\xi = \exp(\lambda t)$ and substituting it into Eq. (3.8). As a result we get

$$\lambda = -\rho \exp(i\theta)\exp(-\lambda\tau)\,. \tag{3.9}$$

Let us define the non-dimensional quantities $z = \lambda\tau, \bar{\rho} = \rho\tau$, thus

$$z = -\rho\tau \exp(i\theta)\exp(-z) \tag{3.10}$$

hence

$$z = -\bar{\rho}\exp(i\theta)\exp(-z). \tag{3.11}$$

As a result, the transcendental characteristic equation is

$$z + \bar{\rho} \exp(i\theta) \exp(-z)) = 0. \tag{3.12}$$

Taking the exponential of Eq.(3.12)

$$\exp(z) \exp(\bar{\rho} \exp(i\theta) \exp(-z) = 1 \tag{3.13}$$

and introducing the new complex variable $\mu = \exp(-z)$, we obtain

$$\frac{1}{\mu} \exp(\bar{\rho} \exp(i\theta)\mu) = 1 \tag{3.14}$$

i.e. following the ideas of the previous chapter we have to analyse the holomorphic function

$$F(\mu) = \mu - \exp(\bar{\rho} \exp(i\theta)\mu). \tag{3.15}$$

To determine the zeros of Eq.(3.15) we now apply the theory of complex functions which means we take $\mu = \exp(i\phi)$, ϕ changes from 0 to 2 π and evaluate

$$F(i\phi) = \exp(i\phi) - \exp(\bar{\rho} \exp(i\theta) \exp(i\phi)). \tag{3.16}$$

Let us first examine when θ and $\bar{\rho}$ take different values, we will show that condition (2.3) is indeed the correct stability condition. For the first case suppose that we fix θ and $\bar{\rho}$ varies. Figure 3.2 shows contour plots for $\theta = 0$ and three different values of $\bar{\rho}$. In figure 3.2(a) and (b) the curve does not surround the origin, which implies the argument does not change. Finally for the largest value in figure 3.2(c) the curve surrounds the origin twice. Thus the stability changes between $\bar{\rho} = 1.5$ and $\bar{\rho} = 1.66$. Such a result is perfectly consistent with Eq.(3.3) as the change of stability is expected to occur at $\bar{\rho} = \pi/2$.
Let us perform a similar analysis at a fixed value for $\bar{\rho} = 0.196$ and varying θ. We clearly see that the curve in Figure 3.3(a) does not surrounds the origin, therefore the argument does not change. In Figure 3.3(b) the curve surrounds the origin once. Hence the stability changes between $\theta = 1.399$ and $\theta = 2$. Again the result is consistent with the findings. As $\bar{\rho}$ increases ($\bar{\rho} = 10$) shown in Figure 3.3 (c) it leaves the instability region, and we see the number of eigenvalues increase.

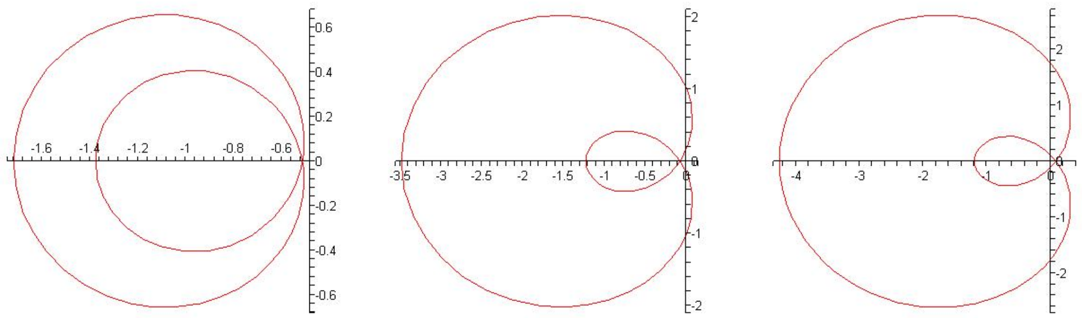

Figure 3.2: Contour plots for eq.(3.16 for different parameter values: (a) $\bar{\rho} := 1$, $\theta := 0$ (b) $\bar{\rho} := 1.5$, $\theta := 0$ (c) $\bar{\rho} := 1.66$, $\theta := 0$

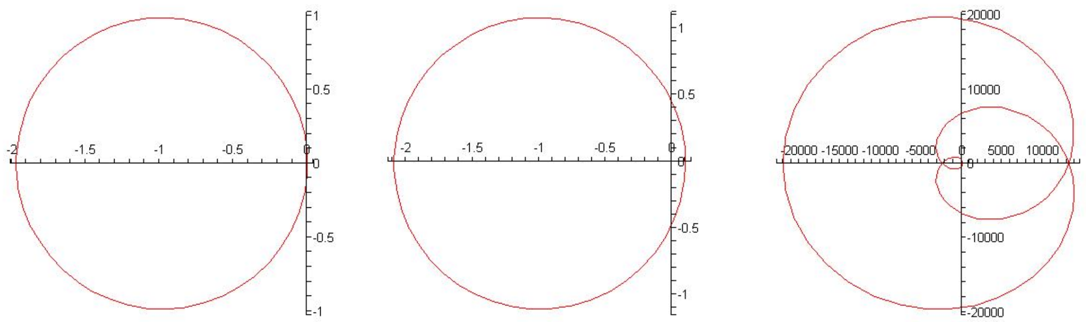

Figure 3.3: Contour plots for Eq.(3.16 for different parameter values: (a) $\bar{\rho} := 0.196, \theta :=$ 1.399 (b)$\bar{\rho} := 0.196$, $\theta := 2$, (c) $\bar{\rho} := 10$, $\theta := 0$

Chapter 4

Stability domains for a second order system

This chapter extends our earlier work on first order delay differential equation as we move up to second order delay differential equations. With the knowledge of the previous chapter we can now address the more ambitious equation

$$\alpha\ddot{x}(t) + \beta\dot{x}(t) + \chi x(t) = \delta x(t-\tau)\,. \tag{4.1}$$

where in physicists' terms, the coefficients have the following meaning: $\alpha = mass$, $\beta = damping$, $\chi = force$. The solution of Eq.(4.1) can be written in terms of exponential functions. Hence if we use $x(t) = \exp(\lambda t)$ in Eq.(4.1) it reduces to an algebraic equation

$$\alpha\lambda^2 + \beta\lambda + \chi = \delta(-\lambda\tau). \tag{4.2}$$

Rewriting the Eq.(4.2) using the abbreviation $z = \lambda\tau$ we obtain

$$\frac{\alpha}{\beta\tau}z^2 + z + \frac{\chi}{\beta} = \frac{\delta\tau}{\beta}\exp(-z) \tag{4.3}$$

hence

$$cz^2 + z = a + b\exp(-z) \tag{4.4}$$

which is the transcendental characteristic equation where $c = \alpha/(\beta\tau)$, $a = -\chi/\beta$, $b = delta\tau/\beta$.

We will be taking a different approach with regards to the complex contour integrals which could be used for providing stability. The main problem is that it is quite tedious to write the characteristic equation Eq.(4.4) in terms of the multiplier

$$\mu = \exp(-z). \tag{4.5}$$

In fact, it is unlikely that we will again end up with a holomorphic function, an essential ingredient for the complex contour integration technique, as it is hard to avoid complex logarithms, i.e., $z = \ln(\mu)$. To get around such a problem we just recall that we are mainly interested in that part of the parameter region where Eq.(4.4) has solutions with negative real parts only. Such a region is bounded by curves, and on that curves the real part of z changes sign, i.e., either $z = 0$ or $z = i\phi$. Hence to obtain the boundaries of the stability region one can directly deal with the z variable and analyse the expression

$$cz^2 + z = a + b\exp(-z) \tag{4.6}$$

for $z = 0$ and $z = i\phi$. For the first choice we obtain the boundary

$$0 = a + b\,. \tag{4.7}$$

In the latter case we end up with

$$c(i\phi)^2 + i\phi = a + b\exp(-i\phi)\,. \tag{4.8}$$

Eq.(4.7) yields a straight line in the a-b parameter plane, which gives a boundary of the stability domain. The region below the line is stable and the region above the line is unstable. Eq.(4.8) when written in real and imaginary part yields a parametric representation for the other part of the boundary

$$\begin{aligned} -c\phi^2 &= a + b\cos(\phi) \\ \phi &= -b\sin(\phi). \end{aligned} \tag{4.9}$$

To illustrate the behavior of the solutions of Eq.(4.7) and (4.9), i.e. the boundary between the stable and unstable regions in the a-b parameter plane, we will discuss how the boundary changes when c is varied. We begin our numerical analysis by considering first the case $c < 0$.

Figure 4.1 displays the stability boundaries for $c = -0.01$, we observe that the stability region is bounded by the red and blue line. The blue line is obtained by the condition $z = 0$ and the red curve by $z = i\phi$. It has the similar stability region for first order delay differential equations ($c = 0$) shown in Figure 4.1 (a). No qualitative change in the stability domain is visible. We can also spot from Figures 4.1, 4.2 and 4.3 that decreasing c yields an increase of the stability region, a larger curvature of the red boundary, and an extension of the stability region to the right half of the parameter plane. The green and the orange curve change their shapes but they are located in the unstable region.

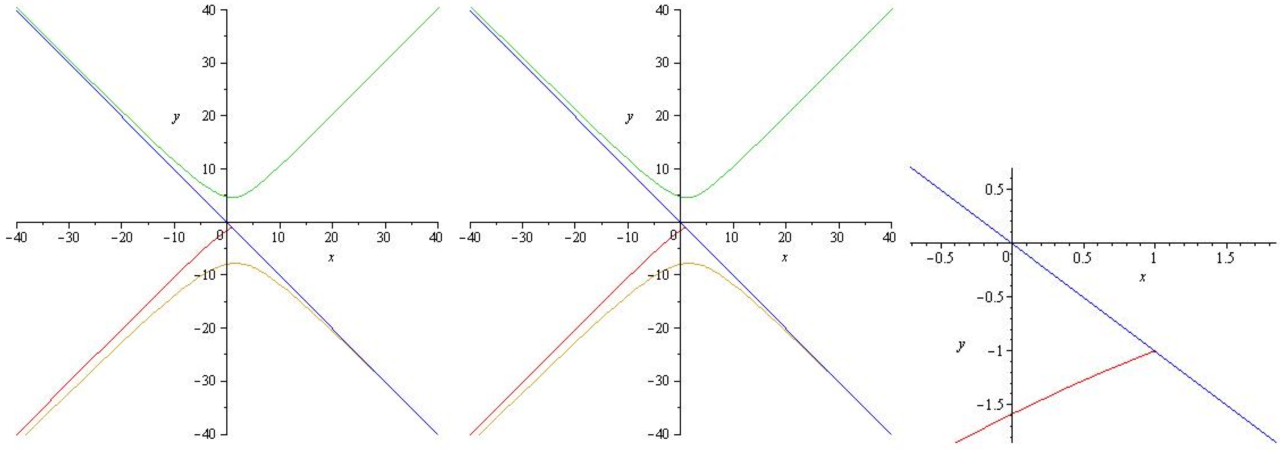

Figure 4.1: Stability domain of the second order delay differential equation, $c = 0$ (left), $c = -0.01$ (right). The inset (the rightmost graph) shows the case $c = -0.01$ on a smaller scale.

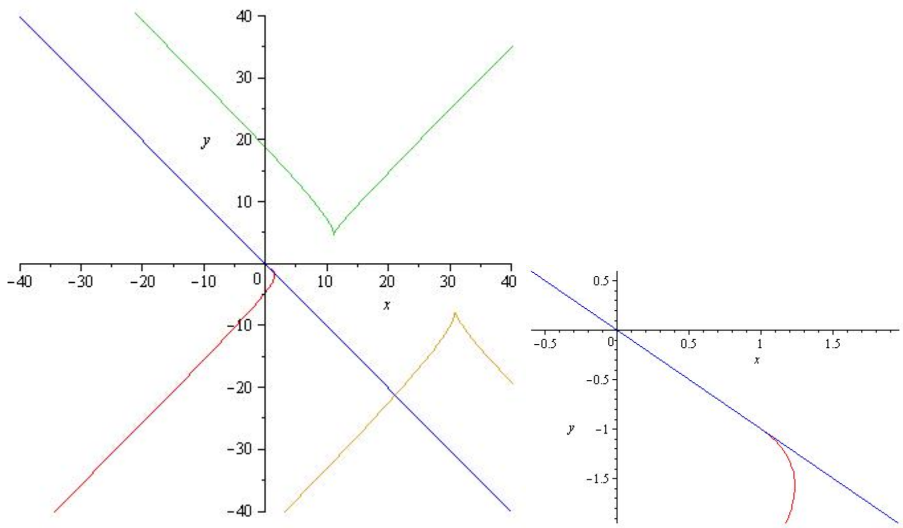

Figure 4.2: The stability domain of the second order delay differential equation is affected for negative values of c, here $c = -0.6$.

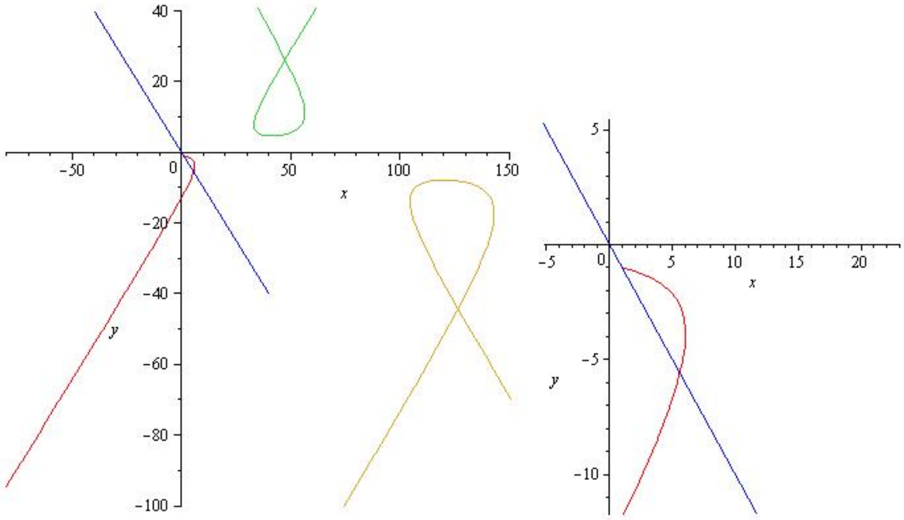

Figure 4.3: As c values get more negative the stability area increases i.e. $c = -1.5$. The inset shows the boundaries close to the origin.

We will further comment on the influence of parameter values and we will see that a qualitative change of the boundary occurs. Varying parameters, the boundary changes continuously as shown in the figures above. In Figure 4.1(b) the stable domain has a triangular shape and both boundaries, which meet close to the origin meet at a finite angle. For Figure 4.2 a change occurs as now both boundaries, those caused by $z = 0$ and a purely imaginary pair, become tangential. Finally in Figure 4.3 both boundaries have changed their orientation. Such qualitative changes are often referred to as bifurcations, here a bifurcation in the shape of the stability boundary. Bifurcation takes place when a qualitative change in the nature of the solution occurs if a parameter passes through a critical point [7]. For instance, in this example, the bifurcation occurred at $c = -0.6$

For the case $c > 0$ we can examine from the diagram below that each curve slightly shifts to the left as the positive value of c gets bigger, as a result the instability region gets larger.

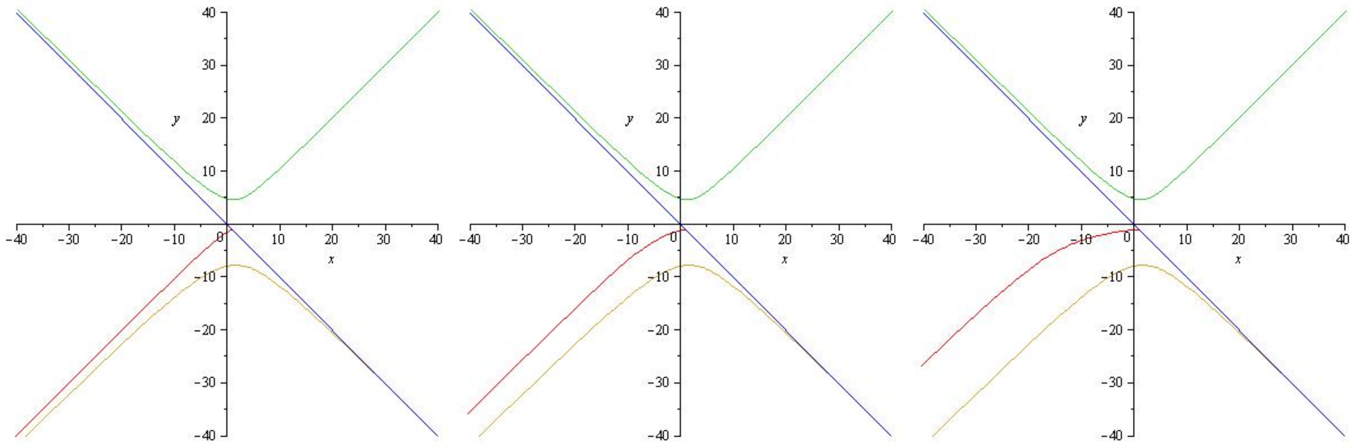

Figure 4.4: As c values gets more positive the stability area decreases i.e. (a) c =0.01 , (b) c=0.5, (c) c=1.5

Chapter 5

Summary and Conclusion

A numerical approach of the stability analysis of linear delay differential equations has been presented. A theorem of complex function theory, more precisely the argument principle, has enabled us to analyse stability properties of delay differential equations. The change in the argument has been demonstrated in the contour plots. The use of such a concept was illustrated by studying an example in chapter 2. The dependence of the stability properties on the parameters of the system, cf. Equation (2.11), has been investigated, varying b we were able to numerically approximate one stability domain $b \in (-\pi/2, 0)$. It would be impossible to numerically obtain all the - potentially infinite - stability intervals ourselves, however this has become possible analytically by the Lambert function which gives us the full stability region. Curiosity has lead us to investigate and extended to higher dimensions for instance a two dimensional system. This was presented in a paper by [6]. Here we were able to verify the stability interval. This was again demonstrated by contour plots. Our investigations lead us to more advance equations such as second order delay differential equations where our previous method no longer applied. We took a new approach of analysing stability boundary by varying a parameter c and monitoring the changes of the boundaries of the stable domain in parameter space. As we have seen the stability analysis of linear differential difference equations result in transcendental characteristic equations which involve powers and exponentials. For further study one could introduce quasi-polynomials.

$$P(z) = \sum_{m=0}^{r} \sum_{n=0}^{s} a_{mn} z^m \exp(nz). \tag{5.1}$$

The purpose of Eq.(5.1) is to determine when all the roots of such a characteristic equation are in the left half-plane. A major result of Pontryagin gives the necessary and sufficient conditions for all the zeros of $P(z)$ to have negative real parts, that is, analytical stability conditions [8].

By Pontryagin's investigation we are now able to study the stability of delay differential equation of any order. This indeed would save a lot of tedious calculations, as we experienced in our second order equation. We came across difficulty in rewriting the equation in terms of multipliers to obtain the change of argument. Fortunately we are able to use Eq. 5.1 to overcome these difficulties. However, one should not assume that this is an easier option as the application of Pontryagin's stability criteria require tedious calculations as well.

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