The Lyapunov spectrum of a simple Dynamical Systems with Time Delay

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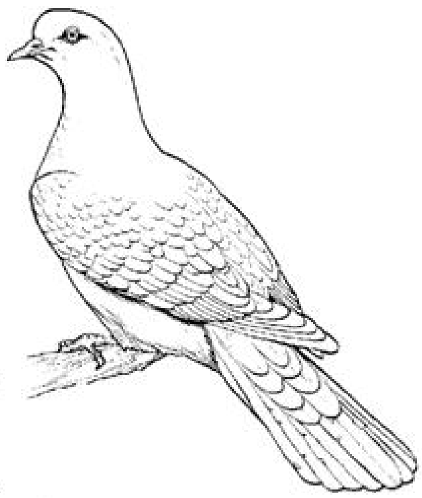
September 10, 2009

Abstract

In this work we shall study one of the most interesting aspects of Dynamical Systems, systems which involves the factor of the time delay. Time delays are often sources of complex behavior in dynamical systems. Yet its complexity needs to be further explored which may exceed our goals in this project. In this brief study, we deal with a simple one dimensional map with a special emphasis on Lyapunov exponents. The idea of Lyapunov exponents is to define characteristic numbers for the dynamical system that allow to classify the behavior of the system in a concise manner while the time delay is taken as adjustable parameter to study its effect on the dynamics of the system. Analytical computations for large time delay τ, have been performed for maps with constant slope, $f(x) = ax|mod1$, while the universal character of such results can be confirmed by numerical simulations for a larger class of systems. And numerical analysis via the Maple diagrams and Lyapunov exponents computation are carried out to show the actual time delay effect. Both the results obtained by the two analysis show that the time delay plays a very important role in inducing chaos.



Acknowledgement

I am deeply indebted to my supervisor Dr. Wolfram Just whose help, stimulating suggestions, encouragement and subtle direction of my efforts helped me all times of the preparation of this project. And I also owe my dear wife whole respect and appreciation for her support and hard efforts that encouraged me to continue my studies.

Contents

Chapter 1

Introduction

Dynamical system is a system that changes over time according to a set of fixed rules that determine how one state of the system moves to another state. A dynamical system could be predictable or unpredictable. Predictable such as we know that the sun will rise tomorrow or adding milk to the tea will not cause an explosion. unpredictable systems are those which are chaotic or random. This project is set to study one branch of the unpredictable dynamical systems which involves the factor of the time delay. It is quite well established that dynamics including time delay yields high dimensional phase spaces. Within the project such a topic is addressed on the basis of simple one dimensional map. For that purpose the time discrete dynamics given by: $x_{n+1} = (1-\epsilon)F(x_n) + \epsilon F(x_{n-\tau})$. Computation of Lyapunov exponents is employed to study the behavior of such system.These numbers should account for exponential convergence or divergence of trajectories that start close to each other. This study is organized as follows. In Chapter 2, an introduction of chaotic dynamical systems is produced in addition to defining the Lyapunov exponent and its significance. In Chapter 3, we introduce the time delayed map by a convex combination and then we compute the Lyapunov exponent to obtain the characteristic equation, $\mu = (1-\epsilon)a + \epsilon a\mu^{-\tau}$, which forms the essence of our analysis. Also we shed light on some special cases where the the impact of the delay is null and weak , i.e. when $\epsilon = 0$ and $\epsilon = 1$ respectively and also we consider the case when the time delay is small ($\tau = 1$). In chapter 4, we produce some figures with the aid of Maple software and we include some numerical analysis which clarify the results from chapter 3. Finally in chapter 5, we consider the case when the delay is large,i.e. $\tau \gg 1$. As it is difficult to obtain solutions for the polynomial of order $\tau + 1$, we recall the asymptotic expansion for the characteristic equation which helps to describe the behavior of the function in a limiting situation. To summarize the essential idea, we expect that the chaosity of the system increases when the delay is large.

Chapter 2

Dynamical Systems

2.1 Unpredictable Dynamical Systems

A dynamical system might be unpredictable for two reasons, either it might be random or it might be chaotic. To identify between these two types of systems, we will just give a breif definition for both.

• **Chaotic system:** Chaotic is a description of a dynamic system that is very sensitive to initial conditions and may evolve in wildly different ways from slightly different initial conditions. In addition to that and because our proposed system in this study is mostly of chaotic behavior, we find it very useful to offer the reader a list of the most common properties of the chaotic dynamical systems. So in general chaotic systems are characterized by:

- There are orbits of all periods.
- All periodic orbits are unstable.
- Periodic points are dense.
- There is sensitivity to initial conditions.
- There is a positive Lyapunov exponent.

• **Random system:** a random dynamical system is a dynamical system with an element of ”randomness”.

In this project we will focus on a low dimensional dynamical system of chaotic behavior and we will try to study the effect of the time delay on that system. It is good to mention that we could find a very simple system even those of one variable may behave unpredictably such as the stock market system.

One of the simple examples of low dimensional dynamical systems is the *Bernoulli shift map* which is the model of our project.

2.2 Bernoulli shift map

Consider the following one-dimensional map:

$$x_{n+1} = ax_n \bmod 1 \tag{2.1}$$

this is the general form of the *Bernoulli shift map*. To check how this map look like, let's consider an example of this map at $a = 2$, i.e.

$$F(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x < 1 \end{cases} \tag{2.2}$$

The graph of $F : [0, 1) \to [0, 1)$ is shown in Figure 2.1. We can see from Figure 2.1 that the Bernoulli shift map consists of two linear segments.

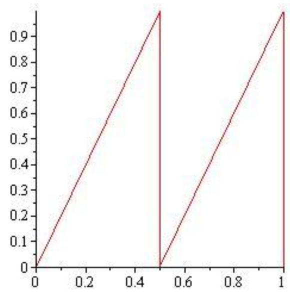


Figure 2.1: The Bernoulli shift map

Also we find it quite useful to list some properties of the *Bernoulli shift map* without proofs just to give a taste of the general conduct of this map.

- The Bernoulli shift map has periodic orbits of all periods.
- There is no **stable** periodic point for the Bernoulli shift map.

- The Bernoulli shift map exhibits sensitivity to initial conditions.

Now we can step in towards our goal for investigating the behavior of the *Bernoulli shift map* with the existence of time delay factor. So first tool we might think of is the use of *Lyapunov exponent.*

2.3 Lyapunov Exponent

Lyapunov exponents are numbers which describe exponential convergence or divergence of trajectories that start close to each other.

2.3.1 Definition for one dimensional maps

Let f be a piecewise smooth map and let $(x_0, x_1, x_2, ...)$ denote the orbit with initial condition x_0. If the limit:

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f^{'}(x_k)| \tag{2.3}$$

exists, the value Λ is called the Lyapunov exponent of the orbit.

2.3.2 Example

For the Bernoulli shift, eq.(2.2) we have $F^{'}(x) = 2$, $\forall x \in [0, 1)$, $x \neq \frac{1}{2}$, hence $\Lambda(x) = \ln 2$ at these points. So here the Lyapunov exponent is the same for almost all x, because the map is uniformly expanding.

2.3.3 Significance of Lyapunov Exponents

The Lyapunov exponent Λ, is useful for distinguishing among the various types of orbits. It works for discrete as well as continuous systems. The main significance of Lyapunov exponent lies on the ability to identify the chaosity of most of the dynamical systems,i.e, if:

- $\Lambda < 0$: The orbit attracts to a stable fixed point or stable periodic orbit. Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems (the damped harmonic oscillator for instance). Such systems exhibit asymptotic stability; the more negative the exponent, the greater the stability.

- $\Lambda = 0$: The orbit may be a neutral fixed point (or an eventually fixed point). A Lyapunov exponent of zero indicates that the system is in some sort of steady state mode. A physical system with this exponent is conservative. Such systems exhibit Lyapunov stability. There are other cases which are more complicated (so called strange nonchaotic attractors) which exhibit complicated motion without sensitive dependence on initial condition.

- $\Lambda > 0$: The orbit is unstable and chaotic. Nearby points, no matter how close, will diverge exponentially. The system shows sensitive dependence on initial conditions.

Chapter 3

Systems with time delay

Consider the general form of a delayed map with a single delay:

$$X_{n+1} = F(X_n, X_{n-\tau}) \tag{3.1}$$

where τ is an integer delay value. Such a dynamical system does not constitute any longer a one-dimensional map as the right hand side of eq.(3.1) depends on X_n and $X_{n-\tau}$. In fact, if one uses some kind of vector notation to write down the equations of motion, then it is obvious that the phase space is given by vectors of the type $(X_n, X_{n-1}, \ldots, X_{n-\tau})$. As a consequence the phase space has dimension $\tau + 1$ and the motion is now characterized by a collection of $\tau + 1$ Lyapunov exponents. The model (3.1) is still far to general to study the influence of time delay. Thus we will specialize the setup further.

3.1 Time delay introduced by a convex combination

In our model, the Bernoulli shift map with the time delay factor can be introduced by a convex combination of an instantaneous and of a delayed term:

$$x_{n+1} = (1-\epsilon)F(x_n) + \epsilon F(x_{n-\tau}) \tag{3.2}$$

where ϵ governs the strength of the delay term in the system and we choose $\epsilon \in [0, 1]$. The special structure of equation (3.2) ensures that the dynamics is well defined, i.e. x_n stays in the interval $[0, 1]$.

We can prove this as follows: assume that x_ℓ is contained in $[0, 1]$ for all $\ell \leq n$. Then properties of the simple shift map ensure that

$$0 \leq f(x_n) \leq 1, \quad 0 \leq f(x_{n-\tau}) \leq 1 \tag{3.3}$$

Furthermore, by definition, we know that $0 \leq \epsilon \leq 1$ holds. In order to show that x_{n+1} is contained in $[0, 1]$ we proceed in two steps. The properties of ϵ and eq.(3.3) guarantee that

$$(1-\epsilon) \geq 0, \quad F(x_n) \geq 0 \quad \Rightarrow \quad (1-\epsilon)F(x_n) \geq 0 \tag{3.4}$$

and

$$\epsilon \geq 0, \quad F(x_{n-\tau}) \geq 0 \quad \Rightarrow \quad \epsilon F(x_{n-\tau}) \geq 0 \tag{3.5}$$

These results, eqs.(3.4) and (3.5), together with eq.(3.2) ensure that $x_{n+1} \geq 0$.

A similar pattern may be used to show $x_{n+1} \leq 1$. We know

$$0 \leq F(x_n) \leq 1 \quad \Rightarrow \quad (1-\epsilon)F(x_n) \leq (1-\epsilon) \tag{3.6}$$

and

$$0 \leq F(x_{n-\tau}) \leq 1 \quad \Rightarrow \quad \epsilon F(x_{n-\tau}) \leq \epsilon\,. \tag{3.7}$$

Thus eq.(3.2) yields $x_{n+1} \leq 1 - \epsilon + \epsilon = 1$. Therefore our model (3.2) is well defined.

3.2 Lyapunov exponents and the variational equation

Now to study the chaotic behavior of our model which involves the effect of time delay, we need to compute the Lyapunov exponent in order to quantify the chaosity of the system. For that purpose we consider two orbits which differ by a small amount, say x_n and $x_n + \delta x_n$. The equation for the small increment δx_n is easily obtained from eq.(3.2) by linearisation.

$$\delta x_{n+1} = (1-\epsilon)F^{'}(x_n)\delta x_n + \epsilon F'(x_{n-\tau})\delta x_{n-\tau} \tag{3.8}$$

Using $F'(x) = a$, eq.(3.8) yields:

$$\delta x_{n+1} = (1-\epsilon)a\delta x_n + \epsilon a \delta x_{n-\tau}\,. \tag{3.9}$$

As mentioned above the Lyapunov exponents govern the rate of exponential increase/decrase of the small increment. Therefor we are going to solve eq.(3.9) using $x_n = \mu$. The Lyapunov exponents are then obtained via the relation $\exp(n\Lambda) = |\delta x_n| = \exp(n\text{Re}\mu)$, i.e. as the real parts of the multiplier μ

$$\Lambda = \text{Re}\mu \tag{3.10}$$

Using $\delta x_n = \mu^n$, $\delta x_{n+1} = \mu^{n+1}$ and $\delta x_{n-\tau} = \mu^{n-\tau}$ eq.(3.9) yields

$$\mu^{n+1} = (1-\epsilon)a\mu^n + \epsilon a\mu^{n-\tau}\,. \tag{3.11}$$

If we now divide by μ^n we get the characteristic equation

$$\mu = (1-\epsilon)a + \epsilon a\mu^{-\tau} \tag{3.12}$$

This characteristic equation of the system with effect of time delay is a very important result as it links the Lyapunov exponents with the delay τ and the strength of the delay ϵ. It enables us to investigate how the chaotic behavior of the system is affected by the delay when it is small, medium,large or very large delay.

In the next section we will shed a light on special cases where we can inspect the system manually.

3.3 Special Cases

3.3.1 Case ($\epsilon = 0$)

When ($\epsilon = 0$) then equation(3.12) becomes $\mu = a$ which means that the system turns out to be one dimensional map, i.e, there is one positive Lyapunov exponent and consequently the system shows chaotic behavior.

3.3.2 Case ($\epsilon = 1$)

When ($\epsilon = 1$) then equation (3.12) becomes: $\mu = a\mu^{-\tau} \Rightarrow \mu^{\tau+1} = a$, so equation(3.12) reduces to $\tau+1$ distinct copies of the map F acting on the time scale $\tau+1$. In particular, there are $\tau+1$ different multipliers.

$$\mu_\ell = a^{1/(\tau+1)}\exp(i2\pi\ell/(\tau+1)), \quad \ell = 0, 1, \ldots, \tau\,. \tag{3.13}$$

which is difficult to compute all solutions especially when the delay is large,i.e. $\tau >> 1$. Therefore it is useful in this case to use good approximation method by employing an *asymptotic expansion* to the eq.(3.13) which will describe the behavior of the function in a limiting situation. We will discuss this further in the chapter 5.

3.3.3 Case ($\tau = 1$)

When the delay is small like $\tau = 1$, we can check the chaosity of the system by computing the Lyapunov exponent. In this case equation (3.12) becomes:

$$\mu = (1-\epsilon)a + \epsilon a \mu^{-1} \tag{3.14}$$

$\Leftrightarrow$

$$\mu^2 - (1-\epsilon)a\mu - a\epsilon = 0 \tag{3.15}$$

The roots of this of this quadratic equation are given by:

$$\mu_{\pm} = \frac{(1-\epsilon)a \pm \sqrt{(1-\epsilon)^2 a^2 + 4a\epsilon}}{2} \tag{3.16}$$

which is an important result, which links the Lyapunov exponent with the values of a and ϵ. Also we can work out the critical values of ϵ in terms of a which will indicate the number of positive Lyapunov exponents. First let us check whether $\mu_+ > 1$.

We have:

$$(1-\epsilon)a + \sqrt{(1-\epsilon)^2 a^2 + 4a\epsilon} > 2 \tag{3.17}$$

$\Leftrightarrow$

$$\sqrt{(1-\epsilon)^2 a^2 + 4a\epsilon} > 2 - (1-\epsilon)a \tag{3.18}$$

$\Leftrightarrow$

$$(1-\epsilon)^2 a^2 + 4a\epsilon > (2 - (1-\epsilon)a)^2 \tag{3.19}$$

$\Leftrightarrow$

$$(1-\epsilon)^2 a^2 + 4a\epsilon > 4 - 4(1-\epsilon)a + (1-\epsilon)^2 a^2 \tag{3.20}$$

$\Leftrightarrow$

$$0 > 4 - 4a \tag{3.21}$$

which is true $\forall a > 1$.
Similarly for μ_-. Here we will use this result to obtain the formula of the critical value

of ϵ in terms a. We start with:

$$\mu_{=}\frac{(1-\epsilon)a-\sqrt{(1-\epsilon)^2a^2+4a\epsilon}}{2}=-1 \tag{3.22}$$

$\Leftrightarrow$

$$(1-\epsilon)a-\sqrt{(1-\epsilon)^2a^2+4a\epsilon}2=-2 \tag{3.23}$$

$\Leftrightarrow$

$$(1-\epsilon)a+2=\sqrt{(1-\epsilon)^2a^2+4a\epsilon} \tag{3.24}$$

$\Leftrightarrow$

$$(1-\epsilon)^2a^2+4+4(1-\epsilon)a=(1-\epsilon)^2a^2+4a\epsilon \tag{3.25}$$

$\Leftrightarrow$

$$1+a-\epsilon a=\epsilon a \tag{3.26}$$

$\Leftrightarrow$

$$\epsilon=\frac{1}{2}(1+\frac{1}{a}) \tag{3.27}$$

which is the critical value of ϵ where the delayed map has only one positive Lyapunov exponent. The importance of this value of ϵ is that it indicated the number of the positive Lyapunov exponents, i.e, at this critical value or less the system has only one positive Lyapunov exponent. While for values greater than that of eq.(3.27), the system will have **two positive** Lyapunov exponents and the system becomes more chaotic. To illustrate this result, lets consider the following example: we use $a = 1.5$. we can work out the critical value of ϵ for this particular value of a. $\epsilon = \frac{1}{2}(1+\frac{1}{1.5}) = \frac{1}{2}(1+\frac{2}{3}) = \frac{5}{6} \approx 0.83$

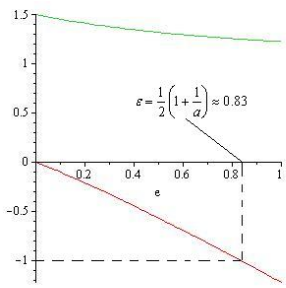


Figure 3.1: Multiplier μ, when $a = 1.5$ cf.eq.(3.16)

At this critical value of ϵ or below, the system has one positive Lyapunov exponent and for greater than that, the system will have two positive Lyapunov exponent. We can see the critical value of ϵ in the figure(3.1) where $\mu_- = -1$.

Chapter 4

Numerical Solutions

Unlike the case $\tau = 1$ it is not completely straightforward to discuss the solutions of eq.(3.12) in dependence on a and ϵ. Thus, to get some qualitative overview we will first resort to a numerical analysis and show graphs with the dependence of the Lyapunov exponents $\Lambda = \text{Re}\mu$ on ϵ for different values of a and τ.

4.1 Example of system with small delay

To begin with we first concentrate on the simple case $\tau = 1$ where the explicit expressions (3.16) are available.

Figure 4.1 shows the multiplier μ graph when $a = 2$. We clearly see that μ_+ gives rise to a branch with positive Lyapanov exponent. while μ_- branch is decreasing until ϵ reaches its critical value at $\epsilon = \frac{1}{2}(1 + \frac{1}{a})$. In this particular example the critical vale of $\epsilon = 0.75$ for $a = 2$. Same discussion as in figure (3.1) follows for this example.

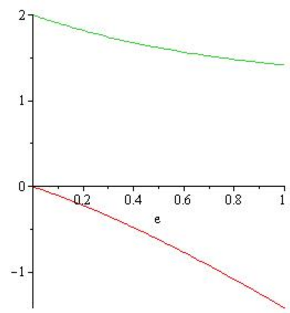
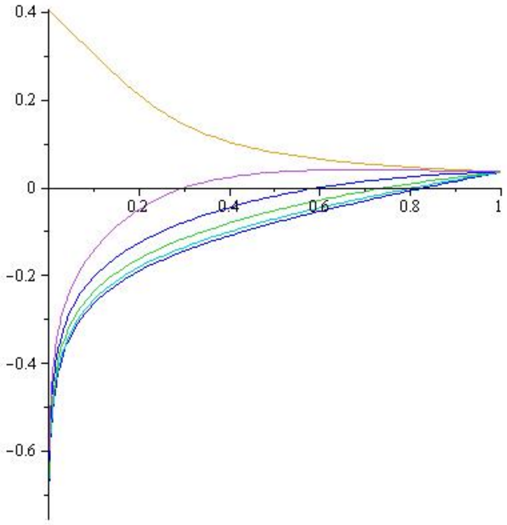


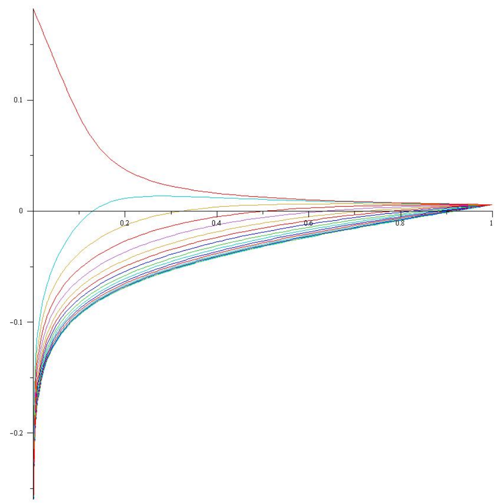
Figure 4.1: Multiplier μ, for $\tau = 1$ and $a = 2$ cf.eq.(3.16)

4.2 Examples of system with medium, large and very large delay

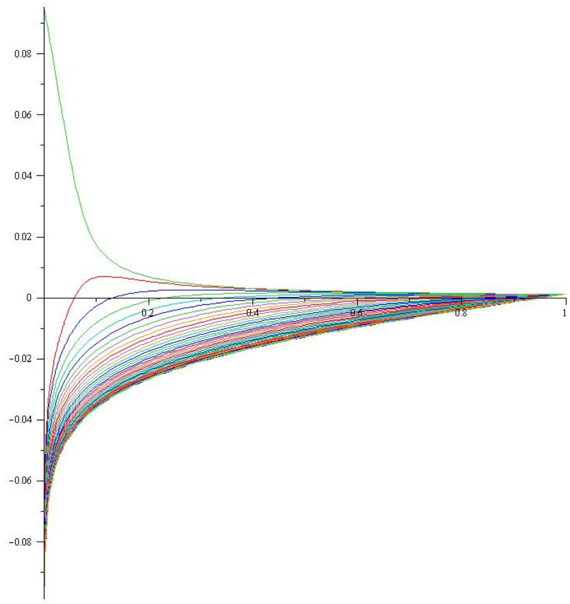
In this example, we consider the case when the delay is medium, large and very large i.e. $(\tau = 10, 30, 80)$



(a) $(\tau = 10)$



(b) $(\tau = 30)$



(c) $(\tau = 80)$

Figure 4.2: Lyapanuv exponent in Medium, Large and very large delay system

From the above three figures, one can notices that there is a strong correlation between the number of Lyapunov exponents and the delay factor,τ, i.e. when the delay is bigger, the number of Lyapunov exponents increases. Also it is clear to see in all three figures that as the strength of the delay ϵ gets closer to one, the more positive Lyapanuv exponents there are. In another words after a particular value of ϵ, all the Lyapanov exponents turns positive. From analytical angle, we can notice that there is a region which can be called the $\epsilon-$region or the threshold of ϵ where the number of positive Lyapanuv exponents is gradually increasing as the value of ϵ gets closer to one, which is what we have called the critical value of ϵ in the previous chapters. And there the system becomes extremely chaotic.

Chapter 5

Limit of the large time delay

5.1 Asymptotic Expansion of Lyapunov Exponent

It is quite difficult to obtain the solutions of eq.(3.12) for general delay τ in analytical terms, unlike the case $\tau = 1$. However, for very large delay one may employ a kind of asymptotic expansion to explain some of the features discovered in the previous chapter by numerical means. For that purpose we will discuss three different cases.

First let us study solutions which are outside the complex unit circle. To be slightly more precise, let us focus on solutions of eq.(3.12) which obey the inequality $|\mu| \geq 1 + \delta$ where δ is some small number, independent of τ. Then, for τ sufficiently large we have $\mu^{-\tau} = \frac{1}{\mu^{\tau}} \ll 1$ and the corresponding term in eq.(3.12) can be discarded. Thus by considering the large delay of the Bernoulli shift map, i.e when $\tau \gg 1$ then the characteristic equation (3.12) can be reduced to be as follows:

$$\mu = (1 - \epsilon)a \tag{5.1}$$

From equation (5.1) and the constraint $\mu > 1$, we can find the threshold of the ϵ-region where the solution (5.1) is valid $(1 - \epsilon)a = 1$, i.e.

$$\epsilon = 1 - \frac{1}{a} \tag{5.2}$$

This value of ϵ indicates that there is one positive real Lyapunov exponent.

Let us now consider the second possibility, i.e., exponents which are strictly inside the unit circle. For that purpose assume $|\mu| \leq 1 - \delta$. Then, for sufficiently large delay the contribution $\mu^{-\tau}$ to eq.(3.12) becomes large in modulus, which results in a contradiction (for nonzero values of ϵ). Thus the characteristic equation does not admit solutions which are strictly inside the unit circle.

Eq.(3.12) constitutes a polynomial of order $\tau + 1$. So far we have found one real solution, eq.(5.1), in a parameter region given by eq.(5.2). All the other solutions must be contained in an annulus of the complex unit circle. The special case discussed previously, eq.(3.13), suggests that we may try to write these solutions for large delay in the following asymptotic form:

$$\mu = (1 + \frac{\rho}{\tau}) \exp(i\phi\tau) \,. \tag{5.3}$$

In particular, it means that for sufficiently large delay we have

$$\mu^{-\tau} = (1 + \frac{\rho}{\tau})^{-\tau} \exp(-i\phi\tau) \simeq \exp(-\rho) \exp(-i\phi\tau) \tag{5.4}$$

Therefore equation (3.12) becomes

$$(1 + \frac{\rho}{\tau}) exp(i\phi\tau) = (1 - \epsilon)a + \epsilon a exp(-\rho) exp(-i\phi\tau) \tag{5.5}$$

$\Rightarrow$

$$\frac{exp(i\phi) - (1 - \epsilon)a}{\epsilon a} exp(\rho) = exp(i\phi\tau) \tag{5.6}$$

Now taking the absolute value of both sides we get:

$$|\frac{exp(i\phi) - (1 - \epsilon)a}{\epsilon a}| exp(\rho) = 1 \tag{5.7}$$

which can be solve for ρ:

$$exp(\rho) = \frac{|\epsilon a|}{|exp(i\phi) - (1 - \epsilon)a|} = \frac{|\epsilon a|}{\sqrt{(\cos\phi - (1 - \epsilon)a)^2 + \sin^2\phi}} \tag{5.8}$$

$\Rightarrow$

$$\rho = \ln \frac{|\epsilon a|}{\sqrt{\cos^2\phi - 2(1 - \epsilon)a\cos\phi + (1 - \epsilon)^2 a^2 + \sin^2\phi}} \tag{5.9}$$

Therefore,

$$\rho = \ln \frac{|\epsilon a|}{\sqrt{1 + (1 - \epsilon)^2 a^2 - 2(1 - \epsilon)a\cos\phi}} \tag{5.10}$$

and finally we obtain:

$$\rho(\phi) = \ln|\epsilon a| - \frac{1}{2}\ln(1 + (1 - \epsilon)^2 a^2 - 2(1 - \epsilon)a\cos\phi) \tag{5.11}$$

although eq.(5.11) does not yield an explicit formula for the solutions of eq.(3.12), it gives us valuable information. In fact, eq-(5.11) tells us how the complex phase and the absolute values of the roots are related to each other. For instance, that means $\rho(\phi) < 0$ (for all ϕ) implies negative Lyapunov exponents only, while $\rho(\phi) > 0$ means entirely positive Lyapunov exponents.

Therefore, we can use this function to obtain the threshold for the small values of ϵ and consequently locate the thresholds of the $\epsilon-$region. As mentioned it becomes much easier to estimate the values of ϵ where the delayed map starts to show increase of its chaosity, i.e, when the number of the *positive Lyapunov exponents* starts to increase until it reaches the maximum. In this section, we will discuss these situations analytically. to enable us find the thresholds of $\epsilon-$region, we need to know the conditions of $\rho(\phi)$ where is it at *maximum* and where is it at *minimum*. So we differentiate eq.(5.11), we get:

$$\frac{d\rho}{d\phi} = \frac{(\epsilon - 1)a \sin\phi}{1 + (1-\epsilon)^2 a^2 - 2(1-\epsilon)a\cos\phi} \tag{5.12}$$

we also need to compute the *second derivative* of $\rho(\phi)$. So differentiating eq.(5.12), we get:

$$\frac{d^2\rho}{d\phi^2} = \frac{(\epsilon - 1)a\cos\phi(1 + (1-\epsilon)^2 a^2 - 2(1-\epsilon)a\cos\phi) - 2(1-\epsilon)^2 a^2 \sin^2\phi}{(1 + (1-\epsilon)^2 a^2 - 2(1-\epsilon)a\cos\phi)^2} \tag{5.13}$$

Now $\frac{d\rho}{d\phi} = 0$ when $\sin\phi = 0$, i.e, when $\phi = 0, \pi$. These two values of ϕ, will specify the *maximum* and the *minimum* of $\rho(\phi)$. In addition to that the second derivative of $\rho(\phi)$ also relies on the sign of a , i.e. whether $a < 0$ or $a > 0$. All of these conditions will provide us with four cases to consider on our map. we can summarize these cases as follows:

- case 1: $a > 0$ and $\phi = 0 \Rightarrow \rho_{max}$.
- case 2: $a > 0$ and $\phi = \pi \Rightarrow \rho_{min}$.
- case 3: $a < 0$ and $\phi = 0 \Rightarrow \rho_{min}$.
- case 4: $a < 0$ and $\phi = \pi \Rightarrow \rho_{max}$.

5.2 Comparison

In this section and for the sake of abridgement, we will just consider one or two cases from the above cases in order to compute the critical values of ϵ which we talked about in the previous sections.

5.2.1 Case 1 for ρ_{max}.

Considering the case where $a > 0$ and $\phi = 0$ we get:

$$\rho_{max} = \ln|\epsilon a| - \frac{1}{2}\ln(1 + (1-\epsilon)^2 a^2 - 2(1-\epsilon)a) \tag{5.14}$$

to obtain the value of ϵ at the beginning threshold of the $\epsilon-$region, we solve for ϵ at $\rho_{max} = 0$. We have:

$$\ln|\epsilon a| = \frac{1}{2}\ln(1 + (1-\epsilon)^2 a^2 - 2(1-\epsilon)a) \tag{5.15}$$

$\Leftrightarrow$

$$\epsilon^2 a^2 = 1 + (1-\epsilon)^2 a^2 - 2(1-\epsilon)a \tag{5.16}$$

$\Leftrightarrow$

$$\epsilon^2 a^2 = 1 + a^2 - 2a^2\epsilon + \epsilon^2 a^2 - 2a + 2a\epsilon \tag{5.17}$$

$\Leftrightarrow$

$$\epsilon(2a - 2a^2) + a^2 - 2a + 1 = 0 \tag{5.18}$$

$\Leftrightarrow$

$$\epsilon = \frac{-a^2 + 2a - 1}{2a - 2a^2} = \frac{a^2 - 2a + 1}{2a^{2-2a}} = \frac{(a-1)^2}{2a(a-1)} \tag{5.19}$$

$\Leftrightarrow$

$$\epsilon = \frac{a-1}{2a} \tag{5.20}$$

which is the critical value of ϵ at the lower threshold. See figure (5.1)

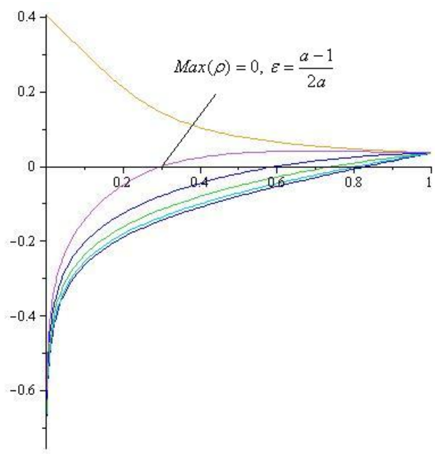


Figure 5.1: ϵ- value at the lower threshold region

5.2.2 case 2 for ρ_{min}.

Now if we consider the case where: $a > 0$ and $\phi = \pi$ we get:

$$\rho_{min} = \ln|\epsilon a| - \frac{1}{2}\ln(1 + (1-\epsilon)^2 a^2 + 2(1-\epsilon)a) \tag{5.21}$$

By following the same pattern as in the previous example and solving for ϵ we will obtain the following:

$$\epsilon = \frac{a+1}{2a} \tag{5.22}$$

which represents the ϵ-value at the upper threshold. See figure (5.2)

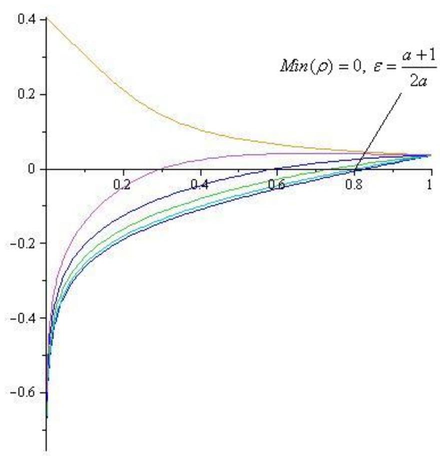


Figure 5.2: ϵ- value at the upper threshold region

5.3 Region thresholds of ϵ.

In this section we go through several numerical and graphical examples on the cases which we have covered in the previous section. And that could provide a better presentation for our results. Also we will explain how the critical values of ϵ, show the threshold region where the number of Lyapunov exponent is increasing gradually within this region until it reaches its maximum at the upper threshold and there where the system becomes more chaotic. So we will list some figures which shows the $\rho(\phi)$ at different numerical values of a and ϵ.

5.3.1 Example

In this example we use the following numerical values: $a = -1.5$ and $\epsilon = 0.1$. Figure(5.3) shows the $\rho(\phi)$ graph which does not cut the axis as the value of ϵ is still less than the critical value of the lower threshold. This mean we still have only one positive Lyapunov exponent.

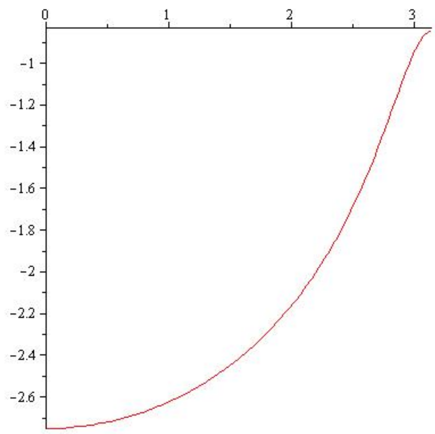


Figure 5.3: $\rho(\phi)$ for $\epsilon < \frac{a-1}{2a}$.

5.3.2 Example

In this example we use the following numerical values: $a = -1.5$ and $\epsilon = 0.2$. Figure(5.4) shows the $\rho(\phi)$ graph which does cuts the axis as the value of ϵ is greater than the critical value of the lower threshold. So the number of the positive Lyapunov exponent has increased.

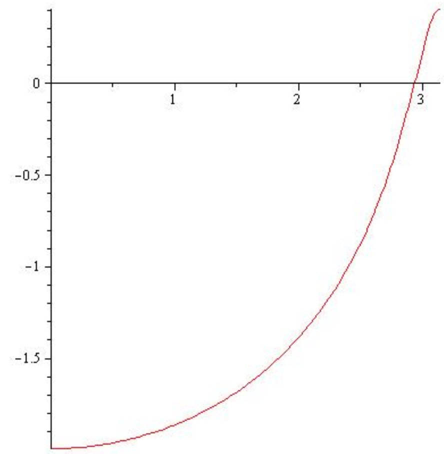


Figure 5.4: $\rho(\phi)$ for $\epsilon > \frac{a-1}{2a}$.

5.3.3 Example

Finally in this example, we use the following numerical values: $a = -1.5$ and $\epsilon = 0.9$ which exceeds the critical value of ϵ at the upper threshold, i.e, $\epsilon > \frac{a+1}{2a}$. Figure(5.5)shows the $\rho(\phi)$ graph thats shows that all the Lyapunov exponents are positive as the value of ϵ is greater than the critical value of the upper threshold.

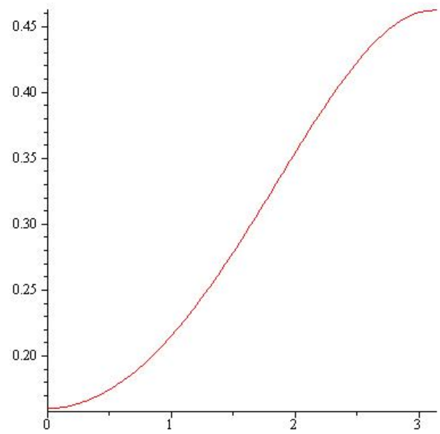


Figure 5.5: $\rho(\phi)$ for $\epsilon > \frac{a+1}{2a}$.

Chapter 6

Conclusion

The main purpose of this study is concentrated on the effects of time delay on the dynamical behaviors of the system.The formulas for computing the critical values of time delay's strength,ϵ are given for different combinations of system parameters and time delays. Also the chaotic behavior of this system is investigated via the computation of the Lyapunov exponents in different phase spaces. In our particular model the effect of time delay changes the behavior of the entire system. In chapters 3 and 4, we have been able to show that there is strong correlation between the number of Lyapunov exponents and the time delay τ. In chapter 5, we discovered that the number of positive Lyapunov exponents depends on the impact of the delay term ϵ.For instance for low impact there is just one positive exponent as for the original map while as we have seen in figures (5.1) and (5.2) for large impact,i.e, when ϵ gets closer to one, all the exponents are positive. Also in chapter 5 we have worked out the critical values of ϵ,i.e. $\epsilon = \frac{1}{2}(1 + \frac{1}{a})$ which forms the threshold between the less and extreme chaotic regions of the system. The results obtained in this study suggest that the time delays play a very important role in the analysis of the dynamics behaviors of system. On the one hand, correct choices of the time delays can change the stability of system. On the other hand, the time delays can induce chaos of system.

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