How the symbolic dynamics depends on the Markov partition

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Introduction

The mutual relationship among Markov partitions is investigated for one-dimensional piecewise monotonic map. It is shown that if a Markov partition is regarded as a map-refinement of the other Markov partition, one can uniquely translate a set of symbolic sequences by one Markov partition to those by the other or vice versa. However, the set of symbolic sequences constructed using Markov partitions is not necessarily translated with each other if there exists no map-refinement relation among them. By using a tent map it will be demonstrated how the resultant symbolic sequences depend on the choice of Markov partitions [4].

The question of how Markov partition can symbolize a given dynamical system without losing any information of complexity in dynamics is one of the most intriguing subjects in analysing information processing in dynamical systems [4].

Among several symbolization schemes, Markov partition provides one of the most natural means to symbolize the dynamical system. By constructing the Markov partition, one can symbolize the original dynamical system and construct its shift space, that is, a set of all possible symbolic sequences constructed from a given Markov partition [4]. The shift space enables us to extract several important properties of the dynamical system such as topological entropy [1].

Systems that admit only finite types of Markov partition must have zero topological entropy. Since most dynamical systems are chaotic, they can have infinitely many Markov partitions. However, the mutual relationship among different Markov partitions has not been well-revealed [4].

For one-dimensional piecewise monotonic map, tent map, the properties of mutual relationship among Markov partitions will be investigated.

It will be shown that if a Markov partition has a certain relationship we call "map refinement of the other Markov partition," the shift spaces corresponding to these two Markov partitions are topologically the same. If this relationship does not hold, the Markov partitions are not necessarily the case [4].

By using a tent map as an illustrative, typical example of one-dimensional piecewise linear map, it will be demonstrated how the choice of Markov partitions affects the resultant shift space

1. Markov partition in the case of one-dimensional piecewise monotonic map

The decimal expansion real numbers, familiar to us all, has a generalization to representation of dynamical system orbits by symbolic sequences. The natural way to associate a symbolic sequence with an orbit is to track its history through a partition. But in order to get a useful symbolism, one needs to construct a partition with special properties. In this work we develop a general theory of representing dynamical systems by symbolic systems by means of so-called Markov partition [9]. To fully understand the process we need to know what the Markov partition is. This chapter will familiarise us with definition of Markov partition and definition of transition matrix A which properties are related with partition, we will also find out how to transfer properties of transition matrix A onto the outsplitting graph.

1.1 Markov partition

Definition 1.0:

Partition P is a collection of intervals $\{I_{0,}I_{1,\dots,}I_{N-1,}\}$ only and only if:

a) $I = \bigcup_{k=0}^{N-1} I_k$ or

b) int(I_k) ∩ int(I_l)= Ø if k ≠l (int=interior) [1]

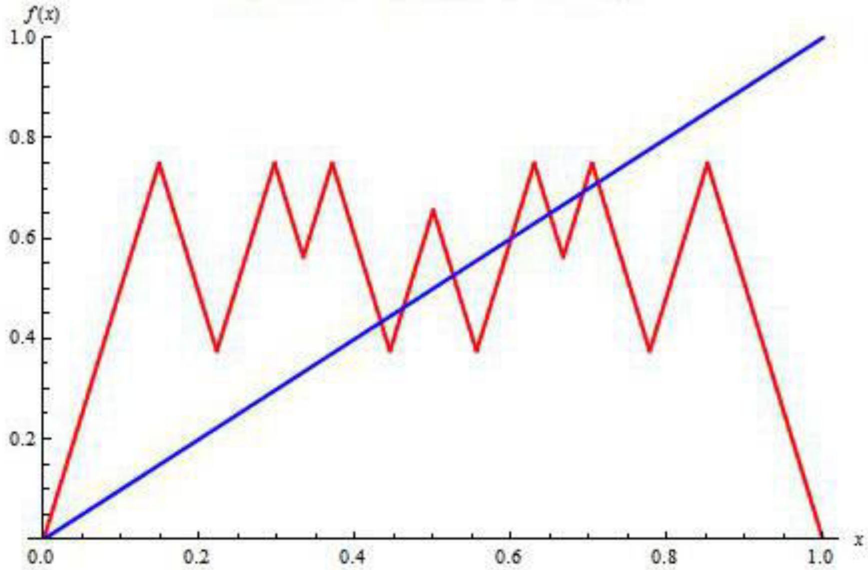
For the interval [0, 1] {[0, 1/3], [1/3, 1]} is a partition and {[0, 2/3], [1/3, 1]} is not.

Definition 1.1:

Let *f*: [0,1]-[0,1] be a piecewise monotonic map, this means that there exists a partition P of interval [0,1] into finitely many pairwise disjoint open intervals, such that for every $I_i \in I$ the map f is continuous [2] and strictly monotonic.

Example 1.0:

The tent map f is finite monotonic map, that is, there is a finite number of intervals where f is decreasing or increasing [10]:



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Knowing meaning of partition we can introduce definition of Markov partition for the interval *I.*

Definition 1.2:

A map $I \to I$ is called Markov map if there is a partition of the closed interval *I:* $\{I_0, I_1, \ldots, I_{N-1}\}$ a so called *Markov partition* [1] such that for all k, k=0,…,N-1 either

a) $f(\mathrm{int}(I_k)) \cap \mathrm{int}(I_l) = \emptyset$
b) Or $\mathrm{int}(I_l) \subseteq (\mathrm{int}(I_k))$

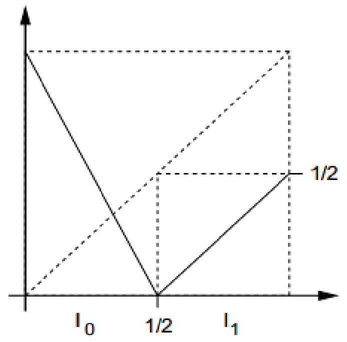
For any piecewise monotonic map the requirement of the partition being Markov reduces to the following: first, the subintervals I_i do not overlap with each other, at most they may only have common end points; second, a subinterval I_i goes into an union of some subintervals under the map f .[3]

Let's consider the simplest example of Markov partition for the interval [0,1]:

Example 1.2:

Let f: [0, 1] →[0, 1]

$$F(x) = \begin{cases} 2-2x \; for \; x \in [0, \frac{1}{2}] \\ x - \frac{1}{2} \; for \; x \; \in (\frac{1}{2}, 1] \end{cases}$$



I_0 =[0, ½] int(I_0)=(0, ½)

I_1 = [½, 1] int(I_1)=(½, 1)

P= {I_0 , I_1} is a partition.

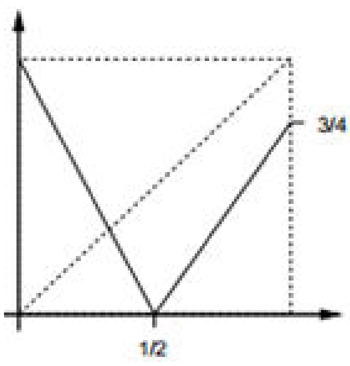
f(int(I_0))=(0,1) so f(int(I_0))⊃ int(I_0) and f(int(I_0))⊃ int(I_1)

f(int(I_1))=(0,1) so f(int(I_1))⊃ int(I_0) and f(int(I_1))∩int(I_0)=∅

Another example shows partition which is not a Markov partition:

Example 1.3:

Let f: [0, 1] →[0, 1]



I_0 =[0, ½]

I_1 =[½,1]
is not a Markov partition since f(int(I_1))∩int(I_1)=(1/2, 3/4)≠∅ but int(I_1) is not contained in f(int(I_1))[1].

Is there another partition which is Markov partition?

Markov maps map boundary points of the partition on boundary points[1].

1.2 Topological transition matrix *A*

For every Markov partition we can construct topological transition matrix *A* which dimension corresponds to the partition of the interval *I*. The topological transition matrix *A* is useful to characterize topology of the shift space. Thus,

Definition 1.3:

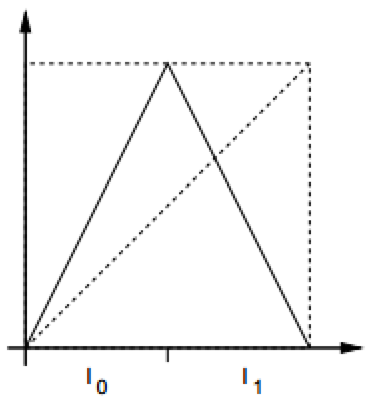
The N x N matrix *A* is defined by:

$$A_{kl} = \begin{cases} 1 \; if \; f(int(I_k)) \supseteq int(I_l) \\ 0 \;\; iff(int(I_k) \cap \; int(I_l) = \emptyset \end{cases}$$

We say that $A_{kl} = 1$ if the transition $I_k \to I_l$ is permitted[1]. Every topological transition matrix *A* has at least one 1 in each row and column.

Example 1.4

Topological transition matrix for tent map with P=2 is of the form:



$I_0 \to I_0$

$I_0 \to I_1$

$I_1 \to I_0$

$I_1 \to I_1$

$$\underline{\underline{A}} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

For tent map all transitions are permitted that is why in each row and each column there are only 1. In this case we say that Markov transition matrix is full, it describes a full shift on two symbols [11].

Markov transition matrices are extremely useful for computing allowed and forbidden symbol sequences [11]. They are also useful for computing topological entropy but this will be described later on.

1.3 Outsplitting grapf and admissible symbol sequences

Transition matrix A is also used to determine whether given symbol sequences is admissible or not. We can start with following theorem:

Theorem 1.0:

F –Markov map with Markov partition $\{I_{0,}I_{1,\dots,}I_{N-1,}\}$ and symbols $\sigma\epsilon\{0,1,\dots,N-1\}$. Assigning symbol sequence $.\sigma_0\sigma_1\sigma_2 \dots$ to $x_n \in I$ according to the rule $x_n = f^{(0)}(x_0) \in I_{\sigma_k}$ leads to the fact that x_n has symbol sequence.$\sigma_n\sigma_{n+1}$ [1].Collection of all possible symbolic sequences forms an abstract space of symbolic sequences.

Let's assume that we have Markov partition with only two symbols: 1 and 0. Each point x_n in I can be written as binary fraction: $x = \frac{1}{2}\sigma_0 + \frac{1}{4}\sigma_1 + \frac{1}{8}\sigma_2 + \cdots$ (= $0.\sigma_0\sigma_1\sigma_2 \dots$) called symbol sequences.

Definition 1.4:

A symbol sequence $.\sigma_0\sigma_1\sigma_2 \dots$ is called admissible if for transition matrix A of a Markov map $A\sigma_k\sigma_{k+1} = 1, k \gg 0$ [1]

Following example shows admissible symbol sequences for tent map with partition P=2:

Example 1.2:

$$\underline{A} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- 0.0000 fixed point, admissible
- 0.010101 is admissible
- 0.1010101 is admissible

Tent map is classified as a surjective map which means that the whole interval I maps onto itself. For this kind of maps all symbolic sequences made of two symbols 0 and 1 are admissible. Admissibility condition is needed to tell whether a given sequence is allowed or not [3].

Next example shows symbol sequence which is not admissible. Let's consider following transition matrix A of a Markov map:

Example 1.3:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

- 0.010101 is admissible
- 0.110010 is not admissible since A_{11}=0 and all symbol sequences which contain a pair "11" are not admissible.

Admissible symbol sequences can be generated by walks on the directed graph $G_p = (\mathcal{V}, \mathcal{E})$ which has n vertices $\mathcal{V}= \{v_1, v_2, \dots, v_n\}$, corresponding to n intervals $\{I_1, I_{2,} \dots, I_n\}$ and directed edges going from v_i to v_j when $A_{ij} = 1$. We denote the set of edges that emanates from v_i by $\mathcal{E}_i$. Suppose that we have a composition of $\mathcal{E}_i$ defined by $\mathcal{E}_i = \bigcup_{j=1}^{k_i} \mathcal{E}_i^j$ where $\mathcal{E}_j^i$ are a subset of $\mathcal{E}_i$ satisfying $\mathcal{E}_i^j \cap \mathcal{E}_i^{j'}$=$\emptyset$. Associated with each decomposition of $\mathcal{E}_i$, let each vertex v_i divide into $\{v_i^1, v_i^2, \dots, v_i^{k_i}\}$. One then can construct a graph whose vertices V^i are defined by $V_i = \{v_1^1, v_1^2, \dots, v_1^{k_1}\} \cup \{v_2^1, \dots, v_2^{k_2}\} \cup \dots \cup \{v_n^1, \dots, v_n^{k_n}\}$ and edges $\mathcal{E}'$ are by $\mathcal{E}' = \left\{\left(v_i^j \to v_{i'}^{j'}\right)/(v_i \to v_{i'}) \in \mathcal{E}_i^j, 1 \ll i, i' \ll n, 1 \ll j \ll k_i, 1 \ll j' \ll k_{i'}\right\}$. All possible graphs that are essentially the same as this graph are called "outsplitting graph" of G with respect to the decomposition $\mathcal{E}_i = \bigcup_{j=1}^{k_i} \mathcal{E}_i^j$, which is here denoted by G' [4].

Outsplitting graph can be created for any Markov partition. To demonstrate how this works we will use tent map as an example.

Let's start with partition when P=2:

Transition matrix A is of the form:

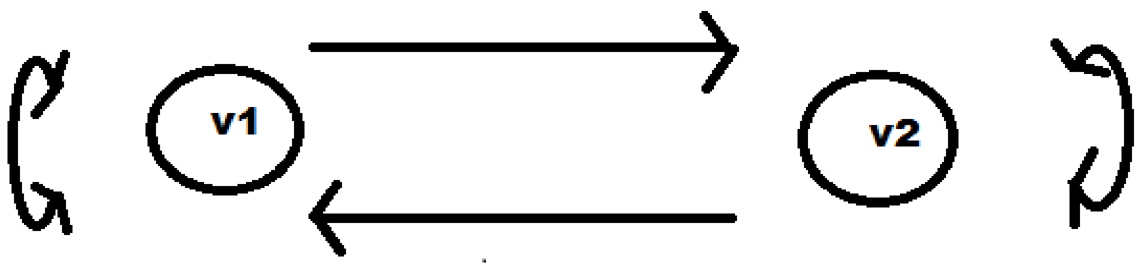
$$\underline{\underline{A}} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Set of vertices $\mathcal{V}= \{v_1, v_2\}$ as we have two intervals: $I_0=[0,1/2]$ and $I_1 =[1/2,1]$ and 4 directed edges from $I_0 \to I_0$, $I_1 \to I_1$, $I_1 \to I_0$, $I_0 \to I_1$ so:

$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ where $\mathcal{E}_1 = \{(v_1 \to v_2), (v_1 \to v_1)\}$ and $\mathcal{E}_2 = \{(v_2 \to v_1), (v_2 \to v_2)\}$.

We draw an arrow from v_1 to v_2 because $f(v_1) \supset v_2$.

Directed graph for Markov partition with P=2 is given by:



In general, let f be a map defined on interval I which has a partition $\{I_0, I_1, \dots, I_n\}$, i.e., $I = I_0 \cup I_1 \cup I_2 \cup \dots \cup I_n$ and $I_0, I_1, \dots, I_n$ are disjoint closed intervals (except at the end points), then we draw an arrow from I_0 to I_1 in the transition graph, if and only if $f(I_0) \supset I_1$ [12].

2. Dependencies of graph topology on Markov partitions

In this section, using the tent map as an illustrative example, it will be shown how the resultant shift space changes when new periodic orbits are added to the Markov partition P. We will show that if a partition P is a Markov partition, any partition P^ constructed by adding new elements to P is also a Markov partition. This means that if we have a partition P we can further "split" interval I and the new constructed partition remains Markov partition. We will introduce definition of partition P^ as a f-refinement to the partition P. We will show that if a Markov partitions have a certain relationship, we call "map-refinement of the other Markov partition," the shift spaces corresponding to these two Markov partitions are topologically the same [4]. Using tent map, we will also show that to each point x_n of the map we can assign a proper symbol sequence and all those sequences create orbit of a given period. Further, we will demonstrate definition of topological entropy and we will show that for the tent map it has always the same value, no matter how the interval I is partitioned or if a partition P^ is a f-refinement to the partition P.

2.1 P^ as f-refinement of P

Let's start with the following definition which will be used to show what happens when interval I is split into "smaller" intervals, when new elements are added to already existing Markov partition P:

Definition 2.0:

P, P^- Markov partitions. We call P^ as f-refinement of P if P $\notin$ P^ and f(P^) =P [4]

Using that fact we can show that:

Lemma 2.0:

If a partition P is a Markov partition, any partition P^ constructed by adding new elements to P is also a Markov partition [4].

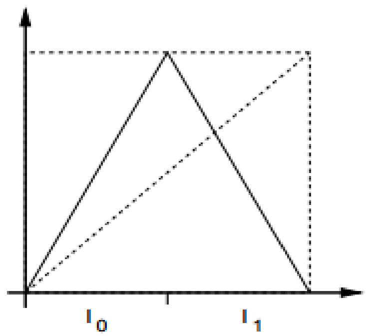
Proof:

Let I^ be an interval with respect to the partition P^, where P^ is f-refinement of P. F restricted to I^ is homomorphism from I^ onto f(I^) since there exists iϵ{1,…,n} such that I^$\subseteq I_i$ and the restriction of f to I^ is homomorphism f(I^). F maps the two boundary points of the interval I^ to those of $f(I^\wedge)$, which are also elements of P^ because of the fact that f(P^)⊆P^. This means that $f(I^\wedge)$ is a connected union of the intervals of P^ between the two boundary points of the interval P^. Given Markov partition P, there are some cases where Markov partition P^ satisfy P∉P^ and f(P^)⊆P^. The Markov partition of a certain finite number of elements P can be divided into two parts: Q composed of periodic orbits and R composed of all the rest. By adding elements of periodic orbits to the Q part, one can construct P^ if there are periodic orbits that are not taken as elements of Q. If the set f^{-1}(P)\P is not empty , one can add any elements of the set to the R part [4]. The resultant partition P^ forms a Markov partition for both cases.

To illustrate this we will use tent map as an example.

Example 2.0:

Let's start with partition for P=2:



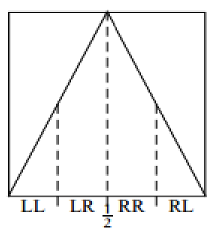
$$I_0 \rightarrow I_0$$
$$I_0 \rightarrow I_1$$
$$I_1 \rightarrow I_0$$
$$I_1 \rightarrow I_1$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

I_0 =[0, ½]

I_1 = [½, 1]

It is easy to see that the above partition is Markov partition. Now acting on the same map we can construct partition for P=4:



Interval I has been divided into 4 parts, 4 intervals namely:

LL= [0, 1/4] int(LL)=(0,1/4)

LR= [1/4, 1/2] int(LR)=(1/4,1/2)

I_0 =LL ∪ LR

I_1 = RR ∪ RL

RR =[1/2,3/4] int(RR)=(1/2,3/4)

RL= [3/4,1] int(RL)=(3/4,1)

Is this still Markov partition?

int(LL)=(0,1/4) f(int(LL))=(0,1/2)

int(LR)=(1/4,1/2) f(int(LR))=(1/2,1)

int(RR)=(1/2,3/4) f(int(RR))=(1/2,1)

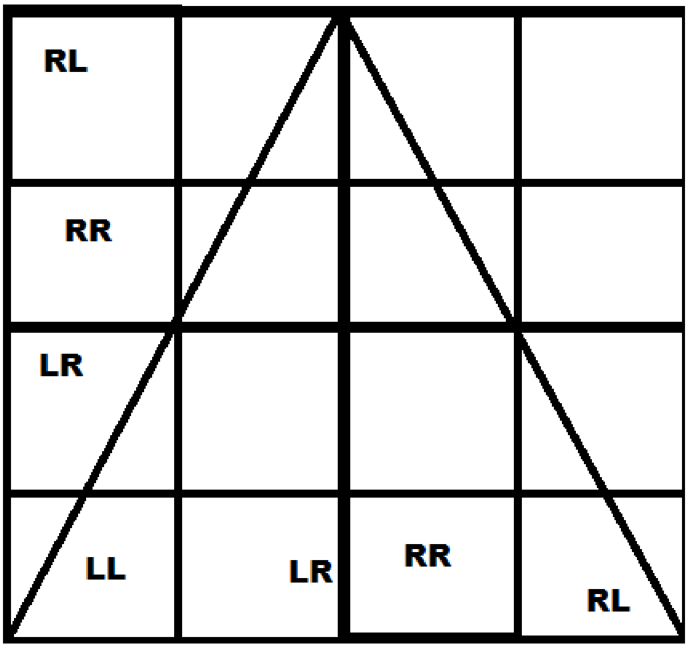
int(RL)=(3/4,1)) f(int(RL))=(0,1/2)

this is Markov partition as either a) or b) applies.

We can notice that partitioning interval I into smaller parts does not change properties which Markov partition has.

Knowing that P=$\left\{0,\frac{1}{2},1\right\}$ and P^=$\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}$ are Markov partitions we can show that P^ is an f-refinement of P since F(1/4)=1/2 and F(3/4)=1/2 and F(P^)⊆P.

Now let's use the tent map with Markov partition P^=$\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}$to construct $G_{p'}$. Because P^ is an f-refinement of P it will be demonstrated how directed graph looks like when the same tent map has 4 intervals.



In this case $\mathcal{V}$=$\{v_1, v_2, v_3, v_4\}$ as there are 4 subintervals.

$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \ \cup \ \mathcal{E}_3 \ \cup \ \mathcal{E}_4$ where:

1= v_1^1 2=v_1^2 3=v_2^1 4=v_2^2

and

$\mathcal{E}_{\mathbf{1}} = \{(v_1^1 \to v_1^1), (v_1^1 \to v_1^2)\}$,

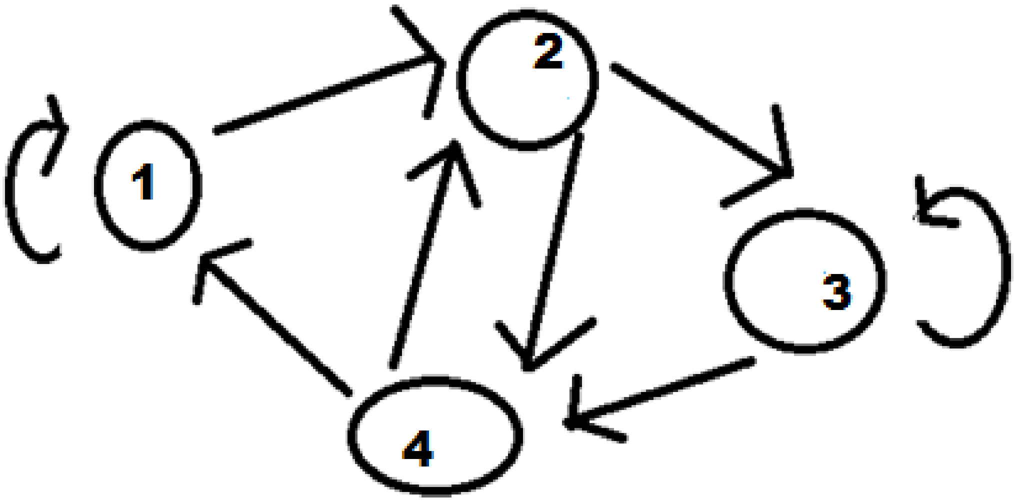
$\mathcal{E}_2 = \{(v_1^2 \to v_2^1), (v_1^2 \to v_2^2)\}$,

$\mathcal{E}_3 = \{(v_2^1 \to v_2^2), (v_2^1 \to v_2^1)\}$,

$\mathcal{E}_4 = \{(v_2^2 \to v_1^2), (v_2^2 \to v_1^1)\}$

Interval I is divided into 4 subintervals $LL = v_1, LR = v_{2,} RR = v_3, RL = v_4$ which is shown on the figure above. This map shows that the interval LL is mapped onto interval LL and LR, LR is mapped onto RR and RL, RR is mapped onto RR and RL and RL is mapped onto LL and LR. Mathematically, we could write that f(LL) ⊃ LL ∪ LR. These relations are summarized below in a transition graph. The arrow from LL to LR says that f(LL) covers LR.

Following picture presents those relations:



Those relations are also very conveniently summarized in a Markov transition matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

2.2 Fixed points, periodic points and symbol sequences

Using Markov partition we can establish one-to-one mapping between the orbits in the system and the set Ω of all allowed symbol sequences. The procedure of mapping the original dynamics of the system to a dynamics in the symbol space Ω is referred to as symbolic dynamics [6]. To understand the process we should first familiarise ourselves with the meaning of fixed point, orbit, symbol sequences and periodic point.

Definition 2.1:

A continuous map is a continuous function between two topological spaces. In some fields of mathematics, the term "function" is reserved for functions which are into the real or complex numbers. The word "map" is then used for more general objects. [12]

A map F: X → Y is continuous if the preimage of any open set is open.

Theorem 2.0 (Fixed point theorem)

Let f be a continuous map on R and I =[a,b] is a closed interval. If $f(I)\supset I$, then there is a fixed point x_0 in I satisfying $f(x_0)= x_0$ [12].

Proof:

Since $f(I)\supset I$, we have $x_1 \in I$, such that $f(x_1) \leq a$, thus $f(x_1)- x_1 \leq 0$. Similarly, we have $x_2 \in I$, such that $f(x_2) \geq b$ and thus $f(x_2)- x_2 \geq 0$. Since f is continuous, we must have a $x_0 \in I$, such that $f(x_0)- x_0 =0$ [12].

It is easy to count many fixed points there are for Tent Map:

Example 2.1: tent map

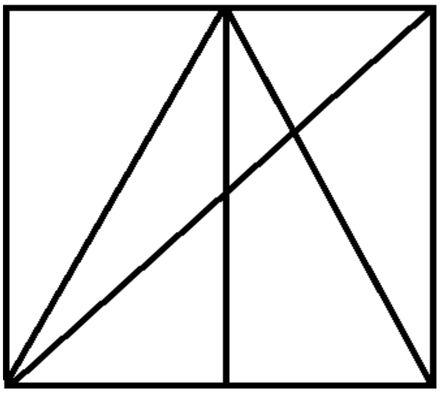


Fig.1

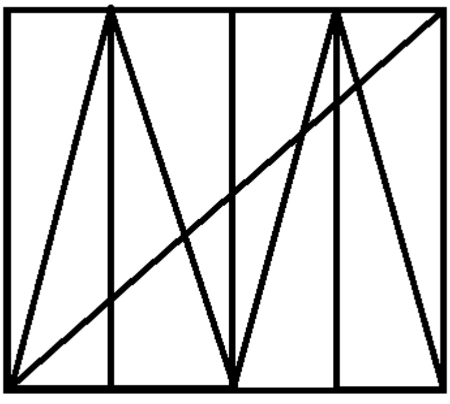


Fig. 2

Figure 1 shows $f(x)$ and figure 2 shows first iterate of the tent map $f^{(2)}(x)$.

The important point of these two graphs is that these two functions intersect the diagonal 2 and 4 times. It requires little stretch of the imagination to believe that $f^{(p)}(x)$ has 2^p fixed points.

The Markov transition matrix can be also used to estimate the number of fixed points of the p^{th} iterate of the map.

Example 2.2:

Markov transition matrix for ten map with P=2 is:

$A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$.

This matrix informs us that there are two fixed points of the original map f and they are in interval I_0 and $I_{1.}$

For the second iterate the Markov transition matrix is of the form:

$A^2=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\times\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$ and it shows straightaway that we have two fixed points in interval I_0 and $I_{1.}$

Definition 2.2:

Periodic point is the point x_0 which has property that $f(x_0)=x_0$ [5].

Let's assume that we have Markov partition with only two symbols: 1 and 0. Each point x_n in I can be written as binary fraction:$x=\frac{1}{2}\sigma_0+\frac{1}{4}\sigma_1+\frac{1}{8}\sigma_2+\cdots$ ($=0.\sigma_0\sigma_1\sigma_2 \ldots$) called symbol sequences.

Symbol sequences allow us to make simple and elegant distinction between different types of orbits. There are essentially two types of orbits: periodic and chaotic. A periodic orbit of period p is one that repeats itself after p steps. In symbol space such an orbit is represented by a symbol sequence of length p which repeats itself forever. A chaotic orbit in phase space is one which is nonrepeating. It is represented in symbol space by a sequence of symbols which is nonrepeating [12].

We will now show that tent map has chaotic orbits but to fully understand the proof we have to familiarise ourselves with definition of Lyapunov exponent of an orbit $(x_1, x_2, \ldots)$ [12].

We will start with the following definitions:

Definition 2.3:

Lyapunov number is the number which is equal to $L(x_1) = \lim_{n\to\infty}|f'(x_1)f'(x_2)\dots f'(x_n)|^{1/n}$. [12]

If this limit exists and assuming that $f'(x_j) \neq 0$ for all j we define Lyapunov exponent as:

Definition 2.4:

Lyapunov exponent is the values which equals to $h(x_1)= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} ln\left|f'(x_j)\right|$. [12]

Definition 2.5:

Let f be a map on R and let $(x_1, x_2, \dots)$ be an orbit. The orbit is called chaotic if:

- It is not asymptotically periodic
- Its Lyapunov exponent $h(x_1)>0$. [12]

Let us now consider the tent map to show that it has chaotic orbits:

1. If the orbit does not contain ½ then $h(x_1)=\ln 2$.

2. If an orbit is asymptotically periodic then it is eventually periodic.

   Eventually periodic means that the orbit becomes periodic after sufficient number of iterations. As an example (1/2,1,0,0,...) is eventually periodic, (1/4,1/2,1,0,0...) and (3/4,1/2,1,0,0...) are also eventually periodic. If an orbit converges to a periodic orbit, it has to be exactly the same as the periodic orbit after some number of iterations.

3. The periodic and eventually periodic orbits are countable.

   For periodic orbits, we can draw figures for $f(x), f^1(x), \dots,$ and then estimate the number of fixed points, period-2 points, period-3 points, etc. This leads to the conclusion that all periodic orbits are countable. For eventually periodic orbits, we first show that all those orbits that become a fixed periodic orbit are countable.

For example, we can list those orbits that become (0, 0,...) as follows:
(1,0,0,...),
(1/2,1,0,0,...),
(1/4,1/2,1,0,...),
(3/4,1/2,1,0,...),
(1/8,1/4,1/2,1,0,...)
In general such orbit can be written as $(x_1, x_2, \dots, x_{p,} 1/2,1,0,0)$
And there are 2^p possibilities. Therefore, they are countable. Next, the union of countable number of sets, with each set countable, is still countable. Thus, all eventually periodic orbits are countable.

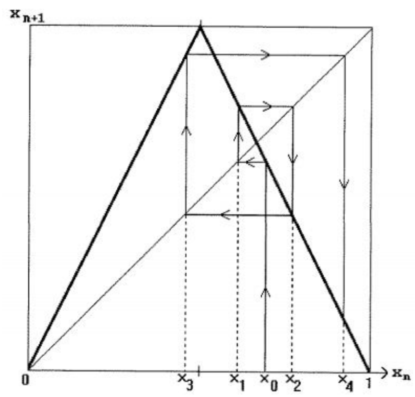
4. The set of all orbits of $f(x)$ is not countable.

   Every number in [0, 1] has an orbit. Two different numbers give different orbits. The set [0, 1] is uncountable.
   Therefore, there are orbits which are not asymptotically periodic and the Lyapunov exponent is ln 2 > 0, so they are chaotic orbits. [12]

All repeated symbol sequences correspond to periodic orbits, for example 0.010101 corresponds to a period 2 orbit and 0.001001001 corresponds to a period 3 orbit. It follows immediately from this that there are infinitely many periodic orbits in the system. There are more ways of constructing period-3 symbol sequences than period-2 symbol sequences, and in general, it can be easily verified that the number of periodic orbits increases exponentially with the period [6].

We can present the above using tent map as an example. In the beginning we will use Markov partition when P=2:

P_2 ={0,1/2} where the subscript of P means the period of periodic orbits used in the Markov partition.



The diagonal line in the above graph is the identity line $x_{n+1} = x_n$ and its intersection with the tent gives the fixed point 2/3 and 0. Every x_n in [0,1] is of the form $x = p/2^n$, p, n integer. There is infinite number of periodic points of the form $x = p/q$ where p and q are integers and q is odd like the period two points 2/5 and 4/5. Any fraction in lowest terms whose denominator contains an odd number as a factor is a preimage of a point in a periodic orbit [7].

The set of all periodic points that participate in orbits of period n can be computed exactly. For x_n, the nthiterate of x_0 is in bin 0 if it lies in [0,1/2] and that is in bin 1 if it lies in (1/2,1].

We want to construct all orbits of period 2. There are 4 possible strings that repeat after two cycles: .0000000; .1111111; .01010101; .10101010. If x is the initial value and it is in bin 0, its value is to be multiplied by 2. If x is in bin 1 its value is to be subtracted from1 and then multiplied by 2. Thus the period two point with orbit represented by .0101010 has initial value $x = 2(1-(2x)) = 2/5$. The period 2 orbit represented by .10101010 starts on $x = 2(2(1-x)) = 4/5$. If we compute the points corresponding to .00000 and .111111 we get 0 and 2/3 respectively, the period one point. Points of higher period are computed in the same way [7].

Thus a fixed point 2/3 is mapped to 2/3:

2/3→2-2*2/3=2/3→2/3

Period 2 orbit:

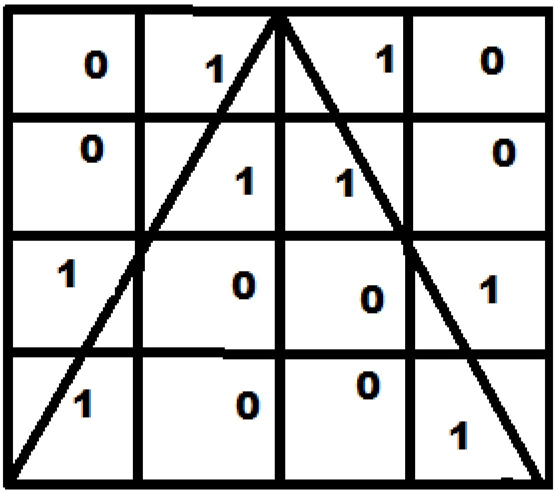
2/5→2*2/5=4/5→2-2*4/5=2/5-→4/5

All fixed points and periodic orbits are unstable.

The number of fixed points that are not fixed points is $2^p - 2$. If p is prime number, then all of those $2^p - 2$ fixed points must lie on periodic orbits of period p. If p is not prime and has integer factors p1, p1…then some of those points will be on orbits of the lower periods p1,p1…. Hence, if p is prime the number of periodic orbits of period p is $N_p = \frac{(2^p-2)}{p}$. This number gets large rapidly: for example N_2 =1, $N_{11} = 168, N_{13} = 630$. For p not prime the number N_p of periodic orbits satisfies $N_p < \frac{2^p-2}{p}$ and is more difficult to obtain [7]. Nevertheless, for large p, we always have that $N_p \simeq 2^p/p$ with a correction which is small compared to N_p [7].

Let's get back to the picture of figure 1 and figure 2. Figure 1 f(x) has two fixed points 0 and 2/3. This tells us that we have two period one orbit. The second iterate $f^{(2)}$(x) has fours fixed points. Two of these are the period one orbits. These have minimal period one. They can masquerade as period two orbits. Therefore only two of the four fixed points belong to an orbit of minimal period 2. Since there are two such points, and both must be on the same period two orbit, there is only one period two orbit [11].

Now let's do the same for the Tent Map but this time for P=4. Constructing the topological transition matrix we got the following picture:



The transition matrix A the above partition is as follows:

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Admissible symbol sequences for this type of partition is:

Since $A_{02}A_{03}A_{10}\ A_{11}A_{20}\ A_{21}A_{32}\ A_{33}$ are equal to 0 then all symbol sequences which contain a pair "02","03","10","11","20","21","32","33" are not admissible.

The remaining symbol sequences which are admissible are for example:

0.00000000

0.13131313

0.22222222

Now, let's try to compute the value for x according to the given symbol sequences:

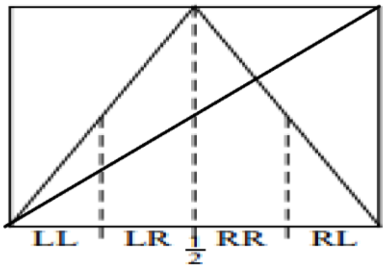
Symbol sequence with period 4: 0.001000100010 corresponds to:

X=0 x 1/2 + 0 x 1/4 + 1 x 1/8 + 0 x 1/16 + 0 x 1/32 + 0 x 1/64 + 1 x 1/128 + 0 x 1/156+...= 1/8 + 1/128+...=1/8 x (1/ (1-1/16))=1/8 x 16/15= 2/15

By symbol shift:

0.00100010 → 0.01000100→0.10001000→0.00010001→0.001000100→ ⋯

2/15 → 4/15→8/15→1/15→2/15→ ⋯



.

In general, N_p denotes the number of periodic points:

$$\begin{aligned} N_p &= \sum_{\sigma_0\sigma_1\ldots\sigma_{p-1}} A_{\sigma_0\sigma_1}A_{\sigma_1\sigma_2}\ldots A_{\sigma_{p-1}\sigma_0} \\ &= \sum_{\sigma_0}(\underline{\underline{A}}^p)_{\sigma_0\sigma_0} = \mathrm{Tr}(\underline{\underline{A}}^p) = \sum_{\nu}\Lambda_\nu^p \simeq \Lambda_{max}^p \end{aligned}$$

Where Λ_v denotes the eigenvalues of Markov transition matrix and Λ_{max} the largest eigenvalue. In particular the number of periodic points grows exponentially with the period p:

$$N_p \simeq \exp(p\ln(\Lambda_{max}))$$

The growth rate $\ln(\Lambda_{max})$, called the topological entropy h_{top} characterises the topological complexity of the system [1].

2.3 Topological entropy

Topological entropy is a positive number assigned to each topological dynamical system, that roughly tells how much chaotic a system is. Let us try to give first an intuitive idea of what it measures. Topological entropy gives the exponential rate of growth of the number of orbits distinguishable with finite but arbitrary precision. There are various equivalent ways of defining topological entropy and sometimes one is more convenient than the others to compute it [8].

We will demonstrate that the values of topological entropy does not change even if we have various types of partition: P=2, P=4, P=3.

For the beginning we will count the value of topological entropy for P=2:

For Tent map we are looking for the eigenvalues of the transition matrix A:

$(1-\lambda)(1-\lambda)-1=0$

$1-2\lambda+\lambda^2-1=0$

$-2\lambda+\lambda^2=0$

$\Delta = 4 - 4$

$\lambda = \frac{2}{1}=2$

$\lambda_{max} = 2$

Thus the topological entropy for Ten map with partition [0, 1/2] and [1/2, 1] is h_{top} =ln$|\lambda_{max}|$ =ln2

What happens if we have partition when P=4?

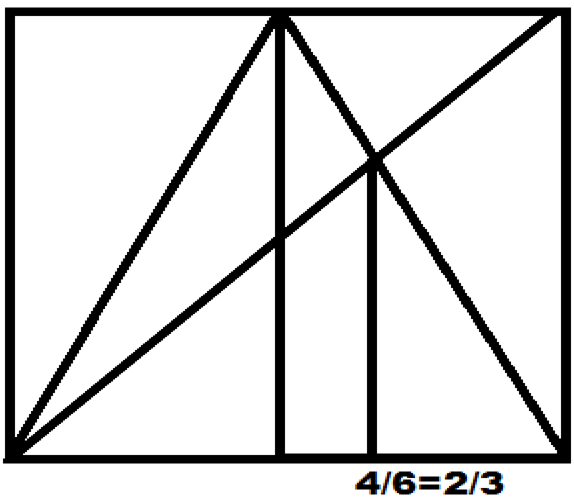
Topological entropy for P=4:

$$A=\begin{pmatrix} 1\,1\,0\,0 \\ 0\,0\,1\,1 \\ 0\,0\,1\,1 \\ 1\,1\,0\,0 \end{pmatrix}=\begin{pmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & 1 & 1 \\ 0 & 0 & 1-\lambda & 1 \\ 1 & 1 & 0 & 0-\lambda \end{pmatrix}$$

Thus the topological entropy for Ten map with partition [0,1/4],[1/4,1/2],[1/2,3/4],[3/4,1] is h_{top} =ln|λ_{max}| =ln2.

2.4 P^ as a not f-refinement of P

We can partition interval I and have P^ which is not f-refinement of P. If P^ is not f-refinement of P then shift space constructed from P^ is not necessarily conjugate to P. What does it mean? This means that, depending on how we symbolize the dynamical system, the structure of the resultant shift space such as number of periodic orbits can be different [4]. So let's see what happens if tent map has Markov partition when P=3:



$I_0 = [0, 1/2]$

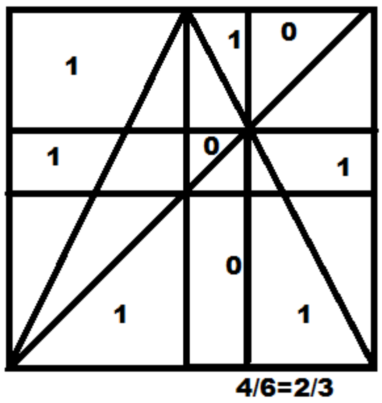
$I_{1=}$ [1/2, 2/3]

$I_2 = [2/3, 1]$

This is still Markov partition.

2/3 is the endpoint of the interval.

Topological transition matrix for the given map is:



$A=\begin{pmatrix} 1\ 1\ 1 \\ 0\ 0\ 1 \\ 1\ 1\ 0 \end{pmatrix}$

It is straightforward to check that P^ is not f-refinement of P since F(P^)≠P.

How does the directed graph look like for the given partition?

In this case $\mathcal{V}$={v_1, v_2, v_3} as there are 3 intervals.

$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ where:

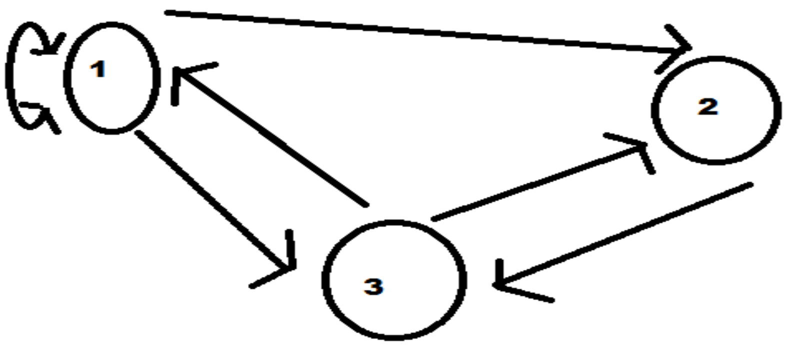
1= v_1^1 2=v_2^1 3=v_2^2

And

$\mathcal{E}_1 = \{(v_1^1 \to v_1^1), (v_1^1 \to v_2^1), (v_1^1 \to v_2^2)\},$

$\mathcal{E}_2 = \{(v_2^1 \to v_2^2)\},$

$\mathcal{E}_3 = \{(v_2^2 \to v_2^1), (v_2^2 \to v_1^2)\}$



The structure of the outsplitting graph $G_{p''}$ for this partition is not topologically equivalent G_p since each incidence matrix has different zero eigenvalues.

What about topological entropy?

Eigenvalues for the transition matrix are equal to 0, -1 and 2 so taking maximum eigenvalue the topological entropy is of course h_{top} =ln$|\lambda_{max}|$ =ln2 for the given Markov partition.

Given information allow us to conclude the following:

Although the number of periodic symbol sequences depends on the way of Markov partitioning, all the Markov partitions have the same topological entropy as that of the original map f^4.

3. Conclusion

First chapter of this work presents definition of Markov partition and other definitions like Markov map, partition which are useful to understand how the interval I can be divided to have Markov partition . I have shown how to construct topological transition matrix A taking information from Markov partition. Giving the definition of admissible symbol sequences I have demonstrated how to build outsplitting graph for the tent map as an example.

Second chapter starts with definition of partition P^ as f-refinement of P and using that fact gives us a proof that partition of already existing interval I into "smaller" parts is also Markov partition. Having partition with P=4 it has been shown how the outsplitting graph looks like in this case. Another part of this chapter includes the definition of fixed point, periodic and chaotic orbit, symbol sequences and information about number of fixed points, orbits, non-periodic points and gives a proof that tent map has chaotic orbits. We can familiarise ourselves with the definition of Lyapunov exponent and we can find out what the topological entropy is and that the value of it does not change when we have different partitions for the same map: tent map. Last part presents example of partition P^ which is not f-refinement of P. Definitions are covered with a proper examples.

I've enjoyed working on this project. I wish to express my gratitude to my project supervisor, Dr Wolfram Just for the guidance he provided during the course of the project as well as for reading through drafts of this report. Thank you also for the guidance he provided during the course of Complex Systems.

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