KURAMOTO MODEL (MTHM038)

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Abstract

Kuramoto was motivated by synchronization, which exists in nature extensively. In the Kuramoto model, each oscillator of a population tunes itself to match its frequency to the common. We exercise the Kuramoto model using the finite number of oscillators, which are two non-identical coupled oscillators, and three coupled oscillators. We deduce to a one-dimensional ordinary differential equation and two-dimensional ordinary differential equations from two-phase differential equations and three-phase differential equations, respectively. Derivation of fixed points and their stability is dealt with the help of the application of techniques.

Key words : synchronization, the Kuramoto model, oscillators, fixed point, stability

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Chapter 1

Introduction

In nature, innumerable flocks of birds fly in the sky and uncountable swarms of fish swim in harmony in the ocean. These sights have always been a breath-taking view for many people. It is not an exaggeration to say that those people have been curious to know how these creatures discipline themselves to form those clean patterns without a commander. No wonder that it has caught the attention of many mathematicians throughout the years. These mathematicians have understood that the phenomenon of *synchronization* governs these spectacular events. Their contributions over many years have led to a valuable link between nature and mathematics. For instance, down the line, Hygen [8] introduced the term synchronization to the literature; Weiner[2] approached it mathematically; Winfree [2] formulated it as an intuitive model and Kuramoto [2] formed it as a solvable model. The permeation of rigorous mathematics into the beauty of nature inspired me to test Kuramoto model.

Synchronization is an organization of events to conduct a system in accordance. Kuramoto is motivated by this term to form his model [8]. The phenomenon of synchronization in large populations of collaborating items is extensively studied in biology, physics, and chemistry [1]. We will look at some specific examples of synchronization in detail in chapter two: specifically, the firework displays by fireflies in biology, and the failure of the Millennium bridge in engineering.

Several authors including Kuramoto have proposed theories based on coupled oscillators. In their models, each member of population is represented as a phase oscillator which runs independently at its intrinsic frequency given they are uncoupled and there is no interaction among them [1]. When each oscillator couples to all the others and interacts with a certain strength, synchronization can be achieved [1].

Further, in chapter two, we establish several important parameters such as coupling strength used in Kuramoto's model. In addition, the relationship between coupling strength and order parameter is depicted in detail with the help of diagrams. We also talk about critical coupling strength and highlight Winfree's interest in the stability of critical coupling strength, to which Kuramoto could not find an answer [3].

In chapter three, we test two non-identical coupled oscillators, which is the easiest case. We form a phase differential equation using the Kuramoto model for each oscillator then we derive a single motion equation by finding the difference of frequencies of phases; this helps us to find the stability of fixed points easily.

In chapter four, we use three identical coupled oscillators and three non-identical oscillators. As in the case of two-oscillators, first, we form a differential equation for each oscillator. In both cases, we derive two motion equations by finding the difference between the frequencies of the phases. Unlike the case of the two oscillators, as this case is difficult, we use identities to derive fixed points and the Jacobian matrix to analyse the linear stability at each fixed point. In the case of non-identical coupled oscillators, we derive two factors which are used to establish fixed points; one of them is solved and the other is difficult to be solved and left for the future work.

Chapter 2

Background

2.1 Synchronization

Synchronization phenomena in large populations of interacting components are widely represented in nature and intensively studied as physical, biological and chemical systems [1]. As we proceed, in this section, we come across some examples of the occurrence of synchronization found in organisms, single cells and inanimate objects in detail. Understanding collective synchronized behaviour is important to derive needed results in all fields such as biology and engineering.

In Biology, examples include networks of pacemaker cells in heart [2, 4], nervous system [1], group of synchronously flashing fireflies [1,2,4]. In [1], Ezio Bartocci mentions that in the case of fireflies, each fire fly represents an internal clock commanding when to flash. Each clock is adjusted in line with the rhythm of other clocks. Over time, the rhythm of all the clocks pass on to rest at the same time, at this point all the fireflies flash together; that is when collective synchronization occurs [1].

Figure 1 is adapted from [13]

Figure 1 Fire display of fire flies

In South east Asia, one could view the flash of synchrony of thousands of male fireflies during night time[4,11].

Further, Steven Strogatz, in his lecture, gives details of glorious examples: birds that flock together or fish swimming in organized schools. They form skilful ballet, even though they are not intelligent creatures. As these creatures are small , synchrony helps to defend themselves against predators; it helps them to swarm to confuse predators. We are used to choreography giving rise to synchrony. These creatures are choreographing themselves[7].

Further he adds that synchrony occurs not only in organisms but also in a single cell. For instance, every beat of our heart depends on the sinoatrial node, which has about 10,000 independent pace maker cells that would each beep, have an electrical rhythm- to send a signal to the ventricles to pump. This population of 10,000 cells have to fire in unison for the pacemaker to work correctly [7].

Today science has started to question about the phenomenon of synchronization and figure out how it works. Especially, synchronization of coupled oscillator dynamics is the basis of many applications in science and engineering [5]. For instance, understanding synchronized collective behaviour is essential in system biology especially for developing methods to dominate the dynamics of systems with favourable qualities [1]. In biology, multiple studies have reported that epileptic brain is defined by increased neuronal synchrony. However, prior to seizure onset the synchrony may decrease [6]. Finding way to make adjustment of synchrony might give more people life and contentment.

Figure 2 is adapted from [10]

Figure 2 Millennium Bridge, London

In engineering, construction of London Millennium bridge became a failure and it was closed as soon as it was opened. The reason behind this unfortunate event was the engineer who designed the bridge was not aware of the synchrony that would produce by the pedestrians on it. Analysis following the event showed that as the bridge is wrapped by thin metal similar to rubber band, and the slight movement of the bridge resulted in pedestrians to adjust their steps to side wards. Consequently that made the bridge swing further and caused a huge movement [7, 9]. If the engineer would have considered the effect of synchrony the money and time could have been saved.

Synchronization is abundance in nature. Awareness of the presence of it and the impact of that on an organism, a single cell or an inanimate object can produce fruitful results.

2.2 Related Works

Kuramoto [3] was motivated by the idea of collective synchronization; in this concept, a large system of oscillators locks to a common frequency even though each natural frequency of oscillators differs from the rest. Before Kuramoto, a considerable number of mathematicians have studied on this concept. According to Choi, in [8], Huygen's seminal observation on synchronization was reported in the middle of the 17^{th} century. He experienced that two pendulum clocks hanging on the same bar, eventually, swing in the same rhythm by the weak interaction regardless of their initial rhythms[8]. The first mathematician who studied collective synchronization, mathematically, was Wiener [2]. However, as his approach was based on Fourier integrals, it was not successful [2].

Later, Winfree's approach was more successful, even though his model was, mainly, based on his intuition [2, 3, 8,14]. He assumed the population of his model is huge. The oscillators are strongly attracted to their limit cycles (see appendix 1). So, the differences their amplitude can be neglected and only the phase differences considered. Additionally, he simplified the model by the assumptions that oscillators are weakly coupled and their natural frequencies are almost identical [2]. Strogatz, in [2,14], mentions that Winfree formulated his model as below :

$$\dot{\theta}_i = W_i + \left(\sum_{j=1}^{N} X(\theta_j)\right) Z(\theta_i), i = 1, \ldots N$$

Where intrinsic frequency of the oscillator i is denoted by W_i and its phase is denoted by θ_i. His model is involved with two functions, a phase-dependent influence, $X(\theta_j)$, and the sensitivity function, $Z(\theta_i)$. Each oscillator j influence, $X(\theta_j)$, on all the other oscillators. The response of the corresponding oscillator depends on its phase, θ_i. However, his model is not solvable [3].

Strogatz mentions that synchronization occurs with respect to the natural frequencies of the oscillators [2,14]. Further, he adds that using numerical simulations and analytical estimations, the population in Winfree's model can display a sudden shift, which is called a phase transition. Each oscillator moves at its intrinsic frequency if the stretch of the frequencies is large compared to the coupling. At that point, the system acts incoherently. If, slowly, the stretch of the frequencies is reduced, at a certain point, a small bunch of oscillators 'freezes into synchrony'; otherwise, incoherence stays until a threshold is reached [2]. In the next section, this special threshold is called a critical coupling strength and discussed using graphs.

2.3 Kuramoto Model

The Kuramoto model is similar to Winfree's model but solvable. As Winfree, he assumed that each oscillator relaxes to a limit cycle and the oscillators are nearly identical and weakly coupled. Kuramoto showed this could be formulated by the oscillator's phase differential equation. He coupled the oscillators by a sine function and assumed that frequency is distributed according to some probability density $g(\omega)$ [2, 14].

Figure 3 is adapted from [12].

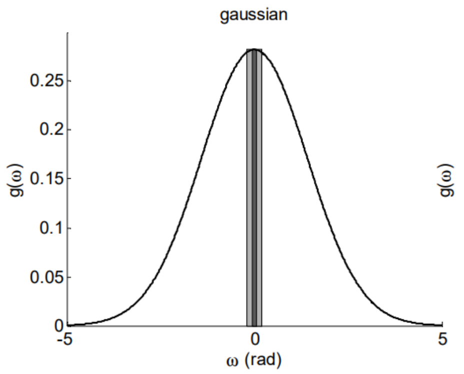

Figure 3 Distribution of natural frequencies

Figure 3 shows that, in [2], how frequencies of the Kuramoto model distributed, where ω is the natural frequency of an oscillator, $\overline{\omega}$ is the average frequency of all the oscillators, which falls at the centre, and $g(\omega)$ is some probability density. Additionally, $g(\overline{\omega} - \omega) = g(\overline{\omega} + \omega)$; this means the distribution is symmetric about the centre. Thus, Kuramoto's model is called by the name[2, 14]. By assuming $\bar{\omega} = 0$, the evolution of the phase of the ith oscillator is given by the following phase equation[2]:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} sin\big(\theta_j(t) - \theta_i(t)\big)$$

The Kuramoto model is an improved version of Winfree's intuitive model. His model of synchronization describes the dynamics of a population of N interacting phase oscillators (θ_i) with natural frequencies, ω_i and initial phases θ_i^0 ; here $i = 1 \dots N$ The phase of a standalone oscillator i evolve in time can be described by $\theta_i(t) = \omega_i t + \theta_i^0$ [1,11]. Each oscillator can be visualized as a point moves along the unit circle of radius one[1].

Each oscillator tends to speed up or slow down according to the phase and the natural frequency of others during interaction. This adjustment of their speed can be interpreted as a process of collective synchronization; the process ends in a total synchronous behaviour. In this special stage, the difference between the phases of the oscillators is constant. In a very special case, this difference is zero. This is when complete synchronization occurs [1].

In his analysis, Kuramoto provides a measure of synchronization by defining the complex order parameters r and ψ as[1]:

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

Where r represents the magnitude of the centroid of the points and ψ denotes the average phase. When phases of the oscillators are spread around the circumference of a unit circle, magnitude of the centroid is almost to zero[2].

Figure 4 is adapted from [2]

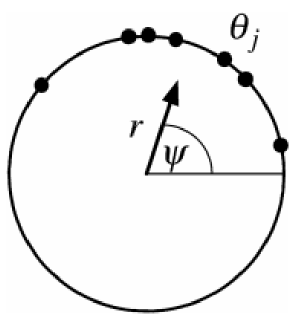

Figure 4 A unit circle with centroid, phases and average phase

According to Strogatz's article in [2], the figure 3 shows a unit circle plotted with the phases θ_j. The arrow represents the magnitude of the centroid at time t which is denoted as $r(t)$; the unified rhythm composed by the total population can be elucidated by the centroid of the phases as a perceptible quantity. $r(t)$ measures the phase coherence, and $\psi(t)$ is the average phase of the oscillators. For example, $r(t) \approx 1$, if all the oscillators run in a single clump and the whole population operates like a giant oscillator. Contrarily, if the oscillators are dispersed around the circle, $r(t) \approx 0$; the individual oscillators act incoherently and there is no unified rhythm produced[2].

The measure of synchronization is represented by r. If all the oscillators are identical, which means their angles $\theta_j\ (t)$ are the same, then $r = 1$[2].

In this subsection, Kuramoto's assumption on the natural frequency distribution of oscillators has been discussed in detail. Additionally, correlation between the magnitude of the centroid of a unit circle and the synchrony has been presented. In the next sub chapter, the important parameter of the Kuramoto model, coupling strength, is discussed.

2.4 Coupling Strength vs Order Parameter

In the above section, we have seen the relationship between order parameter and coherence of the population of oscillators. Now, in this section, we see the connection between the order parameter, r and the coupling strength, K .

Kuramoto mentioned in his lecture that oscillators remain in their natural frequency until they reach a particular coupling strength; beyond this strength they synchronize. This special coupling strength is called critical coupling strength. Winfree was curious about the transition and wanted to know about the stability of the critical coupling strength; however, kuramoto was not able to answer that question[3].

The following figures give the relationship between order parameter, r, coupling strength, K and critical coupling strength, K_c.

Figure 5 is adapted from [2]

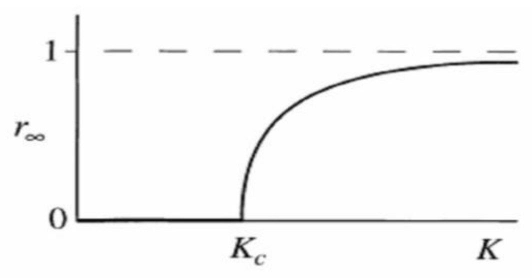

Figure 5 relationship between r and K

The graph above shows when $K < K_c$, there is only one solution, which is $r = 0$. But for $K > K_c$ there are non- zero solutions where K tends to infinity r tends to 1.

Figure 6 is adapted from [2]

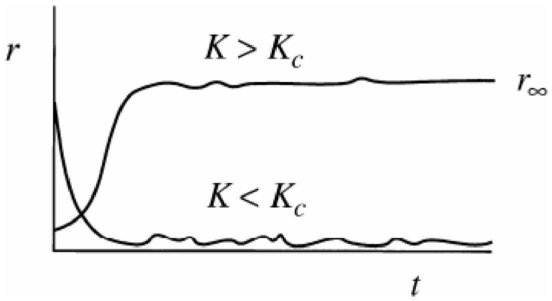

Figure 6 Evolution of r

Moreover, the figure 6 describes how critical coupling strength affects the order parameter. The order parameter decays almost to zero when the coupling strength is below the critical coupling strength; but, when the coupling strength is above the critical coupling strength the order parameter increases quite steeply.

Coupling strength and order parameter are important terms for Kuramoto. We interpret them further in the next section using a unit circle with 500 oscillators.

2.5 The Different Stages of Synchronization

The following figures produced by Matlab give the three cases of 500 oscillators with different coupling strengths. We summarize the facts we discussed in the previous sections, mainly from Strogatz's paper [2], regarding the relationship among coupling strength, critical coupling, the states of phases on a unit circle, order parameter and synchronization with visualization.

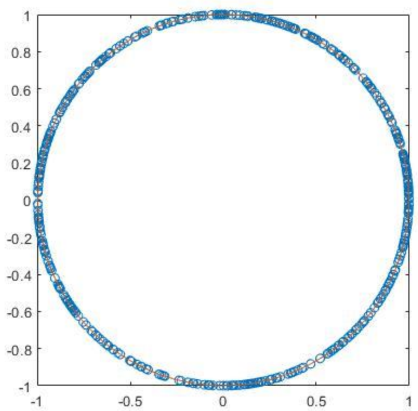

Figure 7 All the oscillators are spread equally

Case 1 : All the oscillators are spread equally along the unit circle. This means natural frequencies are largely spread and magnitude of the order parameter is almost zero; They are desynchronized. Here we could say the coupling strength is below the critical coupling strength. Figure 7 describes this case.

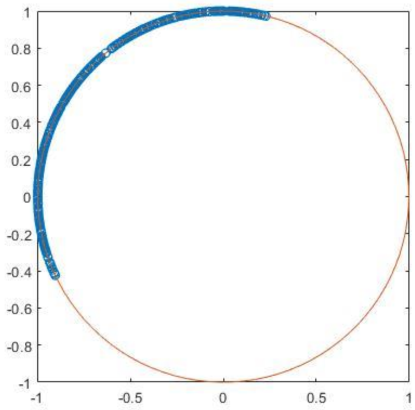

Figure 8 Oscillators clump together

Case 2 : The oscillators flock together and move around the circle. This would happen when the natural frequencies of the oscillators are almost identical. And coupling is above critical coupling strength. Order parameter is very far from zero and near to one. Figure 8 describes this case.

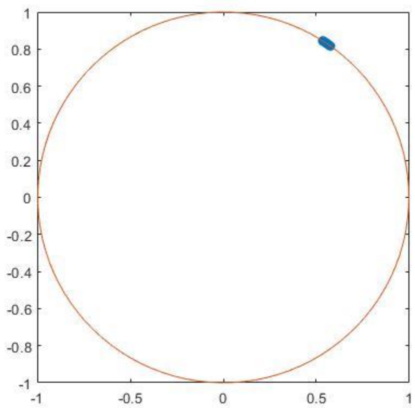

Figure 9 All the oscillators move together as a spot

Case 3 : All the oscillators move almost as a spot; this would happen when all the natural frequencies are almost the same. In this stage order parameter would be almost equal to one, which denotes the oscillators are in complete synchronization. Figure 9 well describes case 3.

This chapter has provided the needed background to exercise the Kuramoto model in chapter three and four. We start with the simplest problem with non- identical two oscillators in chapter three and a harder problem of three oscillators in chapter four.

Chapter 3

Two Non- identical Coupled Oscillators

The Kuramoto model with finite number of oscillators produces interesting results. Therefore, in this chapter, we start with the simplest case with two oscillator system. Let us learn the Kuramoto model for the two oscillators.

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} sin\big(\theta_j(t) - \theta_i(t)\big)$$

Here, N represents two as we consider two oscillators. The frequency of the oscillator, i, at time t is represented by $\dot{\theta}_i(t)$. The right hand side of the Kuramoto equation is governed by the natural frequency of the oscillator, i, denoted by ω_i and coupling strength, K, multiplied by the average of coupling function, which is the sine function of the difference between the phases at time, t, of the two oscillators.

3.1 Derivation of one-dimensional ordinary differential equation

The phase velocity of each of the two oscillators can be studied using the general Kuramoto model mentioned earlier.

$$\dot{\theta}_1(t) = \omega_1 + \frac{K}{2} sin(\theta_2(t) - \theta_1(t)) \tag{1}$$

$$\dot{\theta}_2(t) = \omega_2 + \frac{K}{2} sin(\theta_1(t) - \theta_2(t)) \tag{2}$$

Analysing the above two dimensional system is difficult. Therefore, the phase velocity difference between the two oscillators gives way to form one dimensional system, which is easier to analyse.

The following steps will help us to achieve our need.

1. Subtracting (2) from (1) gives,
   $\dot{\theta}_1(t) - \dot{\theta}_2(t) = \omega_1 - \omega_2 - K\, sin(\theta_1(t) - \theta_2(t))$

2. Introduction of new variables for the following leads to one dimensional ordinary differential equation.

   a. $\dot{\theta}_1(t) - \dot{\theta}_2(t) = \dot{\phi(t)}$, which is the angular velocity difference of two the oscillators

   b. as a consequence of a, we have

      $\theta_1(t) - \theta_2(t) = \phi(t)$, which is the phase difference at time, t.

   c. $\omega = \omega_1 - \omega_2$, which is the natural frequency difference of the oscillators.

Finally, we derive a single motion equation:

$$f(\phi(t)) = \dot{\phi(t)} = \omega - K\, sin\, \phi(t) \tag{3}$$

3.2 Fixed Points

In the previous section, we have brought two dimensional ordinary differential equations to one dimensional ordinary differential equation.

$$f(\phi(t)) = \dot{\phi(t)} = \omega - K\, sin\, \phi(t) \tag{3}$$

This makes our life easier to find the fixed points of the system. Fixed points are independent of time so we can write the equation as shown below.

$$f(\phi^*) = 0 = \omega - K \sin \phi^* \quad (3')$$

ϕ^* represents a fixed point of the system. At fixed points, $f(\phi^*) = 0$.

Therefore, we derive $\sin \phi^* = \frac{\omega}{K}$; this is valid under the condition $K > \omega$.

Using trig identity 2.7 in the appendix , we further derive the following.

$\cos(\phi^*) = +\sqrt{1 - \left(\frac{\omega}{K}\right)^2}$ or $\cos(\phi^*) = -\sqrt{1 - \left(\frac{\omega}{K}\right)^2}$

Therefore, the corresponding fixed points are:

1. $\cos^{-1}\left(+\sqrt{1 - \left(\frac{\omega}{K}\right)^2}\right), 0$
2. $\cos^{-1}\left(-\sqrt{1 - \left(\frac{\omega}{K}\right)^2}\right), 0$

3.3 Linear Stability Analysis of Equation (3)

First, for various K, we graphically analyse the linear stability of the equation.

The plot below produced by matlab presents the number of fixed points, which are produced under different K.

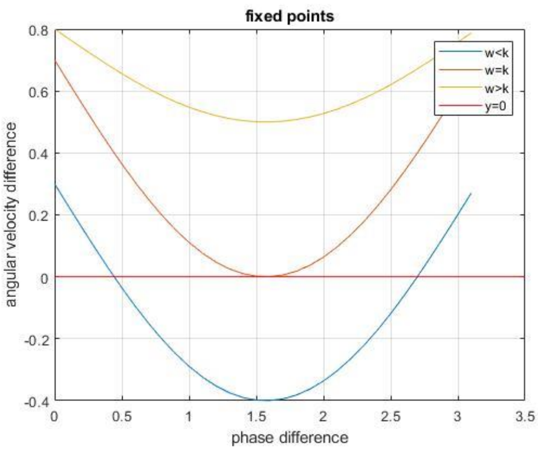

Figure 10 formation of fixed points under different coupling strengths

We notice that when $K > \omega$, two stationary points ϕ_s and ϕ_u exist, where $\phi_s < \phi_u$, ϕ_s being stable and ϕ_u unstable. When $K < \omega$, there are no fixed points; in this stage oscillators are desynchronized(check). When the critical value of K, $K_C = \omega$, where the curve touches the x axis at tangent, saddle node bifurcation is created, this stage is neither stable nor unstable but creates a half stable fixed point at $\phi = \frac{\pi}{2}$. When $K > \omega$ the semi stable fixed-point divides into a stable and unstable fixed point. Stable point attracts all the trajectories as time goes to infinity. At these stationary points, the original oscillators mentioned in the equations (1) and (2) rotate with the same speed; this means the angular difference is a constant for all time.

3.3.1 Linear Stability Analysis for the Case, $K > \omega$

Stability of the fixed point is analysed by the sign of the derivative of the slope of the curve at the fixed point (see appendix 4.3). To classify the stability of the fixed points of equation (3) under the condition, $K > \omega$, we differentiate the right hand side of the equation (3) with respect to ϕ.

$$\frac{d(f(\phi))}{d(\phi)} = -K\, cos\, \phi$$

We analyse the derivatives of the corresponding fixed points.

$$\frac{d(f'(\phi^*))}{d(\phi^*)} = -K\, cos\, \phi^*$$

1. $-K\left(+\sqrt{1-\left(\frac{\omega}{K}\right)^2}\right)$ for the fixed point, $cos^{-1}\left(+\sqrt{1-\left(\frac{\omega}{K}\right)^2}\right), 0$

   The system is stable when the derivative $f'(\phi^*) < 0$; at this point as $\cos\phi^* > 0$, the system is stable when $K > 0$.

2. $-K\left(-\sqrt{1-\left(\frac{\omega}{K}\right)^2}\right)$ for the fixed point, $cos^{-1}\left(-\sqrt{1-\left(\frac{\omega}{K}\right)^2}\right), 0$

   The system is stable when the derivative $f'(\phi^*) < 0$; at this point as $\cos\phi^* < 0$ therefore, the system is stable when $K < 0$.

3.4 Table of fixed Points and Their Stability

Let us summarize the fixed points and the stability for the condition, $K > \omega$

	ϕ^*	Derivative	Stability
1	$-K\left(+\sqrt{1-\left(\frac{\omega}{K}\right)^2}\right)$	For $K > 0$, $f'(\phi^*) < 0$	Stable for $K > 0$.
2	$-K\left(-\sqrt{1-\left(\frac{\omega}{K}\right)^2}\right)$	For $K < 0$, $f'(\phi^*) < 0$	Unstable for $K > 0$ Stable for $K < 0$.

If the strength of the interaction is stronger than ω, the system has a fixed point, $\phi^* = arcsin\frac{\omega}{K}$, at this stage synchronization occurs that means fire flies flash one after another at the same pace; this is called phase locking. If ω is zero, which means the two systems are uniform, complete synchronization occurs; the fire flies flash at the same time.

Chapter 4

Three Coupled Oscillators

In the previous chapter, we have deduced one dimensional ordinary differential equation from two equations of the two oscillators, by subtraction. We found the fixed points and their stabilities by finding the sign of the derivative of the function at each fixed point.

In this chapter, as a fact, stability analysis N oscillators is a difficult process, therefore, we start from the simple case of deriving two equations from three oscillators. Firstly, let us understand the generic Kuramoto model, which describes N oscillators with their intrinsic natural frequencies.

They are connected together with coupling strength, K. The state of oscillator n is characterized by its phase $\phi_n(t)$. Therefore, the dynamics of N oscillator system is regulated by the following equations of motion, specifically we say motion equation of the nth oscillator where $n = 1,2, \dots N$. Each nth oscillator interact with the rest of the N oscillators.

We use the following Kuramoto model formulated for N oscillators.

$$\dot{\phi}_n(t) = \omega_n + \frac{K}{N}\sum_{l=1}^{N} sin(\phi_l(t) - \phi_n(t))$$

4.1 Derivation of Two-dimensional Ordinary Differential Equation

We aim to produce two motion equations by studying the dynamics of the deviation between the phases of the oscillators; firstly, we regulate the Kuramoto equation for the each oscillator labelled 1 to 3.

Therefore, we modify the right hand side of the Kuramoto equation accordingly:

$$\dot{\phi}_n(t) = \omega_n + \frac{K}{3}\sum_{l=1}^{3} sin(\phi_l(t) - \phi_n(t))$$

Here, n denotes the oscillators labelled as 1,2 and 3. Here, K is the coupling strength which is the interaction among the oscillators.

For the oscillator 1, 2, and 3, we have following three phase differential equations:

a. Oscillator 1

We represent $n = 1$ and form the following phase differential equation for the oscillator 1.

$$\dot{\phi}_1(t) = \omega_1 + \frac{K}{3}[sin(\phi_2(t) - \phi_1(t)) + sin(\phi_3(t) - \phi_1(t))] \quad (4)$$

We find the sine function of the difference between the phases of oscillator 2 and 1 and the sine function of the difference between the phases of oscillator 3 and 1. Here we do not include the sine function of the difference between the phase of its own; as this will give zero: $sin(\phi_1(t) - \phi_1(t)) = 0$, which does not contribute to the summation. We average the summation of all the three sine functions by dividing by three. This averaged value is multiplied by the coupling strength and added by the natural frequency of the oscillator 1.

b. Oscillator 2

$$\dot{\phi}_2(t) = \omega_2 + \frac{K}{3}[sin(\phi_1(t) - \phi_2(t)) + sin(\phi_3(t) - \phi_2(t))] \quad (5)$$

For the oscillator labelled as 2, we find the sine function of the difference between the phases of oscillator 1 and 2 and the sine function of the difference between the phases of oscillator 3 and 2.

Here we do not include the sine function of the difference between the phase of its own; as this will give zero: $sin(\phi_2(t) - \phi_2(t)) = 0$, which does not contribute to the summation. We average the summation of all the three sine functions by dividing by three. This averaged value is multiplied by the coupling strength and added by the natural frequency of the oscillator two.

c. Oscillator 3

$$\dot{\phi}_3(t) = \omega_3 + \frac{K}{3}[sin(\phi_1(t) - \phi_3(t)) + sin(\phi_2(t) - \phi_3(t))] \tag{6}$$

For the oscillator 3, we find the sine function of the difference between the phases of oscillator 1 and 3 and the sine function of the difference between the phases of oscillator 2and 3. Here, we do not include the sine function of the difference between the phase of its own; as this gives zero: $\sin(\phi_3(t) - \phi_3(t)) = 0$, which does not contribute to the summation. We average the summation of all the three sine functions by dividing by three. This averaged value is multiplied by the coupling strength and added by the natural frequency of the oscillator 3.

It is vivid that finding fixed points from the three equations is difficult so we reduce them to two differential equations by taking the pairwise differences of phase frequencies among the above three oscillators.

4.1.1 Subtraction of Equations

We use subtraction technique to reduce the three-phase differential equations to two differential equations for our convenience to do the stability analysis later.

By Subtracting equation (5) from (4), we get,

$$\dot{\phi}_1 - \dot{\phi}_2 = (\omega_1 - \omega_2) + \frac{K}{3}(2\,sin(\phi_2 - \phi_1) + sin(\phi_3 - \phi_1) - sin(\phi_3 - \phi_2)) \tag{7}$$

By subtracting equation (6) from (5), we get,

$$\dot{\phi}_2 - \dot{\phi}_3 = (\omega_2 - \omega_3) + \frac{K}{3}(sin(\phi_1 - \phi_2) - sin(\phi_1 - \phi_3) + 2\,sin(\phi_3 - \phi_2)) \tag{8}$$

4.1.2 Introduction of New Variables

We, further, simplify the two equations, (7) and (8), by introducing new variables:

$$\dot{\phi}_1 - \dot{\phi}_2 = \dot{\theta}_1$$
$$\dot{\phi}_2 - \dot{\phi}_3 = \dot{\theta}_2$$
$$(\omega_1 - \omega_2) = \mathrm{w}_1$$
$$(\omega_2 - \omega_3) = \mathrm{w}_2$$

We rewrite the equations (7) and (8) as (7') and(8')

$$\dot{\theta}_1 = \mathrm{w}_1 + \frac{K}{3}(2\,sin(\phi_2 - \phi_1) + sin(\phi_3 - \phi_1) - sin(\phi_3 - \phi_2)) \tag{7'}$$

$$\dot{\theta}_2 = \mathrm{w}_2 + \frac{K}{3}(sin(\phi_1 - \phi_2) - sin(\phi_1 - \phi_3) + 2\,sin(\phi_3 - \phi_2)) \tag{8'}$$

As a consequence of the new variable assigned, $\dot{\phi}_1 - \dot{\phi}_2 = \dot{\theta}_1$, above, we derive $\phi_1 - \phi_2 = \theta_1$ similarly from the assignment of the new variable, $\dot{\phi}_2 - \dot{\phi}_3 = \dot{\theta}_2$, we derive $\phi_2 - \phi_3 = \theta_2$; as a result of these deductions, now, we can derive $\phi_1 - \phi_3 = \theta_1 + \theta_2$. Finally, by substituting the new variables and using identity 2.1 in the appendix, we rewrite our main equations in terms of θ s which forms two functions: $f_{1_{w_1}}(\theta_1, \theta_2)$ and $f_{2_{w_2}}(\theta_1, \theta_2)$.

$$f_{1_{w_1}}(\theta_1, \theta_2) = \ \dot{\theta}_1 = \mathrm{w}_1 + \frac{K}{3}(sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2)) \quad (7'')$$

$$f_{2_{w_2}}(\theta_1, \theta_2) = \ \dot{\theta}_2 = \mathrm{w}_2 + \frac{K}{3}(sin(\theta_1) - sin(\theta_1 + \theta_2) - 2\,sin(\theta_2)) \quad (8'')$$

4.2 Three Identical Coupled Oscillators

In the previous section, we have derived two main motion equations, (7'') and (8''), which we modify for three identical oscillators. We recall the equations for our convenience :

$$f_{1_{w_1}}(\theta_1, \theta_2) = \ \dot{\theta}_1 = \mathrm{w}_1 + \frac{K}{3}(sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2)) \quad (7'')$$

$$f_{2_{w_2}}(\theta_1, \theta_2) = \ \dot{\theta}_2 = \mathrm{w}_2 + \frac{K}{3}(sin(\theta_1) - sin(\theta_1 + \theta_2) - 2\,sin(\theta_2)) \quad (8'')$$

We reduce to simpler equations by dropping the parameters, w_1 and w_2.

This is possible when we chose identical oscillators; they all have the same natural frequencies, which means, their frequencies $\omega_1 = \omega_2 = \omega_3$ Therefore, the difference between the natural frequencies of the oscillators is zero. As a consequence, our right hand side of main equation, $f_{1_{w_1}}(\theta_1, \theta_2)$ loses the term, w_1. Similarly, $f_{2_{w_2}}(\theta_1, \theta_2)$ loses the term w_2 and as a result we have the following differential equations:

$$f_1(\theta_1, \theta_2) = \ \dot{\theta}_1 = \frac{K}{3}(sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2) \quad (7''')$$

$$f_2(\theta_1, \theta_2) = \ \dot{\theta}_2 = \frac{K}{3}(sin(\theta_1) - sin(\theta_1 + \theta_2) - 2\,sin(\theta_2)) \quad (8''')$$

4.2.1 Fixed Points

In section 3, we have computed fixed points for the one-dimensional ordinary differential equation of two oscillators, where we found the fixed points by making the function of the fixed point, $f(\phi^*)$, equals to zero. In this section, we follow the same strategy to find the fixed points of the two-dimensional system. Therefore, we have two differential equations and we equate their functions, $f_1(\theta_1, \theta_2)$ and $f_2(\theta_1, \theta_2)$ to zero at the stationary points. As fixed points are independent of time, we denote $\theta_n(t) = \theta_*$; here, $n = 1{,}2$ and θ_* is a fixed point.

Therefore the right hand side of the equations $f_1(\theta_1, \theta_2)$ and $f_2(\theta_1, \theta_2)$ are set to zero, from which we could derive equilibrium points.

$$sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2) \ = 0 \quad (9)$$

$$sin(\theta_1) - sin(\theta_1 + \theta_2) - 2\,sin(\theta_2) = 0 \quad (10)$$

As finding fixed points for two equations is more difficult than the one dimensional case, we need to use trigonometric identities.

Subtracting (10) from (9) gives the equation, $sin(\theta_2) - sin(\theta_1) = 0$

Using trigonometric identity 2.6, in the appendix, we derive the following equation

$$2cos\left(\frac{\theta_2+\theta_1}{2}\right) sin\left(\frac{\theta_2-\theta_1}{2}\right) = 0 \quad (11)$$

And the factors of the equations are:

i. $sin\left(\frac{\theta_2-\theta_1}{2}\right) = 0$

or

ii. $2cos\left(\frac{\theta_2+\theta_1}{2}\right) = 0$

Using those two factors, we form two cases to find the fixed points.

Case 1

Here, we use the first factor of equation (11) to find the relationship between θ_1 and θ_2, which are used later to find fixed points.

$$sin\left(\frac{\theta_2-\theta_1}{2}\right) = 0$$

$\theta_2 - \theta_1 = n\pi$, where $n = 0,1$

When $n = 0,\quad \theta_2 = \theta_1$ (a)

When $n = 1,\quad \theta_2 = \pi + \theta_1$ (b)

We do not need to consider values of n which are greater than one as they all repeat the same result as above. In detail, we think the system is a vector field of a circle. Consequently, the phases that differ by an integer multiply by 2π are in the same state in a circle.

Using the result in case 1a in the equation (9)

Let us recall the equation (9) :

$$sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2) = 0 \tag{9}$$

Substituting $\theta_2 = \theta_1$,

we get,

$$-sin(\theta_1) - sin(2\theta_1) = 0$$

Using double angle formulae in appendix 2.3 we derive, $sin(\theta_1)(-1 - 2\,cos(\theta_1)) = 0$, thus we have two possible solutions, which are:

a. $cos(\theta_1) = -\frac{1}{2}$

or

b. $sin(\theta_1) = 0$

For $cos(\theta_1) = -\frac{1}{2}$, we have $\theta_1 = \frac{2}{3}\pi \; or \; \frac{4}{3}\pi$

and using case 1a, which is $\theta_1 = \theta_2$ the corresponding fixed points are:

1. $\left(\frac{2}{3}\pi, \frac{2}{3}\pi\right)$
2. $\left(\frac{4}{3}\pi, \frac{4}{3}\pi\right)$

For $sin(\theta_1) = 0$, we have $\theta_1 = n\pi$, where $n = 0,1$

3. When $n = 0$, $\theta_1 = 0$. Consequently, $\theta_2 = 0$. Therefore, the corresponding fixed point is: $(0,0)$

4. When $n = 1$, $\theta_1 = \pi = \theta_2$. Therefore, the corresponding fixed point is: (π, π)

As previously mentioned, we do not need to consider integer values $n \geq 2$ as they all would repeat the same result. Because the values theta differ by the integer values multiply by 2π will give the same phase. For instance, When $n = 2$, $\theta_1 = 2\pi = \theta_2$, Therefore, the corresponding fixed point is: $(2\pi, 2\pi)$, which is in the same state as the fixed point, $(0,0)$.

Using the result in case 1b in the equation (9)

$$sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2) = 0 \qquad (9)$$

Substituting $\theta_2 = \pi + \theta_1$ and using identities 2.4 and 2.3 in the appendix, we derive,

$$-3\, sin\, \theta_1 + sin\, 2\theta_1 = 0$$

$$-3\, sin\, \theta_1 + 2\, sin\, \theta_1\, cos\, \theta_1 = 0$$

$$sin\, \theta_1(-3 + 2\, cos\, \theta_1) = 0$$

Thus, we have two possible solutions : $sin(\theta_1) = 0$ and $cos\, \theta_1 = \frac{3}{2}$, where we discard the later as it cannot produce any solutions as there are no values for $cos\, \theta_1 > 1$.

For $sin(\theta_1) = 0$, we have $\theta_1 = n\pi$, where $n = 0,1$

as $\theta_2 = \pi + \theta_1$ the corresponding fixed points are:

5. $(0, \pi)$ when $n = 0$

6. $(\pi, 2\pi)$ when $n = 1$

Previously, we have seen the reason for not considering certain values of n.

Next, we deal with case 2 in a similar way as in case 1.

Case 2

In this case, we use the second factor of the equation (11) and solve to find the fixed points.

$$2cos\left(\frac{\theta_2 + \theta_1}{2}\right) = 0$$

$$\left(\frac{\theta_2+\theta_1}{2}\right) = \frac{\pi}{2} + n\pi$$

When $n = 0$, $\theta_2 = \pi - \theta_1$

We do not consider $n \geq 1$ as adding on 2π produce repetitive results.

Substituting $\theta_2 = \pi - \theta_1$ in the equation (9) and using identity 2.5 in the appendix, we get,

$$sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2) = 0 \qquad (9)$$

Substitution gives,

$$\sin(\theta_1) = 0$$

we have $\theta_1 = n\pi$, where $n = 0,1$

Again, as we have seen previously, other values of n produce repetitive results.

as $\theta_2 = \pi - \theta_1$ the corresponding fixed points are $(0, \pi)$ and $(\pi, 0)$

We notice that these fixed points are same as the fixed points 5 and 6,which we have gained previously, in case 1. Therefore, we do not include them.

4.2.2 Linear Stability Analysis of Fixed Points

In this chapter, we perform the stability analysis for a more difficult aspect than the two oscillator case. Here, we mainly make use of the signs of eigenvalues (see appendix 4.2 for more detail) to determine the stability. Recall that we have used the signs of the derivatives of the functions of fixed points to determine the stability for the two oscillator case.

For the three identical oscillators, using trigonometric identities and substitution techniques , we have found six fixed points, $\left(\frac{2}{3}\pi, \frac{2}{3}\pi\right)$, $\left(\frac{4}{3}\pi, \frac{4}{3}\pi\right)$, $(0,0)$, (π, π), $(0, \pi)$, and $(\pi, 2\pi)$. In this section, we analyse the stability of each of them.

To analyse the stability of the above fixed points, we compute the Jacobian at those fixed points. The element of Jacobian 2× 2 matrix is denoted by,

$$J_{ij} = \frac{\partial f_i}{\partial \theta_j}$$

Where f_1 is $f_1(\theta_1, \theta_2) = \dot{\theta}_1 = \frac{K}{3}(sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2))$

f_2 is $f_2(\theta_1, \theta_2) = \dot{\theta}_2 = \frac{K}{3}(sin(\theta_1) - sin(\theta_1 + \theta_2) - 2\,sin(\theta_2))$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} \end{bmatrix}$$

$$J = \begin{bmatrix} -\frac{2K}{3}cos\,\theta_1 - \frac{K}{3}cos(\theta_1 + \theta_2) & \frac{K}{3}cos\,\theta_2 - \frac{K}{3}cos(\theta_1 + \theta_2) \\ \frac{K}{3}cos\,\theta_1 - \frac{K}{3}cos(\theta_1 + \theta_2) & -\frac{2K}{3}cos\,\theta_2 - \frac{K}{3}cos(\theta_1 + \theta_2) \end{bmatrix}$$

From the appendix 4.2, we know that multiplication of eigenvalues of a two by two matrix gives a determinant and addition gives a trace. Therefore, if the determinant is positive, the eigenvalues are the same sign. If the determinant is positive and the trace is negative simultaneously, the signs of the real part of both eigenvalues are negative. As a result, we have a stable node or spiral.

1. The Jacobian matrix at the stationary point, $\left(\frac{2}{3}\pi, \frac{2}{3}\pi\right)$:

$$J = \begin{bmatrix} \frac{K}{2} & 0 \\ 0 & \frac{K}{2} \end{bmatrix}$$

The determinant of the matrix is $\frac{K^2}{4}$, which is greater than zero for all values of K. This shows that eigenvalues have the same signs for all values of K.

And the trace is K, which is negative when K is negative; as the eigenvalues are negative under this condition, the system is stable when the values of K are negative.

2. The Jacobian matrix at the fixed point, $\left(\frac{4}{3}\pi, \frac{4}{3}\pi\right)$:

$$J = \begin{bmatrix} \frac{K}{2} & 0 \\ 0 & \frac{K}{2} \end{bmatrix}$$

The determinant of the matrix is $\frac{K^2}{4}$, which is greater than zero for all values of K. This shows that eigenvalues have the same signs for all values of K.

And the trace is K, which is negative for the values of K are negative; the eigenvalues are negative under this condition. The system is stable when the values of K are negative.

3. The Jacobian matrix at the stationary point, $(0, 0)$:

$$J = \begin{bmatrix} -K & 0 \\ 0 & -K \end{bmatrix}$$

The determinant is K^2, which is positive for all values of K.

And the trace is $-2K$, which is smaller than zero for the positive values of K .

Therefore, the system is stable for the positive values of K.

4. The Jacobian matrix at the stationary point, (π, π):

$$J = \begin{bmatrix} \frac{K}{3} & \frac{-K}{3} \\ \frac{-2K}{3} & \frac{K}{3} \end{bmatrix}$$

The determinant of the Jacobian matrix is $\frac{-K^2}{3}$, which is smaller than zero for all values of K.

And the trace is : $\frac{2K}{3}$, which is smaller than zero for the values of $K < 0$

Therefore, the system is unstable.

5. The Jacobian matrix at the stationary point, $(0, \pi)$:

$$J = \begin{bmatrix} \frac{K}{3} & 0 \\ \frac{2K}{3} & K \end{bmatrix}$$

The determinant is $\frac{K^2}{3}$, which is positive for all values of K.

And the trace is $\frac{4K}{3}$, which is negative for the negative values of K.

Therefore, the system is stable for all values negative of K.

6. The Jacobian matrix at the stationary point, $(\pi, 0)$:

$$J = \begin{bmatrix} K & \frac{2K}{3} \\ 0 & \frac{-K}{3} \end{bmatrix}$$

The determinant is $\frac{-K^2}{3}$, which is negative for all values of K.

And the trace is $\frac{2K}{3}$, which is negative for the negative values of K.

Therefore, the system is unstable for all values of K.

4.2.3 Table of fixed Points and Their Stability

In the subchapter 4.1, we have derived two motion equations from the three phase differential equations of the Kuramoto model; and using those equations, in the subchapter 4.2, keeping the natural frequencies the same, we form two motion equations for the three identical oscillators. We find the fixed points of those two differential equations by equating the right - hand side of the functions to zero. Then, using partial derivation, we derive the generic Jacobian matrix for the two main motion equations. We compute the specific Jacobian at each fixed point. And their stability is determined by the sign of their eigenvalues. We use the signs of the determinant and trace as a quick way to decide the stability ; they are summarized in the table below.

	θ_1	θ_2	Determinant	Trace	Stability
1	$\frac{2}{3}\pi$	$\frac{2}{3}\pi$	$\frac{K^2}{4}$	K	Stable for $K < 0$
2	$\frac{4}{3}\pi$	$\frac{4}{3}\pi$	$\frac{K^2}{4}$	K	Stable for $K < 0$
3	0	0	K^2	$-2K$	Stable for $K > 0$
4	π	π	$\frac{-K^2}{3}$	$\frac{2K}{3}$	Unstable
5	0	π	$\frac{K^2}{3}$	$\frac{4K}{3}$	Stable for $K < 0$
6	π	0	$\frac{-K^2}{3}$	$2K$	Unstable

4.2.3 Some Interesting Fixed Points

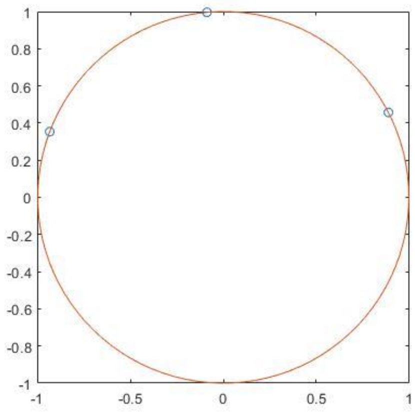

Figure 11 Random three oscillators on a unit circle

Figure 10 shows that the states of phases of random oscillators as three dots. If we imagine the states of the phases of three oscillators on a unit circle, we see some interesting patterns, for the particular fixed points.

In this section, we discuss how these three phases of oscillators fall on patterns on a unit circle for certain fixed points. We discuss the interesting patterns, which are one dot, two opposite dots (dots on diameter) and a star. We relate these patterns with the cases we have seen in section 2.5.

1. **One dot**

Firstly, let us see the speciality of the fixed point $(0,0)$. This implies $\theta_1 = \theta_2 = 0$. In the subsection, 4.1.1, we have seen that $\phi_1 - \phi_2 = \theta_1$ and $\phi_2 - \phi_3 = \theta_2$, by introducing new variables. Consequently, this results in $\phi_1 - \phi_2 = 0$ and $\phi_2 - \phi_3 = 0$. This means all the phases of the three oscillators are the same for all time at this fixed point. We could relate this to the facts we have seen in chapter two with regard to 'coherence' in [2]. We can relate this to the complete synchronization. We imagine a unit circle with a spot; actually, the states of the phases of three oscillators move in an overlapped position. Further, gathering the facts from chapter two, we could relate this to innumerable fireflies flash together as they are just one firefly and created a giant firework display.

2. **Two Dots**

Secondly, let us see the speciality of the fixed point (π, π). Here, we know $\theta_1 = \theta_2 = \pi$. As we know in the explanation given above, $\phi_1 - \phi_2 = \theta_1$ and $\phi_2 - \phi_3 = \theta_2$. Consequently, this results in $\phi_1 - \phi_2 = \pi$ and $\phi_2 - \phi_3 = \pi$. This means, there is always a phase, π, difference between the oscillator one and two, and the phase difference between the oscillator two and three. Consequently, the phases of the oscillator one and three overlap, maintaining the phase gap, π, from the oscillator two. Relating the facts we gathered in chapter two, complete synchronization occurs between the oscillators one and three while the oscillator two oscillate in a constant gap. If we imagine the states of the oscillators on a unit circle as stated earlier, oscillator two always moves opposite to the oscillators one and three; we can notice that the latter oscillators, 1 and 3, are in overlapped position; therefore, they move as one spot.

3. **A Star**

The third interesting fixed point is $\left(\frac{2}{3}\pi, \frac{2}{3}\pi\right)$. Here, we know $\theta_1 = \theta_2 = \frac{2}{3}\pi$. As we know in the explanation given above, $\phi_1 - \phi_2 = \theta_1$ and $\phi_2 - \phi_3 = \theta_2$. Consequently, this results in $\phi_1 - \phi_2 = \frac{2}{3}\pi$ and $\phi_2 - \phi_3 = \frac{2}{3}\pi$. We see that, as a result of this, the phase difference between the three oscillators are the same. Therefore, we have, $\phi_1 - \phi_2 = \frac{2}{3}\pi$, $\phi_2 - \phi_3 = \frac{2}{3}\pi$ and , $\phi_3 - \phi_1 = \frac{2}{3}\pi$. If we represent the states of the phases of the oscillators move on a unit circle as mentioned earlier, they maintain a hundred and twenty degrees angle among them; this produces a star shape. An example of this kind of synchronization is three clocks show different times but maintain the same gap.

4.3 Three Non-identical Coupled Oscillators

In the previous chapter, we considered the three oscillators whose intrinsic frequencies were the same. As a result, we have derived two main motion equations where the right-hand side of the functions have no parameters which represent the pairwise difference of the natural frequencies of the oscillators.

This time, we impose the pairwise difference between the natural frequencies of the oscillators are not zero but the same value with opposite signs.

$$(\omega_1 - \omega_2) = w; \; (\omega_2 - \omega_3) = -w$$

Now, the motion equations are as follows:

$$f_w(\theta_1, \theta_2) \; = \; \dot{\theta}_1 = w + \frac{K}{3}(sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2)) \tag{12}$$

$$g_{-w}(\theta_1, \theta_2) \; = \; \dot{\theta}_2 = -w + \frac{K}{3}(sin(\theta_1) - 2sin(\theta_2) - sin(\theta_1 + \theta_2)) \tag{13}$$

4.3.1 Fixed Points

In the subsection 4.2.1, we found the fixed points for the three identical oscillators. Now, finding fixed points is much more strenuous for the three non-identical oscillators as the motion equations consist of the parameter that is not present in the identical oscillator case. However, we follow the same procedure to calculate the fixed points.

Stationary points can be found by setting $f_w(\theta_1, \theta_2) = g_{-w}(\theta_1, \theta_2) = 0$

$$w + \frac{K}{3}(sin(\theta_2) - 2sin(\theta_1) - sin(\theta_1 + \theta_2)) = 0 \tag{12'}$$

$$-w + \frac{K}{3}(sin(\theta_1) - sin(\theta_1 + \theta_2) - 2\,sin(\theta_2)) = 0 \tag{13'}$$

By adding (12') and (13') we derive the equation,

$$sin(\theta_1) + sin(\theta_2) + 2sin(\theta_1 + \theta_2) = 0 \tag{14}$$

And by subtracting (13') from (12') we derive the equation,

$$sin(\theta_1) - sin(\theta_2) = \frac{2w}{K} \tag{15}$$

Using the identities in the appendix 2.3 and 2.6, the equation (14) becomes as follows.

$$2\,sin\frac{\theta_1 + \theta_2}{2}\left(cos\frac{\theta_1 - \theta_2}{2} + 2cos\frac{\theta_1 + \theta_2}{2}\right) = 0$$

Therefore the factors are :

a. $sin\frac{\theta_1 + \theta_2}{2} = 0$

or

b. $cos\frac{\theta_1 - \theta_2}{2} + 2\,cos\frac{\theta_1 + \theta_2}{2} = 0$

As we did for the three identical oscillators, we can find the fixed points using each factor of the equation (14). Thus we divide into two cases based on the factor, as we did for the identical oscillators.

Case 1

Similar to the section 4.2.1, we find the fixed point for the factor, $sin\frac{\theta_1 + \theta_2}{2} = 0$. Consequently, we have the following relationship between the phases θ_1 and θ_2,

$\theta_1 + \theta_2 = 2n\pi$ where n is a positive integer.

However, as we mentioned in the previous sections, we do not need to consider certain range of values of n. In this case we discard the values, which are greater than zero; all those values would represent the same result as zero. That is they all differ by multiples of 2π.

For $n = 0$ we have $\theta_1 = -\theta_2$ (*)

Substituting this result in the equation (15) gives ,

$$sin\,\theta_1 = \frac{w}{K}$$

Under the condition $w < K$, we use identity 2.7 in the appendix and get the following factors :

$$cos\,\theta_1 = +\sqrt{1 - \left(\frac{w}{K}\right)^2} \;\; or \; -\sqrt{1 - \left(\frac{w}{K}\right)^2} \qquad (**)$$

Additional to the results above , (*) and (**), we use identity 2.2 in the appendix to get the two fixed points.

The fixed points of the case 1 of the non - identical oscillators can be listed as follows :

1. $\theta_1 = cos^{-1}\left(+\sqrt{1 - \left(\frac{w}{K}\right)^2}\right),\; \theta_2 = -\theta_1 = cos^{-1}\left(+\sqrt{1 - \left(\frac{w}{K}\right)^2}\right)$
2. $\theta_1 = cos^{-1}\left(-\sqrt{1 - \left(\frac{w}{K}\right)^2}\right),\; \theta_2 = -\theta_1 = cos^{-1}\left(-\sqrt{1 - \left(\frac{w}{K}\right)^2}\right)$

Case 2

Using one of the factors of equation (14) , we have already derived two fixed points, in case 1.

Now, we use the other factor, $\left(cos\frac{\theta_1 - \theta_2}{2} + 2cos\frac{\theta_1 + \theta_2}{2}\right)$ to find more fixed points.

$$cos\frac{\theta_1 - \theta_2}{2} + 2cos\frac{\theta_1 + \theta_2}{2} = 0$$

$$cos\frac{\theta_1 - \theta_2}{2} = -\,2cos\frac{\theta_1 + \theta_2}{2} \qquad (***)$$

Let us recall the equation (15), in page 22: $sin(\theta_1) - sin(\theta_2) = \frac{2w}{K}$

using the trigonometric identity of 2.6 in the appendix, we derive,

$2sin\left(\frac{\theta_1-\theta_2}{2}\right)cos\left(\frac{\theta_1+\theta_2}{2}\right)=\frac{2w}{K}$

Using the fact in $(***)$, we can further derive,

$2sin\left(\frac{\theta_1-\theta_2}{2}\right)cos\left(\frac{\theta_1-\theta_2}{2}\right)=\frac{2w}{K}$

Finally using the trigonometric identity 2.3 of the appendix,

$$sin(\theta_1-\theta_2)=\frac{2w}{K}$$

Unlike, case 1, we see that finding fixed points for this case is not straight forward. Because of the limited time given for the MSc dissertation for the completion, we leave this problem for a future work.

4.3.2 Linear Stability Analysis at Fixed points

Now, using the fixed points found in case1 of section 4.3.1, we analyse the stability of the motion equations. Let us, first, recall the equations:

$$f_w(\theta_1,\theta_2) \;=\; \dot{\theta}_1=w+\frac{K}{3}(sin(\theta_2)-2sin(\theta_1)-sin(\theta_1+\theta_2)) \qquad (12)$$

$$g_{-w}(\theta_1,\theta_2) \;=\; \dot{\theta}_2=-w+\frac{K}{3}(sin(\theta_1)-2sin(\theta_2)-sin(\theta_1+\theta_2)) \qquad (13)$$

And the generic Jacobian matrix of the two dimensional non-linear system is:

$$J \;=\; \begin{bmatrix} -\frac{2K}{3}cos\,\theta_1-\frac{K}{3}cos(\theta_1+\theta_2) & \frac{K}{3}cos\,\theta_2-\frac{K}{3}cos(\theta_1+\theta_2) \\ \frac{K}{3}cos\,\theta_1-\frac{K}{3}cos(\theta_1+\theta_2) & -\frac{2K}{3}cos\,\theta_2-\frac{K}{3}cos(\theta_1+\theta_2) \end{bmatrix}$$

1. At the fixed point, $\theta_1=cos^{-1}\left(+\sqrt{1-\left(\frac{w}{K}\right)^2}\right),\ \theta_2=-\theta_1$ the corresponding Jacobian is:

$$\begin{bmatrix} \frac{-K\left(2\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)+1\right)}{3} & \frac{K\left(\sqrt{1-\left(\frac{w}{K}\right)^2}-1\right)}{3} \\ \frac{K\left(\sqrt{1-\left(\frac{w}{K}\right)^2}-1\right)}{3} & \frac{-K\left(2\sqrt{1-\left(\frac{w}{K}\right)^2}+1\right)}{3} \end{bmatrix}$$

Determinant is : $\frac{K^2}{3}-\frac{w^2}{3}+6\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)$

Let us decide the sign of the determinant.

For convenience, we bring the above form to the form from which we easily see the signs of its terms : $\frac{K^2}{3}\left(1-\left(\frac{w}{K}\right)^2\right)+6\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)$

Now, we can see the first term is positive and the second term is also positive as it is a positive root.

Therefore, determinant is positive for all values of K.

Trace = $\frac{2K}{3} - \frac{4K}{3}\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)$

= $\frac{2K}{3}\left(1-2\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)\right)$

$\left(1-2\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)\right) < 0$, for $\left|\frac{w}{K}\right| < \frac{\sqrt{3}}{2}$, the trace is negative

Therefore, the system is stable for $\left|\frac{w}{K}\right| < \frac{\sqrt{3}}{2}$

2. At the fixed point, $\theta_1 = cos^{-1}\left(-\sqrt{1-\left(\frac{w}{K}\right)^2}\right)$, $\theta_2 = -\theta_1$, the corresponding Jacobian is:

$$J = \begin{bmatrix} \frac{K\left(2\left(\sqrt{1-\left(\frac{W}{K}\right)^2}\right)-1\right)}{3} & \frac{-K\left(\sqrt{1-\left(\frac{W}{K}\right)^2}+1\right)}{3} \\ \frac{-K\left(\sqrt{1-\left(\frac{W}{K}\right)^2}+1\right)}{3} & \frac{K\left(2\left(\sqrt{1-\left(\frac{W}{K}\right)^2}\right)-1\right)}{3} \end{bmatrix}$$

Determinant is : $\frac{K^2}{3}\left(1-\left(\frac{w}{K}\right)^2 - 2\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)\right)$, which is negative; so clearly the fixed point is unstable.

Trace is : $\frac{\left(4K\sqrt{1-\left(\frac{w}{K}\right)^2}\right)-2K}{3}$

In case 1, we derived two fixed points; however, in case 2, we were not able to find any fixed points as the case is more difficult than case1.

4.3.3 Table of Fixed Points and Their Stability

We summarize the stability of the fixed points of case 1 in the following table:

	θ_1	θ_2	Determinant	Trace	Stability
1	$cos^{-1}\left(+\sqrt{1-\left(\frac{w}{K}\right)^2}\right)$	$-\theta_1$	$\frac{K^2}{3}\left(1-\left(\frac{w}{K}\right)^2\right)+6\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)$	$\frac{2K}{3}\left(1-2\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)\right)$	stable for $\left\lvert\frac{w}{K}\right\rvert<\frac{\sqrt{3}}{2}$
2	$cos^{-1}\left(-\sqrt{1-\left(\frac{w}{K}\right)^2}\right)$	$-\theta_1$	$\frac{K^2}{3}\left(1-\left(\frac{w}{K}\right)^2-2\left(\sqrt{1-\left(\frac{w}{K}\right)^2}\right)\right)$	$\frac{\left(4K\sqrt{1-\left(\frac{w}{K}\right)^2}\right)-2K}{3}$	Unstable

In this chapter, we have exercised linear stability analysis and summarized that a stable and an unstable fixed points exist in the case 1 of non – identical coupled oscillator differential equations.

Chapter 5

Conclusion

Our main task of this project is to find the stability of fixed points derived from phase differential equations which are formed using the Kuramoto model. We have used a variety of techniques such as subtraction, assigning new variables and using identities to bring two differential equations to a one-dimensional ordinary equation and three differential equations to two-dimensional ordinary differential equations.

In chapter three, we introduced the simplest case with two oscillators. Difference between the speed of the phases of oscillators forms a single equation which made our life easier to find the fixed points; we performed the linear stability analysis by finding the sign of the differentiation of the curve at each fixed point.

In chapter four, we performed a linear stability analysis for the three oscillators. We split three oscillator case into further two parts, which were three identical coupled oscillators and three non-identical coupled oscillators. We have seen that this case was harder than the two-oscillator case as we needed to use identities to derive fixed points. Further Jacobian matrix is used to find the stability of the curve at each fixed point as this time we consider two-dimensional ordinary differential equations.

In the identical oscillator case, the pairwise natural frequency difference of oscillators is null but in non-identical case the same but opposite signs; we have denoted that as W and $-W$. The complexity of calculating fixed points arose in the latter case. One of the factors of the equation, which is set to zero is not solved and left for future work.

In the three identical oscillator problem, we found some interesting fixed points where phase differences are constant. The oscillators create a certain pattern on a unit circle. We have observed a star, two opposite dots on diameter and a single dot. These were different synchronized stages.

In further research, we might examine four coupled oscillators. By keeping the natural frequencies the same, four identical oscillators can be dealt with as in three identical oscillators. However, that might still be a complex problem to solve as it might give rise to a three dimensional ordinary differential equation. New techniques may need to be explored, if the current method does not work.

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Appendix 1

Limit cycle
(notes gathered from [11])

A *limit cycle* is an isolated closed trajectory. Isolated means that neighbouring trajectories are not closed; they spiral either toward or away from the limit cycle. If all neighbouring trajectories approach the limit cycle, we say the limit cycle is *stable* (figure 10) . Otherwise the cycle is *unstable*, or in exceptional cases, *half-stable*.

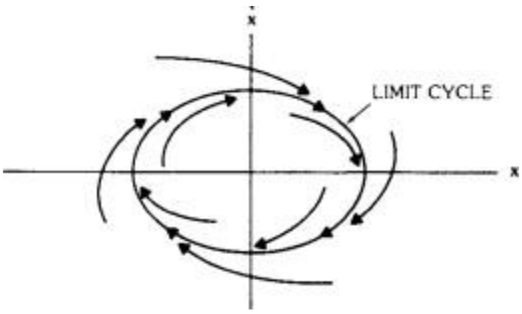


Figure 12 Stable Limit Cycle

Appendix 2

Identities

2.1 $\sin(-x) = -\sin(x)$

2.2 $\cos(-x) = \cos(x)$

2.3 $\sin(2x) = 2\sin x \cos x$

2.4 $\sin(x + \pi) = -\sin(x)$

2.5 $\sin(\pi - x) = \sin(x)$

2.6 $\sin(x) \pm \sin(y) = 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right)$

2.7 $\sin^2 x + \cos^2 y = 1$

Appendix 3

3.1 Matlab code used for figure 9 in chapter three.

```
clear all
x=0:0.1:pi
w=0.3;
k=0.7;
y=w-k*sin(x)
plot(x,y),grid on


hold on
u=0.7;
p=0.7;
l=u-p*sin(x)
plot(x,l),grid on
hold on
h=0.8;
f=0.3;
g=h-f*sin(x)

plot(x,g),grid on
hold off

hold  on
plot(xlim,[0 0],'r'),grid on
hold off
legend('w<k','w=k','w>k','y=0')
ylabel('frequency difference')
xlabel('phase difference')
```

3.2 Matlab code adapted from online and used in figures, 7,8,9 and 11.

```
function f=kuramoto(x,K,N,Omega)

        f=Omega+(K/N)*sum(sin(x*ones(1,N)-(ones(N,1)*x')))';

   end

% Numerical simulation for the Kuramoto model:

% theta_i'=Omega_i + K/N sum_j=1^N sin(theta_j-theta_i)

N=10;    % number of particles.

K=5;     % Coupling strength.

h=0.1;

iter=50;


t=0:h:h*iter;

theta=zeros(N,iter);
```

```
theta(:,1)=2*pi*rand(N,1);

Omega=rand(N,1);

for j=1:iter

k1=kuramoto(theta(:,j),K,N,Omega);

k2=kuramoto(theta(:,j)+0.5*h*k1,K,N,Omega);

k3=kuramoto(theta(:,j)+0.5*h*k2,K,N,Omega);           %4-th order Runge-
Kutta method.

k4=kuramoto(theta(:,j)+h*k3,K,N,Omega);

theta(:, j+1)=theta(:,j)+(h/6)*(k1+2*k2+2*k3+k4);

x=cos(theta(:,j));

y=sin(theta(:,j));

s=linspace(0,2*pi,100);

cx=cos(s);

cy=sin(s);

plot(x,y,'o',cx,cy)
```

Appendix 4

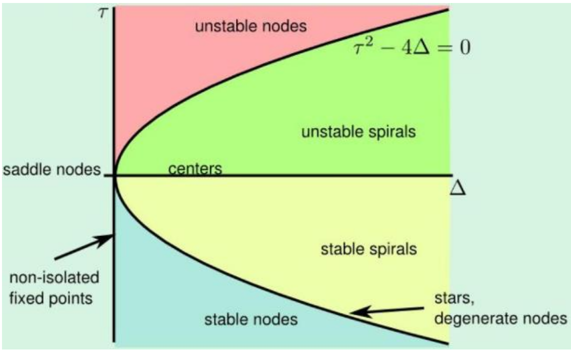
4.1 Fixed Points

(notes gathered from [11])

Fixed points represent equilibrium solutions (sometimes called steady, constant, or rest solutions, since if $x = x^*$initially, then $x(t) = x^*$ for all time). An equilibrium is defined to be stable if all

sufficiently small disturbances away from it damp out in time. Thus stable equilibria are represented geometrically by stable fixed points. Conversely, unstable equilibria, in which disturbances grow in time, are represented by unstable fixed points.

4.2 Stability at a fixed point of two dimensional system



The axis are the trace τ and the determinant Δ of a Jacobian matrix at a fixed point.

All the information in the diagram is implied by the following formulas:

$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$, $\Delta = \lambda_1\lambda_2$, $\tau = \lambda_1 + \lambda_2$.

4.3 Linear Stability Analysis for one-dimensional system

Quantitative measure of stability, such as the rate of decay to a stable fixed point. This sort of information may be obtained by *linearizing* about a fixed point, as we now explain.

Let x^* be a fixed point, and let $\eta(t) = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we derive a differential equation for η. Differentiation yields

$$\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x}$$

Since x^* is constant. Thus $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$. Now using Taylor's expansion we obtain

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2).$$

Where $O(\eta^2)$ denotes quadratically small terms in η. Finally, note that $f(x^*) = 0$ since x^* is a fixed point. Hence

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2)$$

Now if $f'(x^*) \neq 0$, the $O(\eta^2)$ terms are negligible and we may write the approximation

$$\dot{\eta} = \eta f'(x^*)$$

This is a linear equation in η, and is called the linearization about x^*. It shows that the perturbation $\eta(t)$ grows exponentially if $f'(x^*) > 0$ and decays if $f'(x^*) < 0$.

The upshot is that the slope $f'(x^*)$ as the fixed point determines its stability. The slope is always negative at a stable fixed point and inverse is true for when slope is positive.

To perform linearization for two-dimensional system, we extend the linearization technique developed above for one-dimensional systems; detail is given in page 150 in [11].