THE LORENZ EQUATIONS

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ABSTRACT

The project aims to analyse the work of Edward Lorenz. The way dynamical systems with complex behaviour, such as the weather behave after a long period of time. The project will mention the Lorenz equations and what they represent. What happens to the system if we vary one parameter having the other parameters fixed will also be discussed here. In addition it will be discussed about what happens to the dynamical system if we slightly change the initial conditions. Finally it will talk about what is meant by sensitive dependence on initial conditions and the butterfly effect and how these to concepts are related to chaos theory.

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CHAPTER 1
INTRODUCTION

Could anyone think that a tiny error in an input number of a computer program can have huge consequences in the long run behaviour of a complex dynamical system?

Edward Lorenz was a meteorologist and he devoted his work to try to give reasons why it is difficult to give accurate weather forecasting for more than two or three weeks.

Lorenz studied Mathematics at Dartmouth Collage and then had a Masters degree in Harvard University also in Mathematics. He, as well received a SM and a ScD in Meteorology from MIT. Furthermore, he served for the US Army Air Corps in World War II as a weather forecaster and it is then when he chose to do a graduate job in Meteorology. Soon after he took his doctorate in Meteorology. Lorenz was also awarded with many prized such as the Crafoord prize and the Kyoto prize. (MIT, 2008, April 16).

The atmosphere can be modelled by mathematical equations. A small incident during a survey to model the atmosphere led Lorenz to discover chaos. After this event, Lorenz was searching for complex mathematical systems. Such system is the Lorenz attractor or Lorenz equations which will be studied in the project. At the beginning the Lorenz equations were derived to model the atmosphere mathematically.

Firstly, the project will concentrate on deriving the Lorenz equations. What Lorenz considered before the derivation, what mathematical and physical notations were used is something it will also be mentioned in this chapter. It will be explained what the variables and what the parameters of these equations represent.

The next two chapters will look at the different parameter range of these equations. Basically we will adjust one parameter having the other parameters fixed and observe the different behaviours the system takes. If it has stable or unstable points, if it has periodic orbits and even if the system has chaotic behaviour. It will also be discussed why the invariant set of the Lorenz equations is also called Lorenz attractor and about the nontrivial fixed points of the attractor.

After that, the project will talk about the butterfly effect, a term first conceived by Lorenz himself in a paper he wrote in 1972. This chapter will also mention sensitive dependence on initial conditions and how this is related to chaos theory. This will be also explained by the means of some plots.

Finally, chapter 6 presents the conclusions of the project. Who else is involved with chaos theory and where the Lorenz equations are used.

CHAPTER 2

DERIVATION OF THE LORENZ EQUATIONS

Before deriving Lorenz equations, let us first say a few things about what Lorenz did, some mathematical details he considered and about the small experiment he prepared that led him to find the equations the project will talk about.

2.1 The Benard Rayleigh experiment

Lorenz (1963) considered a two-dimensional fluid element, such as water, placed between two plates. The plates are separated by a distance H. The lower plate is then heated uniformly and the upper plate was cooled. The difference of the temperature between the two plates is $\Delta\mathrm{T}$ and this difference is a fixed parameter. The x and y coordinates are parallel to the two plates and the z coordinate is perpendicular to the plates. The equations of motion are given by the Navier Stokes equations for an incompressible fluid $(\nabla \cdot u = 0)$, and the equation of the temperature is given by the heat transfer equation. It is assumed that motion variations are in one direction and are parallel to the two plates. This means that the temperature and the velocity of the fluid depend only on x and z and independent of the y direction. So there are no variations in the y direction.

In his paper, Lorenz (1963), considered systems of deterministic equations used in hydrodynamics. By saying deterministic equation we mean an equation that has the power over a dynamical system without arbitrary forces. He was mostly interested in periodic solutions and we will discuss this later on in the project. To derive his equations, he coped with a phase space Γ in which there was only one trajectory that passed through every point and where time was continuous. In addition the trajectory was bounded with a uniform way. This means that there is a bounded area, say R in which all trajectories eventually stayed in this area. Lorenz organised the trajectories in three ways. The first way was according to the existence of transient properties. Regarding the stability of the trajectories was the second way and the third way was according to the periodicity of the trajectories.

Lorenz (1963) used Saltzman's (1962) equations of free convection to derive his system of three ordinary differential equations. When we say convection we mean the heat flow of a fluid from a hot area to a cold area. Saltzman (1962) rewrote the equations of motion in terms of θ and ψ .He derived his equations using double Fourier transforms in x and z , but we will not discuss this here.

Saltzman's two equations are:

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial\left(x,z\right)} + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x}, \tag{1}$$

$$\frac{\partial\theta}{\partial t} = \frac{\partial\left(\psi,\theta\right)}{\partial\left(x,z\right)} + \frac{\Delta \mathrm{T}}{\mathrm{H}}\frac{\partial\psi}{\partial x} + \kappa\nabla^2\theta, \tag{2}$$

where ψ is a stream function for a two-dimensional motion, θ is the leaving temperature in the state of no convection, g is a constant representing the acceleration of gravity, α is the coefficient of thermal expansion, ν is the kinematic viscosity, κ is the thermal conductivity and $\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial\left(x,z\right)} = \frac{\partial\psi}{\partial x}\frac{\partial\left(\nabla^2\psi\right)}{\partial z} - \frac{\partial\left(\nabla^2\psi\right)}{\partial x}\frac{\partial\psi}{\partial z}$.

These equations are introduced, taking into account that motion is parallel to the $x - z$ plane and there is no motion in the y direction.

At both the boundaries, the vertical velocity is equal to zero. Furthermore at a free boundary the tangential stress is also zero and at a rigid boundary the tangential velocity is zero. At the boundaries this means that

$$\psi = 0, \quad \nabla^2\psi = 0, \quad {\partial\psi}/{\partial z} = 0\,.$$

For the derivation of the Lorenz equations we will only use $\psi = \nabla^2\psi = 0$. Here, we note that these boundary conditions apply to $z = 0$ and $z = H$, where H is the height of the fluid.

2.2 Derivation of Lorenz Equations

The nonlinear system of the partial differential equations (1) and (2) with the boundary conditions mentioned above is difficult to solve. To make things easier, one should choose some approximations to reduce the partial differential equations approximately to nonlinear ordinary differentia equations. For this reason Lorenz (1963) introduced solutions of the following form, as these expressions satisfy the boundary conditions.

$$a\left(1+a^2\right)^{-1}\kappa^{-1}\psi = \mathrm{X}\sqrt{2}\sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right),$$

$$\pi R_c^{-1}R_a\Delta\mathrm{T}^{-1}\theta = Y\sqrt{2}\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}\mathrm{z}\right) - Z\sin\left(2\pi H^{-1}\mathrm{z}\right). \qquad (3), (4)$$

Here, X, Y, Z are only functions of time t, R_a is the Rayleigh number, after the one who invented it. This is equal to $R_a = g\alpha H^3\Delta\mathrm{T}\nu^{-1}\kappa^{-1}$ and $R_c = \pi^4 a^{-1}\left(1+a^2\right)^3$ is the critical value of the Rayleigh number.

Lorenz (1963) substituted equations (3) and (4) into equations (1) and (2) and with some calculations he ended up with the following set of ordinary differential equations, which are known as the Lorenz equations.

$$\dot{X} = -\sigma X + \sigma Y$$

$$\dot{Y} = -XZ + rX - Y$$

$$\dot{Z} = XY - bZ.$$

σ, r, b are constants to be determined and the dot represents derivative with respect to time.

It will now be discussed in some detail how the Lorenz equations are obtained from Saltzman's (1962) partial differential equations.

Now we will substitute (3) into (1). First we will compute the LHS of (1).

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

From equation (3) we see that nothing depends on y so $\frac{\partial^2\psi}{\partial y^2} = 0$ and hence

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} .$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{\left(1+a^2\right)}{a}\kappa X\sqrt{2}\left(\pi a H^{-1}\right)^2 \sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)$$

$$\frac{\partial^2\psi}{\partial z^2} = -\frac{\left(1+a^2\right)}{a}\kappa X\sqrt{2}\left(\pi H^{-1}\right)^2 \sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)$$

and hence, taking common factors

$$\nabla^2\psi = -\frac{\left(1+a^2\right)}{a}\kappa X\sqrt{2}\sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\left[\left(\pi a H^{-1}\right)^2 + \left(\pi H^{-1}\right)^2\right]$$

To compute $\frac{\partial}{\partial t}\nabla^2\psi$ we take into account that the only variable that depends on t is X .

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\left(1+a^2\right)}{a}\kappa\sqrt{2}\sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\left[\left(\pi a H^{-1}\right)^2 + \left(\pi H^{-1}\right)^2\right]\dot{X}$$

For the RHS of (1) we compute the following quantities

$$\frac{\partial\theta}{\partial x} = -\frac{R_c\Delta\mathrm{T}}{R_a\pi}\left[Y\sqrt{2}\left(\pi a H^{-1}\right)\sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\right].$$

$$\nabla^4\psi = \nabla^2\nabla^2\psi = \frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial z^4} + 2\frac{\partial^4\psi}{\partial x^2\partial z^2}$$

$$\frac{\partial^4\psi}{\partial x^4} = \sqrt{2}\,\frac{\left(1+a^2\right)}{a}\kappa X\left(\pi a H^{-1}\right)^4 \sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)$$
$$\frac{\partial^4\psi}{\partial z^4} = \sqrt{2}\,\frac{\left(1+a^2\right)}{a}\kappa X\left(\pi H^{-1}\right)^4 \sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)$$
$$\frac{\partial^4\psi}{\partial x^2\partial z^2} = \frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\psi}{\partial x^2}\right) = \frac{\left(1+a^2\right)}{a}\kappa\sqrt{2}X\left(\pi a H^{-1}\right)^2\left(\pi H^{-1}\right)^2 \sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)$$

$$\nabla^4\psi = \sqrt{2}\,\frac{\left(1+a^2\right)}{a}\kappa X \sin\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\left[\left(\pi a H^{-1}\right)^4 + \left(\pi H^{-1}\right)^4 + 2\left(\pi a H^{-1}\right)^2\left(\pi H^{-1}\right)^2\right].$$

The final term of the RHS of (3) is $\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(x,z)}$ which according to Saltzman (1962) this, as mentioned above, equals to

$$\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(x,z)} = \frac{\partial\psi}{\partial x}\frac{\partial\left(\nabla^2\psi\right)}{\partial z} - \frac{\partial\left(\nabla^2\psi\right)}{\partial x}\frac{\partial\psi}{\partial z} =$$

$$= 2\frac{\left(1+a^2\right)^2}{a}\kappa^2X^2\left(\pi H^{-1}\right)\left[\left(\pi H^{-1}\right)^2 + \left(\pi H^{-1}\right)^2\right]\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\sin\left(\pi a H^{-1}x\right)\cos\left(\pi H^{-1}\right)z +$$
$$2\frac{\left(1+a^2\right)^2}{a^2}\kappa^2X^2\left(\pi a H^{-1}\right)\left(\pi H^{-1}\right)\left[\left(\pi H^{-1}\right)^2 + \left(\pi a H^{-1}\right)^2\right]\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\sin\left(\pi a H^{-1}x\right)\cos\left(\pi H^{-1}z\right).$$

Lorenz (1963), in his paper omitted all other trigonometric terms besides the ones occurring in equations (3) and (4). So we ignore the term $\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(x,z)}$ as it doesn't contain any of the trigonometric terms in the two equations.

So putting all the terms together in $\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(x,z)} + \nu\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x}$ we get

$$-\frac{\left(1+a^2\right)}{a}\kappa\sqrt{2}\sin\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)\left[\left(\pi aH^{-1}\right)^2+\left(\pi H^{-1}\right)^2\right]\dot{X} =$$
$$\nu\frac{\left(1+a^2\right)}{a}\kappa X\sqrt{2}\sin\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)\left[\left(\pi aH^{-1}\right)^2+\left(\pi H^{-1}\right)\right]^2 -$$
$$g\alpha\frac{R_c\Delta\mathrm{T}}{R_a\pi}\left[Y\sqrt{2}\left(\pi aH^{-1}\right)\sin\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)\right]$$

Cancelling the terms that occur in both sides and using the definitions of R_a and R_c, this simplifies to

$$-\frac{\pi^2}{H^2}\left(1+a^2\right)\dot{X} = \nu\left(\frac{\pi^2}{H^2}\left(1+a^2\right)\right)^2 - \nu\frac{\pi^4}{H^4}\left(1+a^2\right)^2$$
$$\Rightarrow \dot{X} = -\nu\frac{\pi^2}{H^2}\left(1+a^2\right)X + \nu\frac{\pi^2}{H^2}\left(1+a^2\right)Y = \frac{dX}{dt}$$

Lorenz (1962), in his paper used $\tau = \pi^2H^{-2}\left(1+a^2\right)\kappa t$. So using the chain rule $\frac{dX}{d\tau} = \frac{dX}{dt}\cdot\frac{dt}{d\tau}$ and with $\frac{dt}{d\tau} = \frac{H^2}{\pi^2}\frac{1}{\left(1+a^2\right)\kappa}$ we get

$$\frac{dX}{d\tau} = \left[-\nu\frac{\pi^2}{H^2}\left(1+a^2\right)X + \nu\frac{\pi^2}{H^2}\left(1+a^2\right)Y\right]\cdot\frac{H^2}{\pi^2}\frac{1}{\left(1+a^2\right)\kappa}.$$

This reduces to $\dot{X} = -\nu\kappa^{-1}X + \nu\kappa^{-1}Y$ and with $\nu\kappa^{-1} = \sigma$ we obtain the first required equation, that is

$$\dot{X} = -\sigma X + \sigma Y.$$

In order to obtain the other two equations we consider equation (2). We substitute equations (3) and (4) into this equation to get the following relationships.

To calculate $\partial\theta/\partial t$ on the LHS of (2), we take into consideration that the only variables depending on t are Y and Z . Therefore we have

$$\frac{\partial\theta}{\partial t}=\frac{\Delta\mathrm{T}R_c}{\pi R_a}\dot{Y}\sqrt{2}\cos\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)-\frac{\Delta\mathrm{T}R_c}{\pi R_a}\dot{Z}\sin\left(2\pi H^{-1}z\right).$$

Now we calculate the RHS of (2). Here we have

$$\frac{\Delta\mathrm{T}}{H}\frac{\partial\psi}{\partial x}=\frac{\Delta\mathrm{T}}{H}\frac{\left(1+a^2\right)}{a}\kappa X\sqrt{2}\left(\pi aH^{-1}\right)\cos\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right).$$

Next we calculate $\nabla^2\theta=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}$. As before, nothing depends on y so we only need to consider $\nabla^2\theta=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}$. After taking common factors, this equals to

$$\nabla^2\theta=\frac{\Delta\mathrm{T}R_c}{\pi R_a}\left[-Y\sqrt{2}\cos\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)\left\{\left(\pi aH^{-1}\right)^2+\left(\pi H^{-1}\right)^2\right\}+Z\left(2\pi H^{-1}\right)^2\sin\left(2\pi H^{-1}z\right)\right]$$

The last term we need to calculate is $\frac{\partial(\psi,\theta)}{\partial(x,z)}$ which as before this equals to

$$\frac{\partial(\psi,\theta)}{\partial(x,z)}=\frac{\partial\psi\partial\theta}{\partial x\partial z}-\frac{\partial\theta\partial\psi}{\partial x\partial z}= \tag{5}$$

$$\frac{\left(1+a^2\right)}{a}2\kappa\left(\pi aH^{-1}\right)\left(\pi H^{-1}\right)\frac{\Delta\mathrm{T}R_c}{\pi R_a}XY\cos\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)\cos\left(\pi H^{-1}z\right)\cos\left(\pi aH^{-1}x\right)-$$
$$\frac{\left(1+a^2\right)}{a}2\kappa\sqrt{2}\left(\pi H^{-1}\right)XZ\cos\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)\cos\left(2\pi H^{-1}z\right)+$$
$$\frac{2\Delta\mathrm{T}R_c}{\pi R_a}\frac{\left(1+a^2\right)}{a}\kappa\left(\pi aH^{-1}\right)\left(\pi H^{-1}\right)XY\sin\left(\pi aH^{-1}x\right)\sin\left(\pi H^{-1}z\right)\sin\left(\pi aH^{-1}x\right)\cos\left(\pi H^{-1}z\right)$$

Now if we take the trigonometric term from the second line from equation (5) we observe that this can be reduced to

$$\cos\left(\pi H^{-1}z\right)\sin\left(\pi H^{-1}z\right)\cos^2\left(\pi a H^{-1}x\right)=\frac{1}{2}\sin\left(2\pi H^{-1}z\right)\left(\frac{1-2\sin\left(2\pi a H^{-1}x\right)}{2}\right)=$$

$$\frac{1}{4}\sin\left(2\pi H^{-1}z\right)-\frac{1}{2}\sin\left(2\pi H^{-1}z\right)\sin\left(2\pi a H^{-1}x\right)$$

Noticing that the second term of the above equation does not appear in equations (3) and (4) we neglect this term as Lorenz (1963) did in his paper.

The trigonometric term in the third line becomes

$$\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\cos\left(2\pi H^{-1}z\right)=\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\left(2\cos^2\left(\pi H^{-1}z-1\right)\right)=$$

$$2\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\cos^2\left(\pi H^{-1}z\right)-\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)$$

As explained before, Lorenz (1963) omitted terms not involving trigonometric terms as those appear in (3) and (4). So we omit the first term of the above equation.

Finally, we take the term in the last line of equation (5), we see that this is

$$\sin^2\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\cos\left(\pi H^{-1}z\right)=\frac{1}{2}\sin\left(2\pi H^{-1}z\right)\left(\frac{1-\cos\left(2\pi a H^{-1}x\right)}{2}\right)=$$

$$=\frac{1}{4}\sin\left(2\pi H^{-1}z\right)-\frac{1}{4}\sin\left(2\pi H^{-1}z\right)\cos\left(2\pi a H^{-1}x\right).$$

As Lorenz (1963) did, we omit the second term of this equation.

Now putting all the terms together, we obtain

$$\frac{\partial\theta}{\partial t}=-\frac{\partial(\psi,\theta)}{\partial(x,z)}+\frac{\Delta \mathrm{T}}{H}\frac{\partial\psi}{\partial x}+\kappa\nabla^2\theta$$

$$\Rightarrow \frac{\Delta \mathrm{T} R_c}{\pi R_a}\sqrt{2}\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\dot{Y} - \frac{\Delta \mathrm{T} R_c}{\pi R_a}\sin\left(2\pi H^{-1}z\right)\dot{Z} =$$

$$-\frac{\left(1+a^2\right)}{a}2\kappa\left(\pi a H^{-1}\right)\left(\pi H^{-1}\right)\frac{\Delta \mathrm{T} R_c}{\pi R_a}\frac{1}{4}\sin\left(2\pi H^{-1}z\right)XY -$$
$$\frac{\left(1+a^2\right)}{a}\kappa 2\sqrt{2}\left(\pi a H^{-1}\right)\left(\pi H^{-1}\right)\frac{\Delta \mathrm{T} R_c}{\pi R_a}\frac{1}{4}\left[\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)\right]XZ -$$
$$2\frac{\Delta \mathrm{T} R_c}{\pi R_a}\kappa\frac{\left(1+a^2\right)}{a}\left(\pi a H^{-1}\right)\left(\pi H^{-1}\right)\frac{1}{4}\sin\left(\pi H^{-1}z\right)XY +$$
$$\frac{\Delta \mathrm{T}}{H}\frac{\left(1+a^2\right)}{a}\kappa\sqrt{2}\left(\pi a H^{-1}\right)\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)X -$$
$$\kappa\frac{\Delta \mathrm{T} R_c}{\pi R_a}\left[\sqrt{2}\left\{\left(\pi a H^{-1}\right)^2 + \left(\pi H^{-1}\right)^2\right\}\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)Y + \left(2\pi H^{-1}\right)^2\sin\left(2\pi H^{-1}z\right)Z\right]$$

Comparing the coefficients of the trigonometric term $\cos\left(\pi a H^{-1}x\right)\sin\left(\pi H^{-1}z\right)$ we have

$$\frac{\Delta \mathrm{T} R_c}{\pi R_a}\sqrt{2}\,\dot{Y} = \frac{1}{2}\frac{\left(1+a^2\right)}{a}\kappa\sqrt{2}\left(\pi a H^{-1}\right)\frac{\Delta \mathrm{T} R_c}{\pi R_a}\left(2\pi H^{-1}\right)XZ + \frac{\Delta \mathrm{T}}{H}\frac{\left(1+a^2\right)}{a}\kappa\sqrt{2}\left(\pi a H^{-1}\right)X -$$
$$\kappa\sqrt{2}\frac{\Delta \mathrm{T} R_c}{\pi R_a}\left[\left(\pi a H^{-1}\right)^2 + \left(\pi H^{-1}\right)^2\right]Y.$$

After some simplifications we get to

$$\dot{Y} = \frac{dY}{dt} = -\left(1+a^2\right)\kappa\frac{\pi^2}{H^2}XZ + \left(1+a^2\right)\kappa\frac{\pi^2}{H^2}\frac{R_a}{R_c}X - \kappa\left[\left(\pi a H^{-1}\right)^2 + \left(\pi H^{-1}\right)^2\right]Y\ .$$

As we did before we use $t = \tau\frac{H^2}{\kappa\pi^2}\frac{1}{\left(1+a^2\right)}$ and $\frac{dt}{d\tau} = \frac{H^2}{\kappa\pi^2}\frac{1}{\left(1+a^2\right)}$ we obtain, using the chain rule

$$\frac{dY}{d\tau} = \frac{dY}{dt}\frac{dt}{d\tau} =$$
$$\left[-\left(1+a^2\right)\kappa\frac{\pi^2}{H^2}XZ + \left(1+a^2\right)\kappa\frac{\pi^2}{H^2}\frac{R_a}{R_c}X - \kappa\frac{\pi^2}{H^2}\left(1+a^2\right)Y\right]\cdot\left[\frac{H^2}{\kappa\pi^2}\left(\frac{1}{\left(1+a^2\right)}\right)\right]$$

With $\frac{R_a}{R_c} = r$ we finally obtain

$$\dot{Y} = -XZ + rX - Y \,.$$

This is the second equation of the Lorenz equations.

Finally we compare the coefficients of $\sin\left(2\pi H^{-1} z\right)$. This is

$$\frac{\Delta \mathrm{T} R_c}{\pi R_a}\dot{Z} = -\frac{1}{2}\frac{\left(1+a^2\right)}{a}\kappa\left(\pi a H^{-1}\right)\left(\pi H^{-1}\right)\frac{\Delta \mathrm{T} R_c}{\pi R_a}XY - \frac{1}{2}\frac{\Delta \mathrm{T} R_c}{\pi R_a}\kappa\left(\pi a H^{-1}\right)\left(\pi H^{-1}\right)XY +$$
$$\kappa\frac{\Delta \mathrm{T} R_c}{R_a}\left(2\pi H^{-1}\right)^2 Z$$

This simplifies to

$$\dot{Z} = \frac{dZ}{dt} = \left(1+a^2\right)\kappa\frac{\pi^2}{H^2}XY - 4\kappa\frac{\pi^2}{H^2}Z \,.$$

As we did before, using the chain rule with $t = \tau\frac{H^2}{\kappa\pi^2}\frac{1}{\left(1+a^2\right)}$ we have

$$\frac{dZ}{d\tau} = \frac{dZ}{dt}\frac{dt}{d\tau} = \left[\left(1+a^2\right)\kappa\frac{\pi^2}{H^2}XY - 4\kappa\frac{\pi^2}{H^2}Z\right]\cdot\frac{H^2}{\kappa\pi^2\left(1+a^2\right)} = XY - bZ \,,$$

where $b = 4\left(1+a^2\right)^{-1}$. So we find the third equation we required. This is

$$\dot{Z} = XY - bZ \,.$$

The system of the three equations is:

$$\dot{X} = -\sigma X + \sigma Y \qquad (6)$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ.$$

The variables X, Y, Z represent respectively the rate of convective overturning, how the temperature varies in the horizontal direction and how temperature varies in the vertical direction.

The three real and positive parameters are:

σ which is called the Prandtl number and it represents the ratio of the viscosity of the fluid of a material to its thermal conductivity, r is the Rayleigh number and it represents the temperature difference between the top and bottom of the system and b represents a ratio of width to height of the container being considered.

Lorenz used $\sigma = 10$, $r = 28$ and $b = 8/3$. In the next chapter we will study what happens if we change the values of these parameters

CHAPTER 3

THE PARAMETER r

After deriving the equations, the next step is to analyse what happens when we vary one of the parameters in the Lorenz equations. We will investigate what happens when we fix the parameters b and σ and vary r. As we said in the previous chapter this parameter is the Rayleigh number. Here we will study what we can observe for small and large r, say $r >> 1$. We will also examine if the system has stable or unstable solutions and in result if periodic orbits exist. Finally, this chapter will state if this system can ever reach chaotic behaviour and if so for what value of r.

3.1 Solutions of the Lorenz Equations

Here we consider a solution of the Lorenz equations, say $X(t), Y(t), Z(t)$ and we are going to study how a nearby solution, say $X(t)+x_0(t)$, $Y(t)+y_0(t)$, $Z(t)+z_0(t)$ will behave. $x_0(t)$, $y_0(t)$ and $z_0(t)$ are small perturbations which depend on τ and so we end up with the following equation.

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}^{\cdot} = \begin{bmatrix} -\sigma & \sigma & 0 \\ -Z+r & -1 & -X \\ Y & X & -b \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \tag{7}$$

Equation (7) allows us to study stability properties of the solution $X(t), Y(t), Z(t)$.

The coefficients of these equations are not all constants. Some of them are variables and change as time changes.

To calculate the (change of phase space) volume of the system we need to calculate the divergence of the vector field. That is, we calculate the trace of the Jacobean

$$\frac{\partial}{\partial x}(-\sigma x+\sigma y)+\frac{\partial}{\partial y}(rx-y-xz)+\frac{\partial}{\partial z}(-bz+xy)$$
$$=-\sigma-1-b=-(\sigma+1+b).$$

Because σ and b are positive numbers, the change in volume is negative and so the volume is contracting. This means all attractors have zero volume. We will discuss this later on in the chapter in more detail.

3.2 Different Values of r

In this section we will investigate what happens for different values of r. At this point we want to study simple stationary. That is solutions that do not depend on time. We shall work within a simply connected and closed region R. This region contains a set $A=\cap_{t\geq 0}\phi_t(R)$. All trajectories will eventually enter this set and will never leave. Therefore this set is an attracting set and has bounded solutions.

We now study the stability of the trivial fixed point $X=Y=Z=0$. For a steady state solution of Lorenz equations we require $X=Y=Z=0$. This is the state where there is no convection. That is, as we said previously, there is no heat flow from hot areas to cold areas.

So for this situation the characteristic equation of the matrix (7) is

$$\det\left(\begin{bmatrix}-\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right)=0$$

$$\begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{pmatrix} = 0$$

$$-(\sigma+\lambda)[(1+\lambda)(b+\lambda-0)]+\sigma r(b+\lambda)+0=0$$
$$-(b+\lambda)\left[\lambda^2+\lambda(\sigma+1)+\sigma(1-r)\right]=0$$

$$(b+\lambda)\left[\lambda^2+\lambda(\sigma+1)+\sigma(1-r)\right]=0\,. \tag{8}$$

Let us take equation (8) and we shall fix σ and b and vary r. First we shall take the case when $r>0$. So we have

$(b+\lambda)\left[\lambda^2+\lambda(\sigma+1)+\sigma(1-r)\right]=0$. This is

$$\lambda=-b \quad \text{or} \quad \lambda_{1,2}=\frac{-(\sigma+1)\pm\sqrt{(\sigma+1)^2-4\sigma(1-r)}}{2}=\frac{-(\sigma+1)\pm\sqrt{(\sigma-1)^2+4\sigma r}}{2}\,.$$

So for $r>0$ the equation has three solutions and all of them are real. The solutions are

$$\lambda_1=-b,\quad \lambda_2=\frac{-(\sigma+1)+\sqrt{(\sigma-1)^2+4\sigma r}}{2},\quad \lambda_3=\frac{-(\sigma+1)-\sqrt{(\sigma-1)^2+4\sigma r}}{2}$$

$\lambda_1=-b$ is negative as b is a positive constant. λ_3 is also negative as the square root is positive and $(\sigma+1)$ is also positive. Since they have a negative sign the expression is negative. λ_2 is positive as $\sqrt{(\sigma-1)^2+4\sigma r}>(\sigma+1)$.

Now, if $r<1$ the three roots are all negative and the origin $(0,0,0)$ is a stable point.

In the next chapter we will see what is happening to the system in the range $r>1$.

CHAPTER 4

NONTRIVIAL FIXED POINTS

In this chapter we will mostly consider the region $r > 1$ and the non trivial fixed points occurring in this region. We note that the nontrivial fixed points correspond to stationary convection. Also, as a remark, a fixed point is a time independent solution.

4.1 The Region $r > 1$

If we take the region $r > 1$ then the matrix (7) has a characteristic equation

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2\sigma b(r - 1) = 0 \qquad (9)$$

Here, we first have to compute the nontrivial fixed point, say X^*, Y^*, Z^*. These numbers are computed from the equations of motion, i.e. equations (6), letting $\dot{X} = \dot{Y} = \dot{Z} = 0$. Then equations (6) give three equations for X^*, Y^*, Z^*, namely $-\sigma^* X^* + \sigma^* Y^*$, $-X^* Z^* + r^* X^* - Y^*$, $X^* Y^* - b^* Z^*$. Solving these equations for X^*, Y^*, Z^* we obtain two extra solutions for the Lorenz equations. We will call them s_1 and s_2. These solutions are fixed points and are $\left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$.

Now we investigate the stability of this fixed point using equation (7) and the associated characteristic equation which is equation (9), for the nontrivial fixed point.

In this case the origin is a saddle point. Such points are unstable but they have a stable and unstable direction.

At $r = \dfrac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)}$ we have

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + \left[\frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)} + \sigma\right]b\lambda + 2\sigma b\left[\frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)} - 1\right] = 0$$

$$(\sigma - b - 1)\lambda^3 + (\sigma - b - 1)(\sigma + b + 1)\lambda^2 + (\sigma(\sigma + b + 3) + \sigma)b\lambda + 2\sigma b(\sigma(\sigma + b + 3) - 1) = 0.$$

At this r value we obtain an imaginary pair of eigenvalues, which indicates the second instability. The eigenvalues are

$$\lambda_1 = -(\sigma + b + 1), \;\; \lambda_{2,3} = \pm i\sqrt{2\sigma(\sigma+1)/(\sigma-b-1)}\,.$$

We can call this critical value of r, r_h. Then for $r = r_h$ we have the so-called Hopf

bifurcation (Guckenheimer, Holmes, 1983, p.93). For $r = r_h$ the fixed points become unstable.

This value of r is a critical value because it shows us if we have stability or instability of steady convection. If $\sigma < b+1$ there exists a stable steady convection and for $\sigma > b+1$ there exists unstable steady convection.

If we now take $\sigma = 10$ and $b = 8/3$ then we have

$$r = r_h = \frac{\sigma(\sigma+b+3)}{\sigma-b-1} = \frac{10\left(10+8/3+3\right)}{10-8/3-1} \approx 24.74\,.$$

For $1 < r < 24.74$ the two solutions s_1 and s_2 are stable. For $r > 1.346$ with $\sigma = 10$ and $b = 8/3$ the eigenvalues become complex and their real parts are negative.

For $r > 24.74$, s_1 and s_2 are not stable. For this condition there is one real eigenvalue which is negative and two complex eigenvalues. The real parts of the complex eigenvalues are both positive.

Lorenz (1963) chose a slightly larger value of r. He fixed $r = 28, \sigma = 10, b = 8/3$ and then used numerical integration for equations (5) with initial condition the origin $(0,0,0)$.

4.2 Time Dependent non Stationary Solutions

This section will concentrate mostly on the time dependent solutions of the Lorenz equations at the standard parameter values, namely the so called Lorenz attractor. As we have seen in the previous sections the Lorenz equations acknowledge stable fixed point solutions for $r < r_h$. If $r > 1$ we obtain a non convective state, for $1 < r < r_h$ the recently generated stable fixed point corresponds to time independent convection. At $r = r_h$ a Hopf bifurcation takes place and in our case no stable time periodic state is generated for $r > r_h$. This is a so called sub critical Hopf bifurcation. In fact for $r > r_h$ aperiodic time dependence is observed and the corresponding attractor has some quite 'strange' figures.

Let us first say a few more details about this so called strange attractor. According to Lorenz (1979) an attractor has basically two sheets and following an orbit these sheets seem to join together. From this, take as a accurate statement, it would follow that pairs of orbits should join together as well and this is not possible. Therefore one sheet is a combination of two sheets which are very close to each other. Consequently, repeating this reasoning, two sheets that are joint together are composed of four sheets four sheets have eight sheets and so on. Hence it appears that there are an infinite number of sheets. The closed set of this kind of sheets is called a strange attractor.

Now, let us examine in slightly more detail how such a strange attractor can occur in a dynamical system. For a value $r \approx 13.926$ there is a global bifurcation a so called homoclinic explosion. This means that the system generates an orbit which links the saddle point at the origin with itself. Roughly speaking the period of such an orbit is infinite. (Sparrow, 1982, p.26).

One can show that for $r < r_a$, because of the homoclinic explosion the equations of motion create an infinite number of periodic orbits which are all unstable.

We note that the two stationary points s_1 and s_2 are still stable at $r \approx 24.06$ and any trajectory starting near s_1 and s_2 will tend to s_1 or s_2. However if we cross the threshold $r = r_h$ no longer a stable fixed point exists and the time dependent solution will ‘bounce’ between all the unstable periodic points

Let us now investigate what happens to the strange attractor in the area $r. > r_a$. The study will concentrate near the area $r = 28$. This is the r value Lorenz (1963) considered in his paper and from this value he discovered the chaotic behaviour of the Lorenz attractor but this will be discussed in more detail later on. To study aperiodic time dependence it is convenient to use a kind of symbolic coarse grained description, a so called symbolic dynamics. One associates a symbol sequence to the orbit under consideration. For the Lorenz system it is convenient to generate the symbol sequence according to the revolution of the two different lopes of the chaotic attractor. One uses two symbols for the coarse grained dynamics. The first is when we have a $'x'$ when the trajectory goes around the point s_2 $(x > 0)$ and we have a $'y'$ when the trajectory goes around s_2 $(x < 0)$. At $r = 28$ we observe that for the sequences $'x'$ and $'y'$ the periodic orbits and trajectories are not in the set. The periodic orbits and trajectories are missing because they are separated from the attractor in homoclinic explosions. These homoclinic explosions occur at a large number of r values. In addition there is an infinite number of different attractors in all the neighbourhoods of all the values of r. This situation concerns the trajectories that stay forever in the attractor of the orbit under consideration. In this strange attractor the orbits and trajectories disappear in homoclinic explosions from the set, they do not leave. This does not happen for all the aperiodic trajectories which disappear. So near the value $r = 28$ there are no stable periodic orbits.

4.3 Period Doubling Scenarios

This section will give attention to stable periodic orbits of the Lorenz attractor. We will conceder values of r much larger than $r = 28$. Here we will see three intervals of r values and investigate their periodic behaviour taking in mind that there are other such intervals smaller than these considered here.

The first interval to analyse is $99.524 < r < 100.795$. According to Sparrow (1982, pp.56-58) there exists a stable periodic orbit in $99.98 \le r \le 100.795$. We still consider the two sequences with $'x'$ and $'y'$ where we have mentioned before. Because of the symmetry of the Lorenz attractor there are stable orbits and these orbits seem to attract trajectories.

For $99.629 < r < 99.98$ there is another stable periodic orbit. (Sparrow, 1982, p.56). As r tends to 99.98, the loops of the orbit will come together at the end. Here we have a period doubling bifurcation. That is when the stable periodic orbit becomes unstable but still exists and it is replaced by a stable periodic orbit with a double period compared to the previous one. There is another stable orbit in the interval $99.547 < r < 99.629$. As the value of r decreases more we observe more period doubling bifurcations.

Moreover, we observe that just below $r = 100.795$ there is a pair of non-symmetric stable periodic orbits and above this value these orbits cannot be seen any more and it is suggested that they disappear in saddle-node bifurcations. (Sparrow, 1982, p.58). This bifurcation happens when on one side of r there is a non-stable periodic orbit and a stable periodic orbit. When the parameter increases or decreases in the direction of r, the two orbits move very close and their periods rends towards the same value. (Sparrow, 1982, p.52). For $r < 99.524$ there exists an infinite number of non-stable periodic orbits.

The second interval is $145 < r < 166$ (Sparrow, 1982, p.59). Here things happen in a similar manner as the above situation but at $r = 160$ there is a symmetric periodic orbit which is stable in the range $154.4 < r < 166.07$. At $r \approx 154.4$ there is a symmetric saddle-node bifurcation. This kind of bifurcation has occurred in the Lorenz system when two saddle non-symmetric orbits and a non-symmetric orbit loses its stability to a pair of non-symmetric orbit combined together to create a stable symmetric orbit. At this value of r the orbit loses its stability to a pair of non-symmetric periodic orbits. (Sparrow, 1982, p.53). These orbits are stable in $148.2 < r < 154.4$. Under the value of $r = 154.4$ this orbit also exists and it is non-stable.

At $r \approx 148.2$ there is period doubling bifurcation. At a value of r greater than $r = 166.07$ we have intermittent chaos (Sparrow, 1982, pp.62-63). Here, although the stable symmetric orbits does not exist, trajectories look to move close to it and then they wonder off and have chaotic behaviour for a short time and then return to periodic behaviour. The length of the chaotic intervals gets bigger as the value of r increases.

The final interval we will mention is $r > 214.364$ (Sparrow, 1982, pp.66-69). Here we have the same circumstances as the previous interval with the difference that the last stable symmetric orbit continues to exist for $r > 313$. Because this time r goes up to infinity and there is no upper limit we cannot observe chaotic behaviour.

In the next chapter we will talk about what we mean by sensitive dependence on initial conditions and how the Lorenz attractor posses chaotic behaviour.

CHAPTER 5

THE BUTTERFLY EFFECT

Lorenz equations were an attempt to describe the atmosphere and weather conditions in a simple manner but then realised that these equations can be formulated to describe a laboratory water wheel. He soon discovered that the Lorenz equations possessed chaotic behaviour as we have seen earlier.

The term ‘butterfly effect’ was first conceived by Edward Lorenz in his effort to describe sensitive dependence on initial conditions and chaos theory. At first the term butterfly effect was used only in meteorology to describe the chaotic behaviour of the weather. Later on chaos theory and the butterfly effect were used in a greater range in science.

Edward Lorenz accidentally discovered the chaotic behaviour of the atmosphere when he was working on a computer program for weather forecasting. Lorenz wished to look again at a specific sequence. Instead of starting from the beginning he decided to start from somewhere in the middle of this sequence because he didn’t want to waste much time. He ran the program and left for a while. When he came back he discovered that the sequence didn’t progress in the same way as the previous one. At the beginning he thought that there was a problem with the hardware but he soon discovered what was going on. The first sequence was run with initial conditions 0.506127. In the second sequence Lorenz typed only 0.506. The difference between the two initial conditions was only 0.000127, which was something very small and yet the two sequences came out so differently. This is how Lorenz discovered the chaotic behaviour of the atmosphere and he started a new search to discover chaotic systems.

It is so amazing that such a tiny difference can cause so huge changes in the long run behaviour. Following this, Lorenz wrote a paper in 1972 with the title ‘Predictability: Does the Flap of a butterfly’s wings in Brazil set off a Tornado in Texas?’

It is surprising that something that seems so small such as the flapping of a butterfly's wings and cannot be felt by anyone, in the long run behaviour can create something as big as like a tornado in the other side of the earth. In the world of meteorologists and other scientists, this phenomenon is also known as sensitive dependence on initial conditions. This is related to chaos theory. A very small variation in initial conditions may alter the behaviour of a complex system, such as the weather in the far future as it is not possible to measure the initial conditions of the atmosphere accurately. The fact that such a miniscule event can create huge differences is one of the reasons we cannot predict weather forecasting accurately for more than one or two weeks.

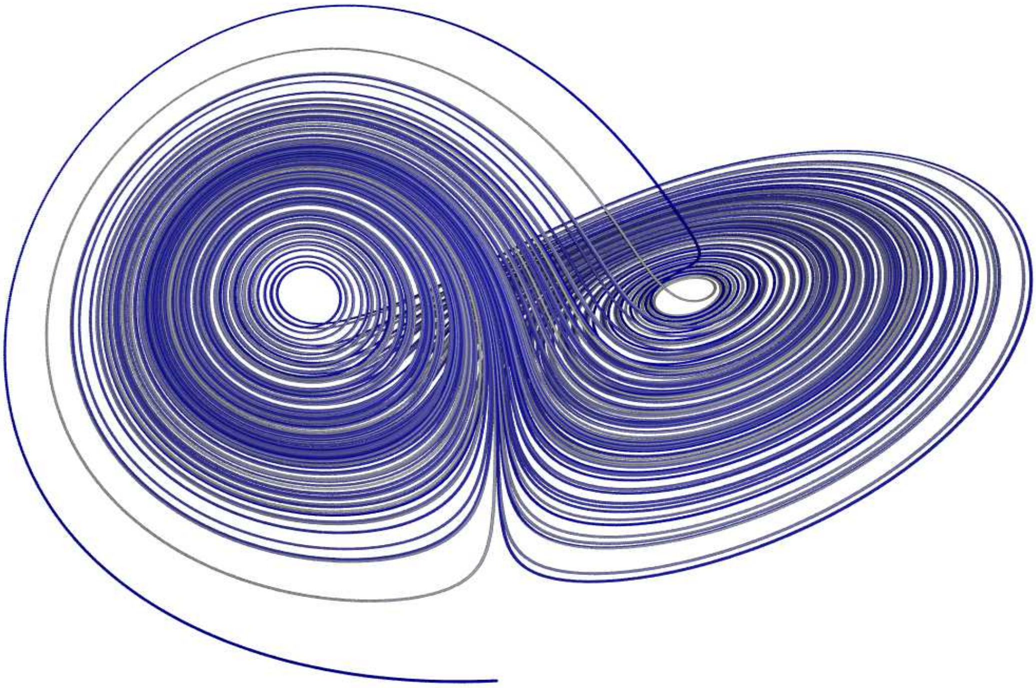

Figure 4.1: The Lorenz attractor. Lorenz equations are always following a spiral. The equations do not settle down to one point and they do not have periodic behaviour as the system never repeats itself. (Valbonesi, I.,2008 April 17).

CHAPTER 6
CONCLUSION

We have seen and discussed about the Lorenz equations. These equations where studied by many scientists such as Haken, Knoblock, Malkus, Yorke and many more (Sparrow, 1982, p.4). We have also seen that the Lorenz equations behave chaotically if we increase the value of the parameter r . At this point, we note that all the results for stable or unstable points, periodic orbits and bifurcations are made using numerical integration with the use of a computer.

Lorenz work to try to find the reason we cannot predict the weather accurately for a long period of time and the butterfly effect was at first only considered by meteorologists. He could never believe that the butterfly effect and chaos theory were impacted by other fields of science such as Mathematics, Physics, Biology. (MIT, 2008, April 16). Lorenz (1972) said that we cannot blame the structure of the weather if we cannot have precise forecasts for more than three weeks. It is our 'incomplete knowledge' of physical principles and some approximations in order to formulate these principles.

Bibliography

Carter, S. (1996). *The Lorenz Mode*l. Retrieved 05/06/2009 from:

http://www.taygeta.com/perturb/node2.html

Ho, A. (1995). Lorenz Attractor. Retrieved 05/06/2009 from:

http://www.zeuscat.com/andrew/chaos/lorenz.html

MIT (2008, April 16). Edward Lorenz, father of chaos theory and butterfly effect, dies at 90. *MIT TechTalk.* 52 (24). Retrieved 05/06/2009 from:

http://web.mit.edu/newsoffice/2008/obit-lorenz-0416.html

Guckenheimer, J. Holmes, P. (1983). *Nonlinear Oscillations Dynamical Systems and Bifurcations of Vector Fields.* New York: Springer-Verlag

Lorenz, E, N. (1963). Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences*, 20, pp.130-141

Lorenz, E, N. (1972). Predictability; Does the Flap of a Butterfly's wings in Brazil Set Off a Tornado in Texas? *American Association of the Advancement of Science*

Lorenz, E, N. (1979). On the Prevalence of Aperiodicity in Simple Systems. *Massachusetts Institute of Technology*. Berlin: Springer, 755, pp.53-75

Saltzman, B. (1962). Finite Amplitude Free Convection as an Initial Value Problem. *Journal of the Atmospheric Sciences*, 19, pp.329-341

School of Mathematics and Statistics, University of St Andrews, Scotland. (2008). *Edward Norton Lorenz*. Retrieved 10/08/2009 from:

http://www.gap-system.org/~history/Biographies/Lorenz_Edward.html

Sparrow, C. (1982). *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. New York: Springer- Verlag

Thompson, S, M, T, Stewart, H, B. (1986). *Nonlinear Dynamics and Chaos.* Chichester: John Wiley and Sons

Tucker, W. (NA). *The Lorenz Attractor Exists: An auto-Validated Proof*, Cornell University. New York: Ithaca

Valbonesi, I. (2008 April 17). *Edward Lorenz: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas***.** Ecopolis. Retrieved 10/08/2009 from:

http://www.ecopolis.org/edward-lorenz-does-the-flap-of-a-butterflys-wings-in-brazil-set-off-a-tornado-in-texas/

Williams, R.F. (1979). The Structure of Lorenz Attractors. *Math IHES*, 50, pp.101-152