Linear stability of oscillators subjected to time delay

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1 Introduction

Dynamical systems can be described by various types of equations of motion. Which type applies may depend on the general context. For example:

(i) Ordinary differential equation is a differential equation in which the unknown function is a function of a single independent variable. It normally describes the motion in a system with few degrees of freedom like pendula.

(ii) Partial differential equation is a differential equation in which the unknown function is a function of multiple independent variables and their partial derivatives. Thus it is used to formulate problems involving functions of several variables such as the propagation of sound, electrodynamics or fluid flow.

(iii) Stochastic differential equation is a differential equation in which one or more of the terms are a stochastic process. SDE are important whenever there is any kind of uncertainty.

(iv) Differential Algebraic equation is a differential equation comprising differential and algebraic terms given in implicit form. Such models are used for studying fundamental problems in abstract mathematics.

(v) Delay differential equation (DDE) is an equation where the evolution of the system at a certain time depends on the state of the system at earlier time. They are relevant when there is time delay.

In this thesis I will be studying the delay differential equations. We shall study its stability properties using analytical computation. In the next three chapters we inspect the stability properties of the first order delay differential equation and the second order delay differential equations. Before we do so, let me first make clear what we actually know about delay differential equations. I will also briefly mention the history of delay differential equations and its various applications.

Delay differential equations (also known as delayed system) are type of differential equation in which derivative of the unknown function at a certain time is given in terms of the values of the function at previous times; they are a special case of a class of differential equations called functional differential equations.

A general form of the time delay differential equation for $x \in \Re$ is given by:

$$\dot{x}(t) = f(t, x(t), x_t)$$

where $x_t = \{x(\tau) : \tau \le t\}$ represents the trajectory of the solution in the past.

Delay Differentials equations could be continuous as well discrete DDEs. A first order continuous delay differential equations have the following general form:

$$\dot{x}(t) = f(t, x(t), \int_{-\infty}^{0} x(t+\tau) d\mu(\tau))$$

A first order discrete delay differential equations could be presented in general as:

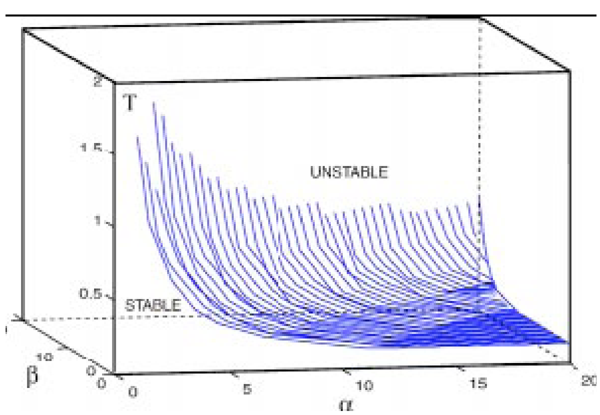
$$\dot{x}(t) = f(t, x(t), x(t-\tau_1), \ldots\ldots.., x(t-\tau_n)) \qquad \text{for } \tau_1 > \ldots. > \tau_n \ge 0$$

The essential difference in ODE and DDE is in the initial value problem. For ODE it is given by a point whereas for DDE it is defined by a function. In particular delay differential equations are defined on infinite dimensional phase space, contrary to the other differential equations. Thus for linear systems the corresponding characteristic equations will have an infinite number of solutions as well, which complicates the analytical computation.

DDE were initially introduced in the eighteenth century by Laplace and Condorcet. Nothing was done throughout the nineteenth century, but only after Second World War. The basic theory concerning the stability system of DDEs was developed by Pontryagin in 1942. Also important works of DDE have been written by Bellmann and Woke 1963 and Hale in 1977.

DDE are notoriously difficult to solve by analytical methods. Thus in application one often applies numerical methods like Runge-Kutta method [3], asymptotic solutions such as perturbation methods, analytical method like Lambert W function and graphical tools. The situation becomes more tractable when studying the linear system. Then the computation of time dependent solution can be reduced to linear or non linear eigenvalue problems.

Several attempts have been made to find an analytical solution for delay differential equations by solving its transcendental characteristic equation under different conditions. Lambert W function (Omega function) is a useful analytical method to solve this transcendental equation of the DDEs. The advantage of this method is that the solution obtained can be compared to the general solution form of ordinary differential equation. Every function of $w(s)$ that satisfies $w(s)\exp(w(s)) = s$ is called a Lambert W function [3]. Such method will give us the stability region of a delay differential equation. For example the diagram below shows stability criteria for a generalized first order DDE [3]



Systems of delay differential equations now occupy a place of central importance in all areas of science particularly, in the biological sciences and also in engineering and economics. An important application of delay differential equations is the reduction of chatter in machining processes. It is also used in population models, economic systems, remote control, urban traffic, electric transmission line, heat exchangers and control systems for nuclear reactors with time delay and manufacturing system.

In this project we will be analyzing the stability of a first order delay differential equation and a second order delay differential equation. Instead of the Lambert W function method we will use less complicated method; which will give us an estimate stability region.

2 Linear first order delay differential equation

The linear first order of delay differential equations are the simplest differential delay equations, even though in general they are quite difficult to solve. But analytical results can be obtained for delay differential equations. Within this chapter we will be studying the linear first order delay differential equation. The linear first order delay differential equation we will be looking at has the following form:

$$\dot{x}(t) = \alpha x(t) + \beta x(t-\tau) \qquad (1)$$

where α and β are real valued parameters and $\tau \geq 0$ denotes the time delay.

Mainly we shall investigate the stability properties of equation (1) with and without time delay. Hence we want to identify those values of α and β such that all the solutions of equation (1) tend to zero when t goes to infinity. Solution can be written down in terms of the exponential function:

$$x(t) = \exp(\lambda t) \qquad (2)$$

where λ are the eigenvalues. But any solution of our linear system can be written as a linear superposition of such exponential; therefore such a choice does not limit generality of our arguments. The general solution can be written as a sum of such exponential type:

$$x(t) = \sum_{i=1}^{\infty} c_i \exp(\lambda_i t) \qquad (3)$$

Thus the stability problem now reduces to the question whether the alleged characteristic values of λ have positive or negative real part. The stability requires that all solutions satisfy the condition that $\operatorname{Re}\lambda < 0$. Which means that if all exponential decay ($\operatorname{Re}\lambda_i < 0$) then the general solution decays as well. Another way of saying is that equation (1) will be asymptotically stable if and only if all the roots of its characteristic equation (cf. section 2.1) have negative real roots.

If there exist a root of the characteristic equation with a zero or positive real part then the trivial equation (1) is not asymptotically stable.

If the region is unstable, then there is at least one eigenvalue say λ_1 with positive real part such that:

$$x(t) = c_1 \exp(\lambda_1 t) + \sum_{i=2}^{\infty} c_i \exp(\lambda_i t) \tag{4}$$

Then the general solution contains at least one exponential which grows when t goes to infinity where $c_1 \neq 0$. Thus the unstable case requires the condition $\operatorname{Re}\lambda > 0$. For simplicity we will use equation (2) to obtain the characteristic equation rather than equation (3).

2.1 Characteristic equation

Equation (1) can be reduced to an algebraic equation using equation (2). Using equation (2) implies that:

$$x(t-\tau) = \exp(\lambda(t-\tau)) \tag{5}$$

$$\dot{x}(t) = \lambda \exp(\lambda t) \tag{6}$$

Then equation (1) becomes:

$$\lambda \exp(\lambda t) = \alpha \exp(\lambda t) + \beta \exp(\lambda(t-\tau))$$

$$\lambda = \alpha \frac{\exp(\lambda t)}{\exp(\lambda t)} + \beta \frac{\exp(\lambda(t-\tau))}{\exp(\lambda t)}$$

$$\lambda = \alpha + \beta \exp(-\lambda\tau) \qquad \text{where } \tau \geq 0 \tag{7}$$

which is the characteristic equation of the linear differential difference equation; also called the transcendental eigenvalue equation. Such equation has normally infinitely many solutions. For later purpose it will be convenient to rewrite equation (7) using the following abbreviations:

$$z = \lambda\tau\,, \quad a = \alpha\tau\,, \quad b = \beta\tau \tag{8}$$

where $\tau > 0$. Then equation (7) simplifies to:

$$z = a + b\exp(-z) \tag{9}$$

We want to investigate the stability properties of equation (1). The true boundary is obtained from the parametric representation of the curve when $\lambda = i\varphi$ in equation (7). But within this project we shall establish an estimate of the stability domain. Solving the transcendental equation (9) for z gives a criterion for the stability of the equation. From equation (9) we need to figure out the part of the (a,b) parameter plane which admits solutions with negative real part only. Thus the region where the real part is negative determines a stable region. The boundary of the stability domain is determined by the condition that the characteristic equation possesses either a solution $z = 0$ or a solution with vanishing real part $z = i\varphi$ where $\varphi \in \Re$.

2.2 Case without time delay

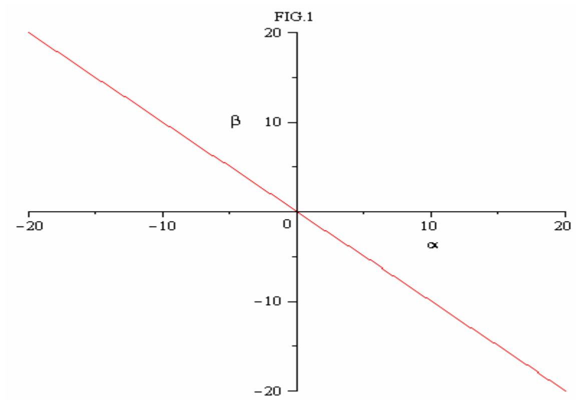
To illustrate some basic idea, lets focus on the trivial case without time delay i.e. the case where $\tau = 0$. Then the characteristic equation (7) simplifies to:

$$\lambda = \alpha + \beta \qquad (10)$$

This means there is just one single real solution. As we mentioned before when $\operatorname{Re}\lambda < 0$, it yields a stable solution. If we let $\operatorname{Re}\lambda < 0$ then that implies that $\alpha + \beta < 0$. Therefore the region $\alpha + \beta < 0$ is stable while the region $\alpha + \beta > 0$ is unstable. The stability domain can easily sketched in the (a,b) parameter plane (cf. figure 1).

The boundary of the stability domain is obviously determined by the solution of the equation $\alpha + \beta = 0$. Therefore even without solving the characteristic equation for λ one could determine one of the boundaries of the stability by selecting $\lambda = 0$. From Figure 1 the diagonal line $a = -b$ gives a boundary of the stability domain and the region below that line is stable.

Figure 1: The stability domain when $\tau = 0$:



2.3 Boundaries of the stability domain

The transcendental equation (9) is quite difficult to solve for z. However the boundaries of the stability domain are determined by the condition that $\mathrm{Re}\,\lambda$ changes sign, i.e. $\mathrm{Re}\,\lambda = 0$. The $\mathrm{Re}\,\lambda$ changes sign by the condition mentioned in the previous paragraph. The two conditions are when:

- A real value changes sign i.e. $z = 0$ on the boundary. (*)
- A complex conjugated pair of solutions crosses the imaginary axis i.e. $z = i\varphi$ and where $\varphi \in \Re$ denotes the imaginary part of the solution. (**)

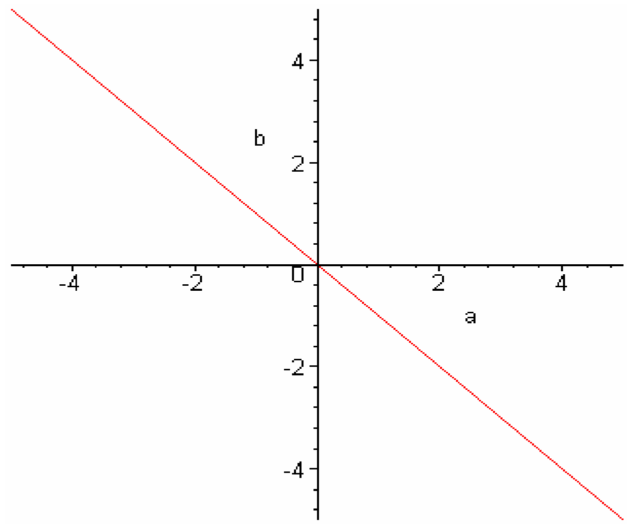
In this paragraph we will examine both cases separately and each case more detailed.

Consider the first condition represented by (*) where the boundary of the stability domain is determined by the condition $z = 0$. This changes equation (9) to:

$$0 = a + b \qquad (11)$$

This stability region is the same as the one in the previous section where we considered the case without time delay; so equation (11) gives us the same boundary. The boundary is displayed in figure 2. Similar to the case without time delay the region below the boundary line is stable.

Figure 2: The stability domain when z = 0:



Now let's consider the second condition which is represented by (**). The boundaries domain are obtained by the condition $z = i\varphi$ where $\varphi \in \Re$. This condition changes equation (9) as follows:

$$i\varphi = a + b\exp(-i\varphi)$$

$$i\varphi = a + b\cos\varphi - ib\sin\varphi \tag{12}$$

Comparing the real and imaginary part on both sides of the equation we obtain:

$$0 = a + b\cos\varphi \tag{13}$$

$$\varphi = -b\sin\varphi \tag{14}$$

Rearranging equation (13) and equation (14) we get:

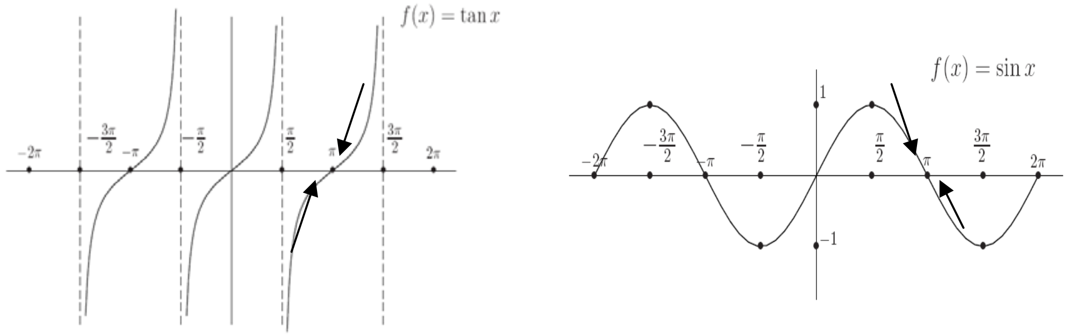
$$a = \frac{\varphi}{\tan\varphi}, \qquad b = -\frac{\varphi}{\sin\varphi} \tag{15}$$

Equation (15) determines the parametric representation of curves in the (a,b) plane.

The expressions which determine a and b develop singularities for $\varphi = k\pi$ where $k = 0,1,2,\ldots.$, since trigonometric functions (cf. fig 3) appear in the denominator of equation (15), thus these equations determine a family of distinct curves (cf. fig 5). So each interval $\varphi \in (k\pi,(k+1)\pi)$ gives rise to a different branch.

A complex conjugated pair of eigenvalues $i\varphi$ and $-i\varphi$ cross the imaginary axis. Hence $z=-i\varphi$ also determines the boundary of the stability domain. But if we put $z=-i\varphi$ into equation (9) we get the same a and b values as in equation (15). Therefore we omit the case when $z=-i\varphi$.

Figure 3: The Trigonometric functions:



In figure 3 The “arrows” indicate whether the trigonometric functions approach zero from above or below when $\varphi \to \pi$.

Investigating each of these curves separately, we observe a pattern on how each curve are represented on the (a.b) parameter plane (cf. figure 5).

Consider the first branch where $\varphi \in (0,\pi)$. In the limit $\varphi \to 0$ the coordinates a and b both tend to a finite limit, since both the numerator and denominator tend towards zero.

Application of L'Hôpital's rule yields the finite limit of a and b i.e.

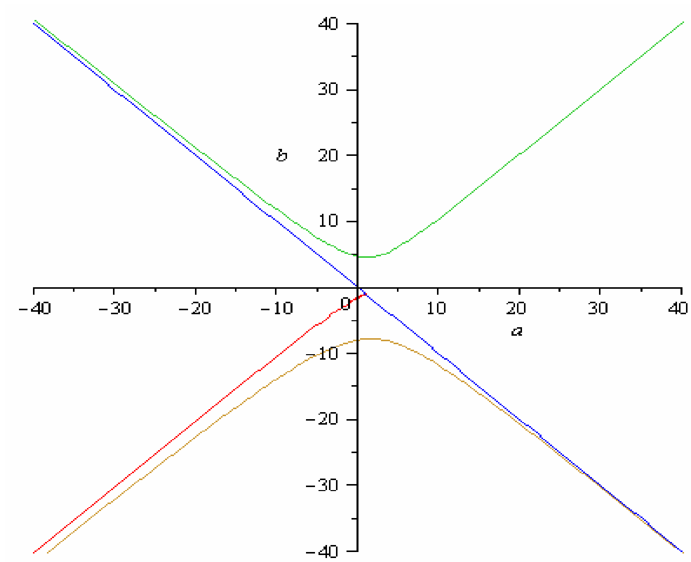
$$\lim_{\varphi\to 0} a = \lim_{\varphi\to 0}\frac{\varphi}{\tan\varphi} = \lim_{\varphi\to 0}\frac{1}{\sec^2\varphi} = 1$$

$$\lim_{\varphi\to 0} b = \lim_{\varphi\to 0} -\frac{\varphi}{\sin\varphi} = \lim_{\varphi\to 0}\frac{-1}{\cos\varphi} = -1 \qquad (16)$$

Hence when $\varphi \to 0$ then $a \to 1$ whereas $b \to 1$ i.e. it touches at the line $a=-b$ when $a=1$ and $b=-1$. Therefore this branch terminates on the boundary given by $z=0$. In the limit $\varphi \to \pi$ both a and b tend to $-\infty$. Since $\tan\varphi$ approaches zero from below as $\varphi \to \pi$ and $\sin\varphi$ approaches zero from above as $\varphi \to \pi$ (cf. fig 3). Hence both parameters a and b diverge.

Now let's consider the second branch where $\varphi \in (\pi, 2\pi)$. In the limit $\varphi \to \pi$ both a and b both tend to ∞. Because $\sin\varphi$ approaches zero from below and $\tan\varphi$ from above where $\varphi > \pi$. For the limit $\varphi \to 2\pi$ a tends to $-\infty$ and b tends to ∞ where $\varphi < 2\pi$. Since $\sin\varphi$ approaches zero from below and $\tan\varphi$ from below. Those two branches when $\varphi \in (0, \pi)$ and $\varphi \in (\pi, 2\pi)$ are sketched in figure 4.

Figure 4: The first two branches of the parametric curves of the first order delay differential equation



where:

- Blue line represent boundary determined by the condition $z = 0$.
- Red curve is the boundary determined by the condition $\varphi \in (0, \pi)$. From figure 4 you can that when $\varphi \to 0$, that the curve terminates on the boundary $z = 0$ at (-1,1) and when $\varphi \to \pi$ both a and b tend to $-\infty$ i.e. $(-\infty, -\infty)$.
- Green curve is determined by the condition when $\varphi \in (\pi, 2\pi)$.
- Yellow curve is determined by the condition when $\varphi \in (2\pi, 3\pi)$.

At first after all this calculation I have sketched the two branches by hand. Then I used Maple to draw the curves more accurate; so I obtained figure 4.

The other branches can be analyzed in similar way where $\varphi \in (k\pi,(k+1)\pi)$ and $k = 0,1,2,3,....$. Whether the value of k is odd or even it is very important. Since it determines the position of the curves in the (a,b) parameter plane. If k is even then the curve is in the region below a-axis and if k is odd it is in the region above a –axis. The limits are always infinite and asymptotes can be worked out from equation (15).

The asymptote is the ratio of the two parameters a and b, which according equation (15) can be written as:

$$\frac{a}{b} = \frac{\varphi}{\tan\varphi} * \frac{-\sin\varphi}{\varphi}$$

Which implies that $$\frac{a}{b} = -\cos\varphi \qquad (17)$$

For example in the limit $\varphi \to \pi$ $\frac{a}{b} \to 1$, which means $a \cong b$, i.e. it yield an asymptote.

In equation (17) taking $\varphi \to k\pi$ where $k = 0,1,2,...$ it will gives two asymptotes, depending on the value of k. If k is odd then it will give us the asymptote $a \cong b$ and if k is even it will give us the asymptote $a \cong -b$. Both asymptotes are represented in figure 5.

By taking $\varphi = k\frac{\pi}{2}$ and k is odd integer i.e. $k = 1,3,5,...$, we obtain the coordinates where each curves cuts on the b-axes.

For example the case when $k = 1,3,5$ we obtain:

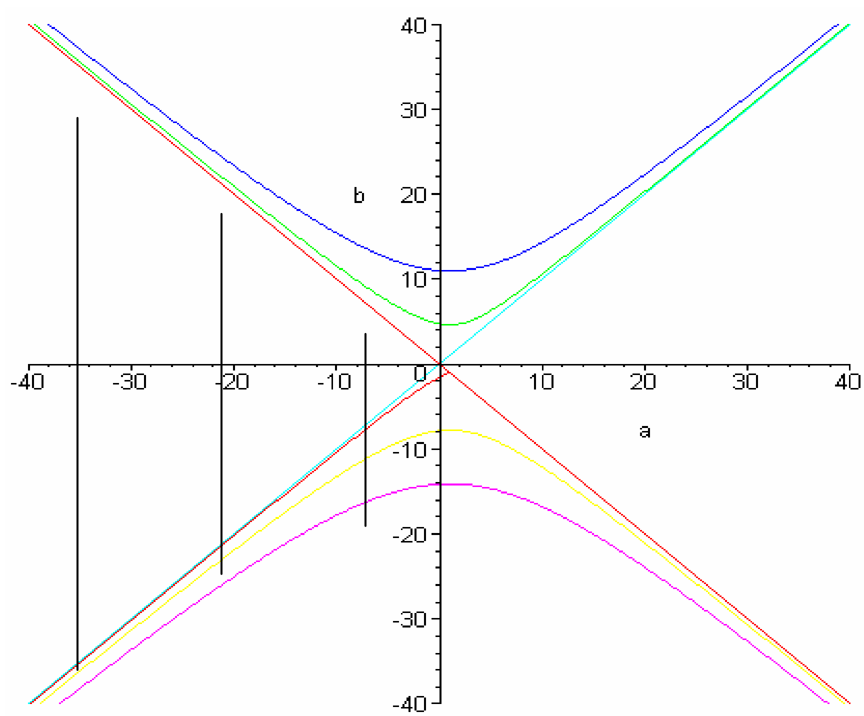
$$\varphi = \frac{\pi}{2} \qquad a = 0 \text{ and } \qquad b = -\frac{\pi}{2}$$

$$\varphi = \frac{3\pi}{2} \qquad a = 0 \text{ and } \qquad b = \frac{3\pi}{2}$$

$$\varphi = \frac{5\pi}{2} \qquad a = 0 \text{ and } \qquad b = -\frac{5\pi}{2}$$

We can see that if $k = 4n+3$ then the coordinates of b are negative. Whereas for $k = 4n+1$ then coordinates of b are positive where $n = 0,1,2...$ But for any value of k the coordinates of a is zero. (cf. figure 5)

Figure 5: The parametric curves of the first order delay differential equation:



I have used Maple to sketch figure 5 which shows two asymptotes and 5 curves.

The curves could be labelled by $k = 0,1,2.....$ Label the first branch $\varphi \in (0,\pi)$ by $k = 0$ (red colour) and label $k = 1$ for the next branch (green colour) etc (cf. fig4).

The true stability domain is bounded by the curves we have computed i.e. it is bounded between the two boundaries. One boundary is a part of the line $a + b = 0$ and the other boundary is the branch where $\varphi \in (0,\pi)$. But the asymptotes are analytical estimate for the stability domain. The red line and the blue line are the two asymptotes. In this thesis we limit our studies to the region between the two asymptotes($a + |b| < 0$). Hence the striped region ($a + |b| < 0$) have to be stable region since it is in the region between the first branch and the line $a + b = 0$. We will verify this conjecture using algebraic estimates in the next chapter.

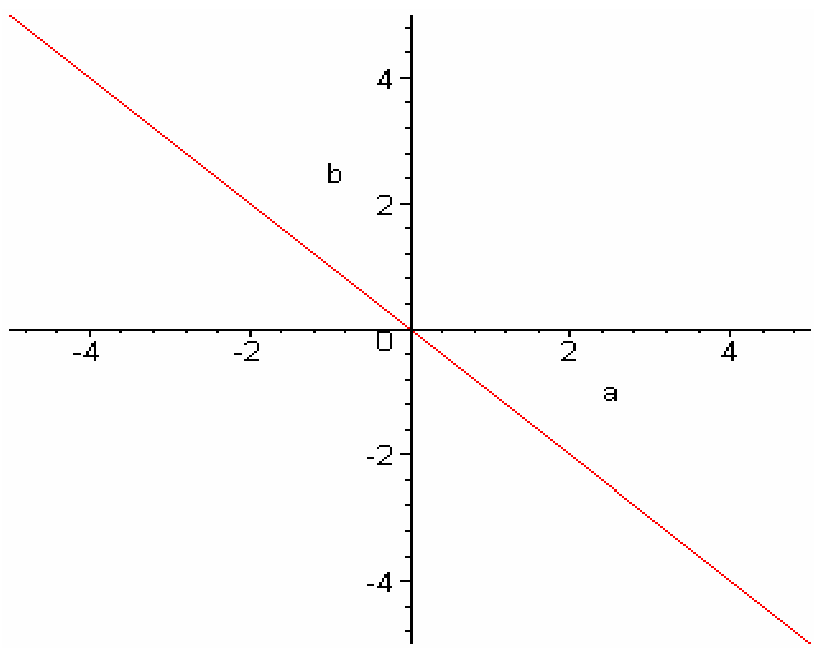
3 Algebraic estimates for the boundaries of stability domain

In the previous chapter we made some conjecture about the stability region. We determined an estimate for the unstable and stable region. In this chapter we shall reformulate these conjectures into two lemmas and proof each lemma separately.

Lemma 1

If $a+b>0$ where a and b are real numbers then the characteristic equation $z=a+b\exp(-z)$ has a solution where $\mathrm{Re}(z)>0$ **(Unstable).**

Figure 6: The estimate unstable region:



where the $a=-b$ represents the boundary and the region above the line is an unstable region.

Proof

We will prove this lemma using the intermediate value theorem. The Intermediate Value Theorem (IVT) states that if $f:[x,y]\to\Re$ is continuous and $f(x)<0$, $f(y)>0$ where $x<y$ then there exist a point $c\in(x,y)$ such that $f(c)=0$.

Using IVT we require showing that we have at least one positive solution i.e. that the transcendental equation has a solution $z>0$ in the region $a+b>0$.
Let $f(z)=z-a-b\exp(-z)$ then:

$$f(0)=0-a-b\exp(0)=-a-b=-(a+b)<0, \quad \text{Since } a+b>0$$

$$f(\infty)=\infty$$

Hence $f(\infty)=\infty$ implies the existence of a value y such that $f(y)>0$ and Intermediate Value Theorem can be applied for $f(0)$ and $f(y)$. Therefore we get:

$$f(z)=0 \qquad \text{for } z>0$$

Thus we have at least one positive solution in the region $a+b>0$. Therefore the region above the line $a=-b$ is unstable.

Lemma 2

If $a+|b|<0$ where a and b are real numbers then the characteristic equation $z=a+b\exp(-z)$ has no solution where $\mathrm{Re}(z)>0$ **(Stable).**

Proof

Suppose that a + | b | < 0. If the Re (z) <0 then the transcendental equation is stable for that region. We shall proof this lemma by contradiction.

Let $z = x + iy$ then the transcendental equation (9) becomes:

$$x + iy = a + b\exp(-x - iy)$$

$$x + iy = a + b\exp(-x)(\cos y - i\sin y) \qquad (18)$$

We are only interested in the real part of equation (18). Hence we obtain:

$$x = a + b\exp(-x)\cos y \qquad (19)$$

Assume that $\mathrm{Re}(z) > 0$ which implies $x > 0$. We know that the function of cosy is bounded between 1 and -1 i.e. $|\cos y| < 1$ and $\exp(-x) < 1$ because we assume $x > 0$. Using what we know we can see that the right hand side of equation (19) is less than the value of $a + |b|$. In other words what we are trying to say is that:

$$a + b\exp(-x)\cos y < a + |b| \qquad (20)$$

Since a + $|$b$|$ < 0 equation (19) becomes:

$$x = a + b\exp(-x)\cos y < a + |b| < 0$$

But we assumed $x > 0$, hence we get a contradiction. Therefore the region $a + |b| < 0$ is a stable region.

Both lemmas show that at least the region above the diagonal $a + b = 0$ is unstable (cf. figure 5 and 6); also that the region bounded by the two diagonal $a = -b$ and $a = b$ is stable. We have obtained an estimate stability region for the linear first order delay differential equation (1).

4 Second order delay differential equations

In this chapter we are looking at more advanced topic compared to the first order delay differential equation. It is more complicated but a similar approach can be applied to the second order differential difference equations which describes oscillators subjected to time delay feedback. The second order delay differential equation which we will study in this chapter has the following form:

$$\alpha\ddot{x}(t)+\beta\dot{x}(t)+\chi x(t)=\delta x(t-\tau) \tag{21}$$

where $\alpha = mass$, $\beta = damping$, $\chi = \ force$

4.1 The transcendental equation

Using $x(t)=\exp(\lambda t)$ in equation (21) we obtain the following transcendental eigenvalue equation

$$\alpha\lambda^2+\beta\lambda+\chi=\delta\exp(-\lambda\tau)$$

$$\frac{\alpha}{\tau^2}z^2+\frac{\beta}{\tau}z+\chi=\delta\exp(-z)$$

$$\frac{\alpha}{\tau}z^2+\beta z+\chi\tau=\delta\tau\exp(-z) \tag{22}$$

The parameter β has an important impact on equation (22). Different values of the parameter β will give us different stability properties. Hence to determine the stability properties of equation (21) we need to consider the following three cases separately

- $\beta > 0$
- $\beta < 0$
- $\beta = 0$

In this thesis, it will take a great deal to cover all the three cases. As a result we will only focus on the case when the parameter $\beta > 0$.

4.2 The stability domain for $\beta > 0$

If $\beta > 0$ then we could divide both side of equation (22) by β. Then we obtain the following equation:

$$\frac{\alpha}{\beta\tau}z^2 + z + \frac{\chi}{\beta} = \frac{\delta\tau}{\beta}\exp(-z) \tag{23}$$

Hence rearranging equation (23) we find:

$$cz^2 + z = a + b\exp(-z) \tag{24}$$

where $c = \dfrac{\alpha}{\beta\tau}$, $a = -\dfrac{\chi}{\beta}$ and $b = \dfrac{\delta\tau}{\beta}$

We shall evaluate equation (24) with the same method we applied in equation (9) section 2.1. But equation (24) is more complicated than the transcendental equation of the first order delay differential equation. There will be three cases that we need to discuss separately which are when $c = 0$, $c > 0$ and $c < 0$.

If we consider the trivial case when $c = 0$ then we obtain the transcendental eigenvalue equation of the first order delay differential equation (cf. chapter 2).

Now let's look the case when $c > 0$. Again to determine the boundaries stability domain we need to consider the conditions $z = 0$ and $z = i\varphi$. When $z = 0$, we get the same result as in section 2.2, but what about when $z = i\varphi$?

Taking $z = i\varphi$ equation (24) becomes:

$$c(i\varphi)^2 + i\varphi = a + b\exp(-i\varphi)$$

$$\Rightarrow \qquad -c\varphi^2 + i\varphi = a + b\exp(-i\varphi) \tag{25}$$

Comparing the real part and the imaginary part on both sides of the equation we have:

$$\varphi = -b\sin\varphi$$

$$-c\varphi^2 = a + b\cos\varphi \tag{26}$$

Rearranging equation (26) gives:

$$a = -c\varphi^2 + \frac{\varphi}{\tan\varphi} \qquad \text{and} \qquad b = -\frac{\varphi}{\sin\varphi} \tag{27}$$

Which gives us another stability boundary in the parameter space of (a,b), where a and b represents parametric equations which you obtain a set of curves. Each of these curves are determined by $\varphi \in (k\pi,(k+1)\pi)$. The first branch is obtained by putting $k = 0$ where a finite limit occurs. Similarly to chapter 2, from equation (27) asymptotes could be determined. The asymptote are worked out by considering the ratio of the two parameter a and b which can be written as:

$$\frac{a}{b} = \frac{-c\varphi^2 \tan\varphi + \varphi}{\tan\varphi} * \frac{-\sin\varphi}{\varphi}$$

$$\Rightarrow \qquad \frac{a}{b} = c\varphi \sin\varphi - \cos\varphi \tag{28}$$

In section 2.3 the equation a/b gave us only two asymptote namely $a = -b$ and $a = b$. But equation (28) is more complicated, so we need to derive an equation which gives us the general form of the asymptotes. Consider φ close to $k\pi$ where k is an integer. Let $\varphi = k\pi + \Delta$, considering Δ to be small, then equation (27) gives us:

$$a = -c(k\pi + \Delta)^2 + \frac{k\pi + \Delta}{\tan(k\pi + \Delta)}$$

$$b = -\frac{k\pi + \Delta}{\sin(k\pi + \Delta)} \tag{29}$$

Using Taylor series we attain $\sin(k\pi + \Delta) \cong (-1)^k \Delta$ and similarly $\tan(k\pi + \Delta) \cong \Delta$. So equation (29) changes to:

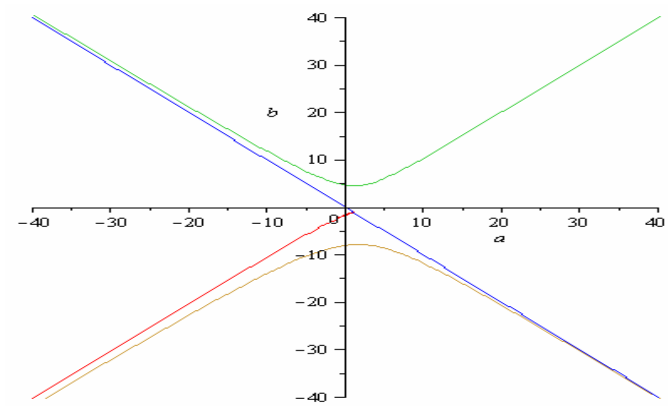
$$b \cong -\frac{k\pi}{(-1)^k \Delta}$$

$$a \cong -c(k\pi)^2 + (-1)^k b \tag{30}$$

This gives us the asymptotes of equation (24). The asymptotes will shift up each time for different values of c where $c > 0$.

Equation (27) is more complicated to sketch then equation (15). I have used Maple to sketch the parametric curves of the equation. We can see in figure 7 which has small positive value of c that the curves look very similar to the curves in figure 5. Hence it has the same stable region as the linear first order delay differential equation (1).

Figure 7: The parametric curves of the second order delay differential equation for a small value of c i.e. c = 0.1:



Comparing figure 8a and 8b we view that each curve shifts to the left as the positive value of c gets bigger. Some of the curves are below the a-axis while the other is above the a-axis depending on φ where $\varphi \in (k\pi, (k+1)\pi)$ (cf. figure 8a and 8b). We see that if k is even then the curve is below the a-axis while if k is odd, the curve is above the a-axis. From figure 8a and 8b, we observe that the region between the two sets of boundaries contains a parabola shaped region which opens to the left. Hence the parabola shaped region should be a stable region. We shall proof this conjecture in the next section.

Figure 8a: The parametric curves of the second order delay differential equation when $\beta > 0$ and c = 0.75:

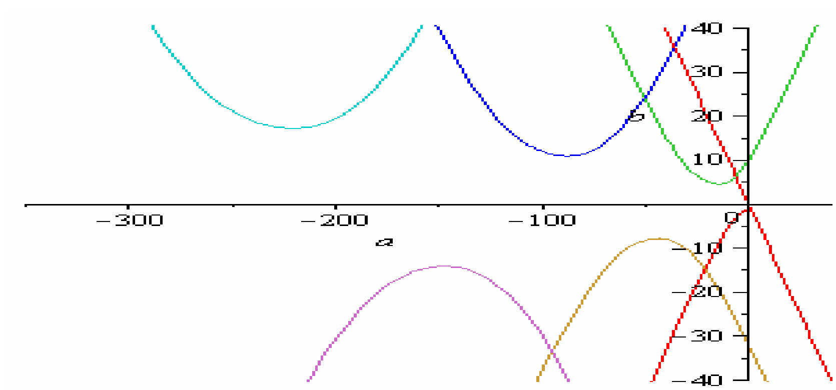


Figure 8b: The parametric curves of the second order delay differential equation when $\beta > 0$ and c = 1.0:

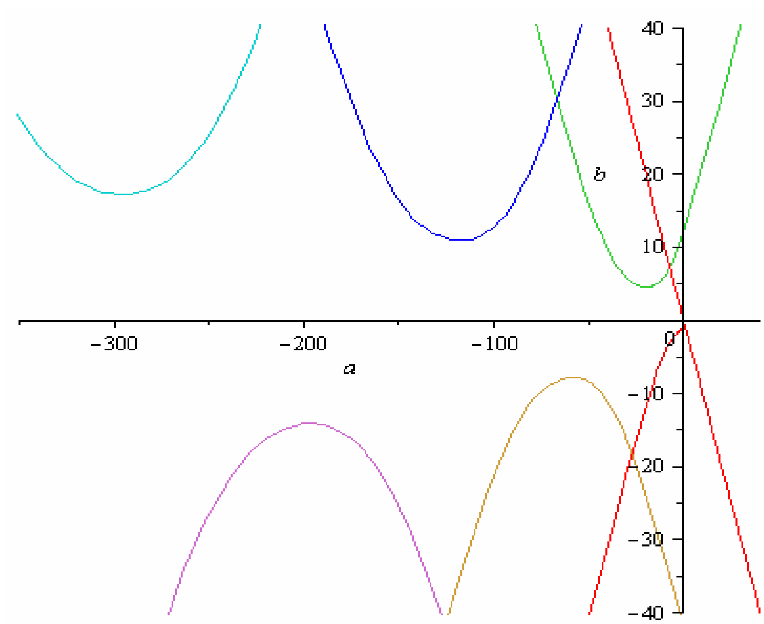


Figure 8a and 8b we see that the distance between the curves gets bigger as c increases; also that the curves become wider.

What about the case when $c < 0$. We could determine the parametric curves of the equation when $c < 0$ using the same method. In order not to replicate the same calculation again, I have used maple to construct the parametric curves of the equation when $c < 0$ (cf. fig 9a, 9b). From figure 9a and 9b we notice that the stable region is bounded by the red and the blue line. Hence this case has very similar stability region as the first order delay differential equation; see the figures below. When $c < 0$, we could use the same stability region as the first order DDE as we only resolve for an estimate stability region in this project. But as c gets more negative we observe that the stability region gets bigger, since the red curve becomes more curved. The green and the orange curves change their shapes because they are in the unstable region.

Figure 9a: The parametric curves of the second order delay differential equation for $\beta > 0$ with negative value i.e. c = -0.5:

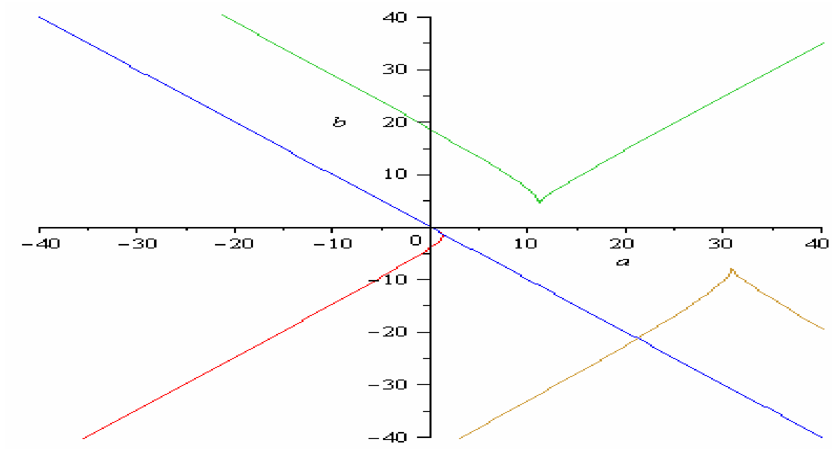
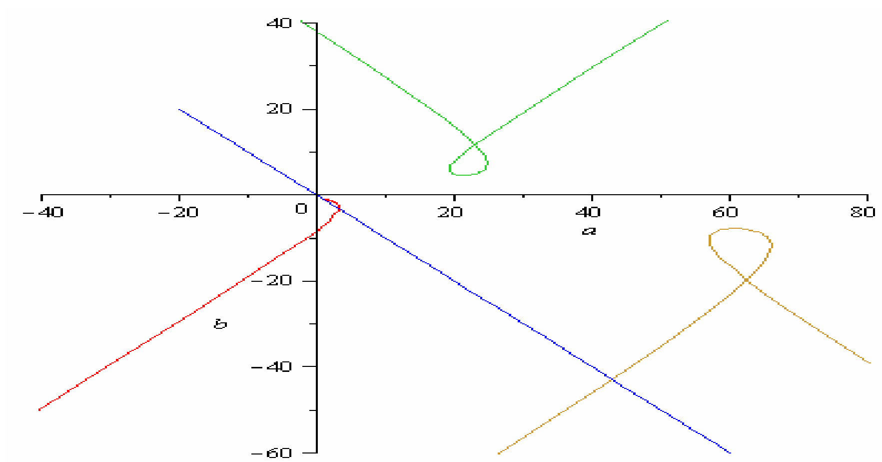


Figure 9b: The parametric curves of the second order delay differential equation for $\beta > 0$ and c = -1.0:



where:

- The blue line represents the boundary obtained by the condition $z = 0$ (i)
- The red curve is obtained by the condition $z = i\varphi \quad \varphi \in (0, \pi)$ and (ii)
- The region bounded between (i) and (ii) is the stable region

4.3 Algebraic estimates

In the previous section, we made conjecture for the stability region for the cases $c > 0$ and $c < 0$. The case $c < 0$, is very similar to the first order delay differential equation and so we shall omit the proof. For the case $c > 0$, we stated that the region which has the parabola shaped region is stable; so will proof this conjecture in lemma 4. The following 2 lemmas are only valid for the case $c > 0$.

Lemma 3

If $a + b > 0$ and $c > 0$ then the transcendental equation $cz^2 + z = a + b\exp(-z)$ has a solution where $\operatorname{Re} z > 0$ (Unstable)

Proof

We will proof this lemma the same way as lemma 1. We need to show that we at least have one positive solution in the region $a + b > 0$

Let $f(z) = cz^2 + z - a - b\exp(-z)$.

$$f(0) = -a - b = -(a+b) < 0 \qquad \text{since } a + b > 0$$

$$f(\infty) = \infty \qquad \text{since } c > 0$$

Hence by intermediate value theorem $f(z) = 0$ for $z > 0$ i.e. there is at least one positive solution in the region $a + b > 0$. As a result the region $a + b > 0$ is unstable

Lemma 4

If $a + |b| + c|b|^2 < 0$ and $c > 0$, a, b and c are real numbers then the transcendental equation $cz^2 + z = a + b\exp(-z)$ has no real solution where $\operatorname{Re} z > 0$ (Stable).

Proof

If the region $a + |b| + c|b|^2 < 0$ is stable then $\operatorname{Re}(z) < 0$. Hence to proof the lemma we need $\operatorname{Re}(z) < 0$. We shall proof this lemma by contradiction.

Suppose that $a+|b|+c|b|^2<0$ and $c>0$.

Let $z=x+iy$ where x, y are real numbers. Then the equation (24) simplifies to:

$$c(x^2+2ixy-y^2)+x+iy=a+b\exp(-x)(\cos y-i\sin y)$$

Then comparing the real and imaginary parts on both sides of the equation we get

$$cx^2-cy^2+x=a+b\exp(-x)\cos y \tag{31}$$

$$2cxy+y=-b\exp(-x)\sin y \tag{32}$$

Assume that $\mathrm{Re}(z)>0$ which implies $x>0$. We rearrange equation (32) to obtain a relationship between y and b.

$$y(2cx+1)=-b\exp(-x)\sin y$$

$$\Rightarrow \quad |y|(2cx+1)<|b|$$

$$\Rightarrow \quad |y|<\frac{|b|}{2cx+1}$$

$$\Rightarrow \quad |y|<|b| \tag{33}$$

Similar to the proof of lemma 2 we conclude that $a+b\exp(-x)\cos y<a+|b|$ since $x>0$ by assumption. Using this statement with equation (33) then equation (31) becomes:

$$cx^2+x<c|b|^2+a+|b|<0 \qquad \text{since } a+|b|+c|b|^2<0$$

But we assumed $\mathrm{Re}(z)>0$ i.e. $x>0$, hence we get contradiction. Therefore $\mathrm{Re}(z)<0$. As a result the region $a+|b|+c|b|^2<0$ is stable and it has parabola shape.

If $a+|b|+c|b|^2=0$ then we find that:

$$a+c(|b|^2+1/c\,|b|)=0$$

$$a+c((|b|+1/2c)^2-1/4c^2)=0$$

$$a+c(|b|+1/2c)^2-1/4c=0$$

$$a=\frac{1}{4c}-c(|b|+1/2c)^2$$

So the equation $a+|b|+c|b|^2=0$ has a parabola shape which is flipped on the b-axis.

5 Conclusion

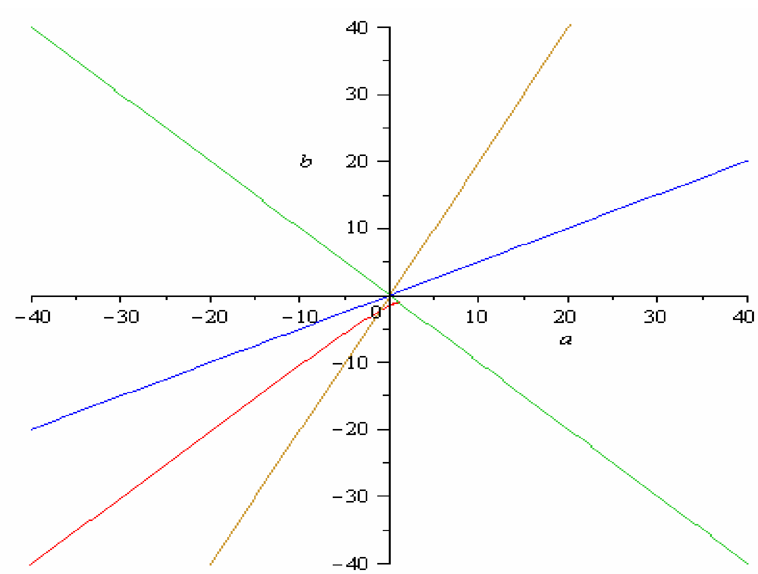
To summarize what we have done so far, we determined an estimate for the unstable and stable regions. We have achieved this by establishing the characteristic equation, from which we determined an estimate of the stability domain; we made a conjecture about the region of our stable estimate and finally proved it as a lemma.

The true boundary can be obtained from the parametric representation of the curve $\lambda = i\varphi$. On that boundary the change from stable to unstable occur. Considering this case would have complicated this thesis; so we can conclude that to obtain an estimate stable region for the delay differential equations we could use the steps mentioned above. But if we need to determine the true boundary which will gives the whole stable region, then we need to use methods like Runge-Kutta or Lambert W functions.

In chapter 2 and 3 the linear first order system; the stability domain has been expressed in terms of rescaled parameters i.e. $a = \alpha\tau$ and $b = \beta\tau$. We could have also investigated how for a fixed values of α and β, the stability could change when the delay time τ increases. According to equation (8) such a change in delay corresponds to a ray in the (a,b) plane originating from the origin. If $a + |b| < 0$ then such a ray never leaves the stability region displayed in figure 5. Hence the time dynamics is stable for all values of the delay τ. If $a + b < 0$ but $a - b > 0$ then the initial part of the ray is in the stability domain, but it will cross the boundary at some value (cf. figure 10). The point where it crossed is called the critical time delay where $\tau = \tau_c$. As a consequence, the dynamics is stable for $\tau < \tau_c$ but unstable for $\tau > \tau_c$. We could also evaluate critical delay time. In critical case the eigenvalue equation has a solution $\lambda = i\varphi$; the condition when $\lambda = i\varphi$ determines the boundary domain i.e. critical line. Thus the solution in the critical case oscillates with frequency φ.

In chapter 4 the second order system for positive mass and damping where $c > 0$ has indicated that the stability domain has a parabola like shape see figure 8. Since any ray in the (a,b) plane will finally cut the parabola like shaped, we then conclude that such system will always have critical delay time and systems with large delay will become ultimately unstable. So higher order systems would be more complicated to investigate because the transcendental equation becomes more complex.

Figure 10: critical time delay:



where:

- The green line corresponds to line $a = -b$
- The red curve is the boundary obtained by $z = i\varphi$ where $\varphi \in (0, \pi)$
- The blue line corresponds to the line when $\alpha = 2$ and $\beta = 1$ i.e. it is the region $\alpha + |\beta| < 0$
- The orange line corresponds to the line when $\alpha = 1$ and $\beta = 2$ i.e. it is in the region $\alpha + \beta < 0$

6 Bibliography

[1] Richard Bellman and K. L. Cooke, Differential difference equation
Volume 6 Academic Press, New York1963

[2] K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics (1992)

[3] Farshid Maghami, Analysis of a system of linear delay differential equations
Volume 125 (Jun 2003)
http://www.psych.ucsb.edu/~janusonis/asl2003.pdf

[4] Liu Ming Zhu, Asymptotic stability for gauss methods for neutral delay differential equations. Volume 2 (march 2002)
http://www.msas.maliwatch.org/msas2002/pdf/msas_pp085_89.pdf

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