Period doubling and renormalisation group approach for piecewise linear maps

by

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Abstract

This thesis describes the period doubling and renormalisation group approach specifically for piecewise linear maps. We start from very elementary concepts and give sufficient arguments to clarify ideas. Moreover, we show in detail how the bifurcation sequence and renormalisation group theory are used to study the period doubling sequence. Although the tent map is mainly discussed by way of illustration, a quadratic map is mentioned towards the end.

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Chapter 1

Introduction

Chaos theory is a fascinating and popular field of science. For instance such frequently used sources as Wikipedia describes the field in following terms: "Chaos theory is a branch of Mathematics which studies the behavior of certain dynamical systems that may be highly sensitive to initial conditions. Sensitivity to initial condition means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. This sensitivity is popularly referred to as the 'butterfly effect'. As a result of this sensitivity, which manifests itself as an exponential growth of error, the behavior of chaotic systems appears to be random."

Linear systems are never chaotic; for a dynamical system to display chaotic behavior it has to be non-linear. Chaotic behavior is also observed in natural systems, such as weather. There are other applications in the field of science and engineering. In our work, we consider a simple dynamical system though such dynamical system can display some chaotic behavior. Here we are mainly concerned with the change of chaotic behavior when a parameter of the system changes. There exists a huge zoo of such bifurcation sequences, and we here focus on the prominent example of period doubling bifurcations. In fact, by a quite elaborate mathematical machinery one can show that there are certain features of such a bifurcation sequence which do not depend on the details of the underlying system, i.e., which are universal. Here we want to give the reader a taste of the corresponding mathematical concepts by studying the tent map in analytical terms.

We use mainly three chapters to discuss our topic. In chapter two, we present some relevant definitions which describe the elementary steps and help us to build our ideas for the later part. In chapter three, we consider specific values of our parameter to work out the impact on the tent map such as chaotic motion and period doubling. Finally, applying the concept of the previous chapter, namely renormalisation, we embark on this idea in more formal terms.

Chapter 2

Basic facts about piecewise linear Markov maps

In this chapter, we give some relevant definitions to understand the later work properly. In what follows we consider simple dynamical systems, namely maps on an interval, $f\colon I \to I$. Such dynamical systems are able to display quite a rich behavior, e.g., instabilities and chaotic motion. Two quite important quantities, the so-called Lyapunov exponent and the invariant density, are useful to characterize chaotic motion. The former quantity measures the exponential rate of nearby trajectories, i.e. the sensitivity of the system, while the latter quantity is able to capture the statistical properties of the motion.

Definition : Let f be a (piecewise) smooth map and let $(x_0, x_1, x_2, \ldots\ldots)$ denote the orbit with initial condition x_0. If the limit

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(X_k)| \tag{2.1}$$

exists, the value Λ is called the ***Lyapunov exponent*** of the orbit.

Definition: Let $f\colon I \to I$ be a (continuous) map. A non-negative integrable function $\rho\colon I \to \mathbb{R}$ is called ***Invariant density*** if

$$\int_I \rho(x)\,\mathrm{d}x = 1 \tag{2.2}$$

and

$$\int_I \rho(x) \times h(x)\,\mathrm{d}x = \int_I h(f(x)) \times \rho(x)\,\mathrm{d}x \tag{2.3}$$

for any integrable function $h\colon I \to \mathbb{R}$.

Definition and Lemma: Let $f\colon I \to I$ denote a (piecewise) smooth map with *invariant density* ρ. Then the density obeys the so called ***Frobenius-Perron equation***

$$\rho(x) = \sum_{Y \in f^{-1}(x)} \frac{1}{|f'(Y)|} \rho(Y)\,. \tag{2.4}$$

For general maps quantities like the Lyapunov exponents or the invariant density, if they exist at all, are difficult to compute by analytical means. There exists, however, a nice class of maps, the so-called piecewise linear Markov maps, where the computation is quite straightforward. The analysis is based on a suitable partition of the interval, i.e., a collection of "disjoint" closed intervals $\{I_0, I_1, \ldots, I_{N-1}\}$ which cover the whole interval I and which nicely comply with the dynamics of the map f.

Definition: A map $f\colon I \to I$ is said to be ***Markov*** if there exists a partition of I, $\{I_0, I_1, ..., I_{N-1}\}$ (the so called ***Markov partition***) such that for all $k \in \{0, 1, ..., N-1\}, l \in \{0, 1, ..., N-1\}$ either

$$f(int(I_k)) \cap int(I_l) = \varnothing \tag{2.5}$$

or

$$int(I_l) \subseteq f(int(I_k))\,. \tag{2.6}$$

The dynamical features expressed by eq.(2.5) and (2.6) can be easily captured by the so-called topological transition matrix.

Definition : The $N \times N$ matrix defined by

$$A_{k\ell} = \begin{cases} 1 & \text{if} \quad f(int(I_k)) \supseteq int(I_\ell) \\ 0 & \text{if} \quad f(int(I_k)) \cap int(I_\ell) = \varnothing \end{cases} \tag{2.7}$$

is called the ***Topological transition matrix*** of the *Markov* map f.

If we now consider Markov maps with constant slope on the elements of the Markov partition we can even solve the Frobenius-Perron equation. For that purpose one introduces:

Definition : The ***Transfer matrix*** is defined by

$$T_{k\ell} = \frac{A_{\ell k}}{|\gamma_\ell|} \tag{2.8}$$

where $A_{\ell k}$ is the ℓk-th entry of the transition matrix A and γ_ℓ denotes the slope of the map f on the element I_ℓ of the Markov partition.

Since the Invariant density is piecewise constant then it can be written as a piecewise constant function,

$$\rho(x) = \sum_{k=0}^{N-1} \rho_k \chi_k(x) \tag{2.9}$$

where $\chi_k(x)$ denotes the characteristic function of I_k i.e.

$$\chi_k(x) = \begin{cases} 1 & \text{if} \quad x \in I_k \\ 0 & \quad Otherwise \end{cases} \tag{2.10}$$

and the vector $(\rho_0, \rho_1, \rho_2, \ldots\ldots\ldots, \rho_{N-1})$ is eigenvector of the Transfer matrix T to the eigenvalue one i.e.

$$\rho_k = \sum_{l=0}^{N-1} \frac{A_{lk}}{|\gamma_l|} \rho_l \tag{2.11}$$

Therefore, by straightforward calculation we can establish a relation between the Frobenius-Perron equation and the Transfer matrix, which is

$$\rho(x) = \sum_{Y \in f^{-1}(x)} \frac{1}{|f'(Y)|} \rho(Y) = \sum_l \sum_k \frac{A_{kl}}{|\gamma_k|} \rho_k \chi_l(x)\,. \tag{2.12}$$

Chapter 3

Period doubling transition of the tent map

We specifically consider a tent map

$$f(x) = 1 - a|x| \tag{3.1}$$

for $x \in [-1, 1]$ with parameter value $a \in [0, 2]$. After differentiation we get $|f'(x)| = a$. Now according to eq.(2.1) the Lyapunov exponent is computed as

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)| = \lim_{n\to\infty} \frac{1}{n} \times n \ln a = \ln a\,. \tag{3.2}$$

The Lyapunov exponent is clearly positive for $a > 1$, i.e., the tent map shows chaotic motion for such parameter values.

For a given value of x the pre-image set is easily computed to be $f^{-1}(x) = \{1/a - x/a, x/a - 1/a\}$. Hence, according to eq.(2.4) the Frobenius-Perron equation reads

$$\rho(x) = \sum_{Y\in f^{-1}(x)} \frac{1}{|f'(Y)|}\rho(Y) = \frac{1}{a}[\rho(\frac{1}{a} - \frac{x}{a}) + \rho(\frac{x}{a} - \frac{1}{a})]\,. \tag{3.3}$$

If the arguments on the right hand side of eq.(3.3) are not contained in the domain of the map, i.e., in the interval $[-1, 1]$. then, of course, the corresponding term does not appear in the Frobenius-Perron equation. We can easily deal with such a constraint if we require the density ρ to vanish outside the domain of the map.

Now we consider two cases for the parameter value a. First considering case $a = 2$. Then we have $f(x) = 1 - 2|x|$. The graph of this particular tent map is shown in figure 3.1.

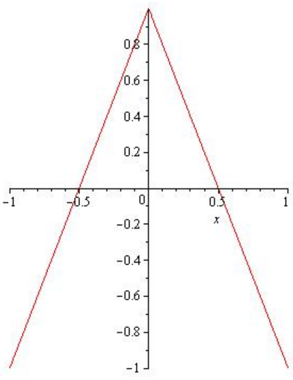

Figure 3.1: Graph of the map $f(x) = 1 - 2|x|$.

It is quite easy to show that the two intervals $I_0 = [-1, 0]$ and $I_1 = [0, 1]$ constitute a Markov partition since the interior of both intervals I_0 and I_1 is mapped to $(-1, 1)$. So, according to eq.(2.7) the topological transition matrix is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{3.4}$$

Clearly the tent map is a piecewise linear Markov map. Hence (cf. eq.(2.8)) the Transfer matrix is

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} . \tag{3.5}$$

Hence using eq.(2.12) one can compute the density. From the characteristic equation, we get

$$\det(T - \lambda I) = 0 \tag{3.6}$$

that means

$$\begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} = (\frac{1}{2} - \lambda)^2 - \frac{1}{4} = \lambda^2 - \lambda = \lambda(\lambda - 1) = 0 \tag{3.7}$$

Computation of the roots is straightforward

$$\lambda = 0 \text{ or } \lambda = 1 \tag{3.8}$$

Now the amplitudes ρ_i of the invariant density are determined by eq.(2.11)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} . \tag{3.9}$$

By simplification, we have

$$\rho_0 = \rho_1 = c \, . \tag{3.10}$$

Thus (cf. eq. (2.9)) the invariant density is

$$\rho(x) = \rho_0\chi_0(x) + \rho_1\chi_1(x) = c(\chi_0(x) + \chi_1(x)) \tag{3.11}$$

where $\chi_0(x)$ and $\chi_1(x)$ denote the characteristic functions of the intervals I_0 and I_1, respectively. To find the value of the constant c, we need to imply the normalisation condition (2.2)

$$1 = c\int_{-1}^{1}\chi_0(x) + \int_{-1}^{1}\chi_1(x) = 2c \tag{3.12}$$

Since both the characteristic functions are contained in the interval $[-1, 1]$ i.e. $\chi_{[-1,0]} = \chi_{[0,1]} = 1$ for $-1 \leq x \leq 1$.

Thus $c = 1/2$ and eq.(3.11) yields for the invariant density the constant function

$$\rho(x) = \frac{1}{2}\chi_{[-1,1]}(x)\,. \tag{3.13}$$

It is in fact quite easy to confirm that the density (3.13) satisfies the Frobenius-Perron equation (3.3) for $a = 2$. As for the right hand side both arguments, $1/2 - x/2$ as well as $x/2 - 1/2$ are contained in the interval $[-1, 1,]$ since $-1 \leq x \leq 1$. Thus

$$\frac{1}{2}\{\rho(-\frac{x}{2} + \frac{1}{2}) + \rho(\frac{x}{2} - \frac{1}{2})\} = \frac{1}{2}(\frac{1}{2} + \frac{1}{2})\chi_{[-1,1]}(x) = \rho(x) \tag{3.14}$$

as required. In conclusion, and as expected, the tent map shows chaotic behavior at $a = 2$ with a uniform distribution of orbit points in the domain of the map.

Now we will work on parameter value $a = \sqrt{2}$ for function $f(x) = 1 - \sqrt{2}|x|$. The graph of this particular tent map is shown in figure 3.2

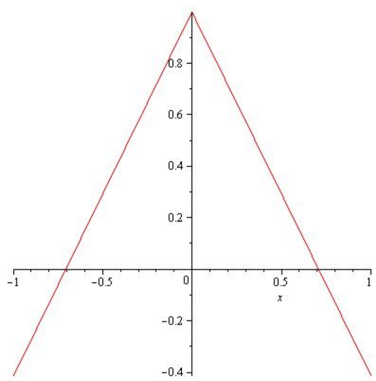

Figure 3.2: Graph of the map $f(x) = 1 - \sqrt{2}|x|$

It is easy to show the intervals $I_0 = [-1, 1 - \sqrt{2}], I_1 = [1 - \sqrt{2}, 0], I_2 = [0, \sqrt{2} - 1], I_3 = [\sqrt{2} - 1, 1]$ represent a Markov partition since the interior of I_0, I_3 is mapped to

$(1-\sqrt{2}, \sqrt{2}-1)$ and interior of I_1, I_2 is mapped to $(\sqrt{2}-1, 1)$
So according to eq.(2.7) the topological transition matrix will be

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} . \tag{3.15}$$

Obviously the tent map is a piecewise linear Markov map. Then (cf. eq.(2.8)) the Transfer matrix is

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} . \tag{3.16}$$

Now using eq.(2.12) it is straightforward to work out the density. From the characteristic equation we have

$$\det(T - \lambda I) = 0 \tag{3.17}$$

that means

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} = \lambda^2(\lambda^2 - \frac{1}{2}) - \frac{\lambda^2}{2} = \lambda^2(\lambda^2 - 1) = 0 \tag{3.18}$$

After a simple calculation we have

$$\lambda = 0, \lambda = \pm 1 \tag{3.19}$$

Then the amplitudes ρ_i of the invariant density are determined by eq.(2.11)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} . \tag{3.20}$$

By simplification, we have

$$\rho_0 = 0, \quad \rho_1 = \frac{1}{\sqrt{2}}\rho_3, \quad \rho_2 = \frac{1}{\sqrt{2}}\rho_3, \quad \rho_3 = \frac{1}{\sqrt{2}}\rho_1 + \frac{1}{\sqrt{2}}\rho_2 \tag{3.21}$$

Using normalization (cf. eq.(2.2)) we find out the value of amplitudes ρ_i

$$1 = \sum_{i=0}^{3} \rho_i |I_i| = \rho_0|I_0| + \rho_1|I_1| + \rho_2|I_2| + \rho_3|I_3| \tag{3.22}$$

where $|I_i|$ is the length of the interval.

Hence

$$\rho_3 = \frac{1}{2\sqrt{2}(\sqrt{2}-1)}, \quad \rho_2 = \frac{1}{4(\sqrt{2}-1)}, \quad \rho_1 = \frac{1}{4(\sqrt{2}-1)} \tag{3.23}$$

From the figure 3.2, we notice that I_1 and I_2 are mapped to I_3 and I_3 is mapped to $I_1 \cup I_2$, So there is no other motion except a period 2 motion.

We can use the second iterate of the function $f^{(2)}(x) = f(f(x)) = 1 - \sqrt{2}|1 - \sqrt{2}|x||$ for $a = \sqrt{2}$ to illustrate the period 2 motion. The graph of the second iterate is shown in figure 3.3. If we choose an initial point in $I_1 \cup I_2$ then such points will be mapped

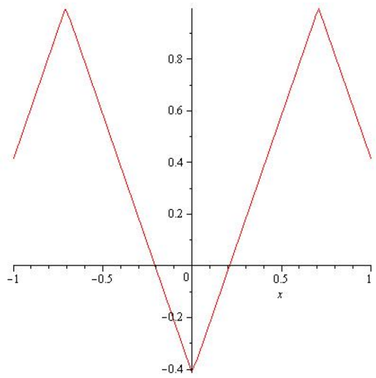

Figure 3.3: Second iterate of the tent map at $a = \sqrt{2}$, $f^{(2)}(x) = 1 - \sqrt{2}|1 - \sqrt{2}|x||$.

by the second iterate into the same interval. In terms of the orbit of the original map it means that every other orbit point is located in $I_1 \cup I_2$. Such a feature reflects the period two chaotic motion of the map. More importantly, if we confine the second iterate to the subset $I_1 \cup I_2$ then the map "looks the same" as the original tent map at $a = 2$ (cf. figures 3.1 and 3.3). In formal terms the two maps are topologically conjugate

$$f^2_{a=\sqrt{2}}(x) \simeq f_{a=2}(x)\,. \tag{3.24}$$

The implication of such a relation between the second iterate and the original map is huge. It basically means that the second iterate of the map behaves "in the same way" as the original map, at a "rescaled" parameter value. Such an observation would imply that the second iterate undergoes a period doubling bifurcation at a lower parameter value, which corresponds to a period four chaotic motion of the original map. Applying the "renormalisation" concept, eq.(3.24), again one concludes that the system displays an infinite sequence of period doubling. In the next chapter, we are now going to elaborate on such an idea in more formal terms.

Chapter 4

Renormalisation group for the tent map

Using the coordinate transformation $y = -\alpha x = h(x)$ and the conjugacy relation $h(f(x)) = g(h(x))$ where α is a scaling factor, we express the relation between the tent map at $a = 2$ and its second iterate at $a = \sqrt{2}$ as

$$h^{-1}(f^{(2)}_{a=\sqrt{2}}(h(x))) = f_{a=2}(x)\,. \tag{4.1}$$

Using the particular form of the conjugacy eq.(4.1) reads

$$\frac{-1}{\alpha} f^{(2)}_{a=\sqrt{2}}(-\alpha x) = f_{a=2}(x) \tag{4.2}$$

that means

$$\frac{-1}{\alpha}(1 - \sqrt{2}|1 - \sqrt{2}\alpha|x||) = 1 - 2|x|\,. \tag{4.3}$$

We can now easily find out the value of α if we recall that for the validity of the conjugacy a neighborhood of $x = 0$ should be considered (cf. the previous chapter and figures 3.1 and 3.3). Thus $|1 - \sqrt{2}\alpha|x|| = 1 - \sqrt{2}\alpha|x|$ holds on the relevant domain and comparing the constant terms in eq.(4.3) yields

$$\alpha = \sqrt{2} - 1\,. \tag{4.4}$$

We have already seen the relation between 2nd and 1st iterate for parameter values $a = \sqrt{2}$ and $a = 2$ (cf. in previous chapter), and the current computation puts this assertion in formal terms.

Now we can generalize the relation using the doubling transformation

$$\frac{-1}{\alpha} f^{(2)}_{\tilde{a}}(-\alpha x) = f_a(x) \tag{4.5}$$

for the tent map at parameter value a and its second iterate at parameter value $\tilde{a}$. Therefore,

$$\frac{-1}{\alpha}(1-\tilde{a}|1-\tilde{a}\alpha|x||) = 1-a|x|\,. \tag{4.6}$$

If we take again into account that the conjugacy can be established only close to $x=0$, i.e. $|1-\tilde{a}\alpha|x|| = 1-\tilde{a}\alpha|x|$, eq.(4.6) simplifies

$$1-\tilde{a}+\tilde{a}^2\alpha|x| = -\alpha + a\alpha|x|\,. \tag{4.7}$$

Equating the terms of the same "order" we obtain

$$\alpha = \tilde{a}-1 \tag{4.8}$$

and

$$\tilde{a}^2 = a \tag{4.9}$$

which is the desired relation between the parameters of a tent map and its 2nd iterate such that both maps are conjugate.

The relation (4.9) may be considered as a map, say g, for parameter values, i.e.

$$a = g(\tilde{a}) = \tilde{a}^2\,. \tag{4.10}$$

We can use this relation to compute the subsequent period doubling values, i.e. $a_0 = 2$, $a_1 = \sqrt{2} = 2^{1/2}$ $a_2 = \sqrt{\sqrt{2}} = 2^{1/4}$, ..., $a_n = 2^{1/2^n}$, ... and these values obviously obey

$$a_0 = g(\tilde{a}_1) = \tilde{a}_1^2, \quad a_1 = g(\tilde{a}_2) = \tilde{a}_2^2, \quad \ldots \tag{4.11}$$

That means the sequence a_0, a_1, ... is an orbit of the map g^{-1}. For historical reasons the map g is called *Renormalisation group transformation*. The name refers to the fact that by rescaling (by α), that is by renormalisation, the 2nd iterate can be cast into the form of the original map.

So from the above relation we can work out whether and how the bifurcation sequence $a_0 = 2$, $a_1 = \sqrt{2}$, $a_2 = \sqrt{\sqrt{2}}$, ... converges, i.e., whether it has a limit, say a_*. Obviously

$$a_* = \lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1/2^n} = 1 \tag{4.12}$$

and Taylor series expansion of the exponential function yields

$$a_n = 2^{1/2^n} = \exp(1/2^n \times \ln(2)) = 1 + \left(\frac{1}{2}\right)^n \ln(2) + \ldots \tag{4.13}$$

for large values of n. Thus the sequence of bifurcation values approaches the limit at an exponential rate with rate 1/2. In terms of the renormalisation group transformation

(4.10) the limit can easily be expressed in terms of the fixed point of the map. If we observe eq.(4.11) and take the limit of large n then we indeed obtain

$$a_* = g(a_*) = a_*^2 \tag{4.14}$$

which is consistent with the result $a_* = 1$ [1]. The speed of convergence, i.e. how the sequence approaches the limit (cf. eq.(4.13)) can be analyzed by a linear stability analysis of this fixed point (i.e. it is a stability analysis in the parameter space). In this case linearisation of eq.(4.10) yields $g'(a_*) = 2$. Since the sequence of parameter values constitutes an orbit of the inverse map, g^{-1}, we obtain for the rate of convergence $1/2$, as predicted by eq.(4.13).

All the properties can be computed from the renormalisation group transformation (4.10) and the fixed point analysis. That general pattern is the power of renormalisation group treatments. Whenever one can establish a relation like equation (4.10) then properties of bifurcation sequences follow from straightforward analysis.

[1] The renormalisation group equation, eq.(4.14), produces a second (the so called trivial) fixed point, $a_* = 0$. This fixed point is related with the non chaotic motion for $a < 1$ and will not be considered here.

Chapter 5

Conclusion

In this thesis we have studied bifurcation scenarios in simple piecewise linear maps, namely the tent map, and we have illustrated how concepts from renormalisation group theory can be employed to study period doubling sequences. In chapter 2, we reviewed some elementary ideas of our topic which guided us to develop the machinery for later work. We presented some definitions and related them when it was necessary. In chapter 3, by considering different parameter values of our tent map we showed its chaotic behavior in a simple dynamical system. Here we also observed the importance of the relation between the 2nd iterate and the original map and defined the concept of renormalisation. Finally in chapter 4, using the concepts of the previous chapter we studied renormalisation group transformation and fixed point analysis which can compute all the necessary properties . Thus computing the properties of bifurcation sequences boils down to straightforward analysis.

In fact the tent map is just a simple and exact illustration of renormalisation group transformations for dynamical systems. A more prominent, and to some extent more typical application concerns unimodal maps with a quadratic maximum, i.e. the logistic family. While a mathematical rigorous setup becomes quite intricate, some basic features can be already figured out, if some approximation is employed.

Here we consider a map with a quadratic maximum,i.e. $f_a(x) = 1 - ax^2$. Now using the doubling transformation, we have

$$\frac{-1}{\alpha} f_{\tilde{a}}^2(-\alpha x) = f_a(x) \tag{5.1}$$

for the quadratic map at parameter value a and its 2nd iterate at parameter value $\tilde{a}$. Therefore,

$$\frac{-1}{\alpha}(1 - a(1 - a(-\alpha x)^2)^2) = 1 - ax^2 \tag{5.2}$$

Now however, it is not easily possible to obtain a renormalisation group transformation (eq. 4.10) without approximation. If one eliminates terms of order x^4, then one can again yield an expression of the form

$$2a^2(a-1) = \tilde{a} \tag{5.3}$$

which is now an approximation for the exact renormalisation group transformation. By studying the fixed points and the stability of this transformation, information about the bifurcation sequences may be computed.

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