School of Mathematical Sciences

MSci final year project

The Fundamental Theorem of Calculus: an example to illustrate the need for rigour in modern mathematics

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ABSTRACT

On the following pages this project will investigate the well known Fundamental Theorem of Calculus. It will first focus on what the theorem means today. Later, it will look at the specific conditions that need to be met for the application of the theorem and it will analyse different approaches of some mathematicians throughout the 17th and the 18th century, and how that has affected the evolution of the understanding of this important concept. The topic is especially interesting since after a detailed scrutiny it can be realised how important mathematical rigorousness is, even over a spectrum of hundreds of years.

INTRODUCTION

The analytical geometry of Descartes and the calculus of Newton and Leibniz have expanded into the marvelous mathematical method more daring than anything that the history of philosophy records of Lobachevsky and Riemann, Gauss and Sylvester. Indeed, mathematics, the indispensable tool of the sciences, defying the senses to follow its splendid flights, is demonstrating today, as it never has been demonstrated before, the supremacy of the pure reason.

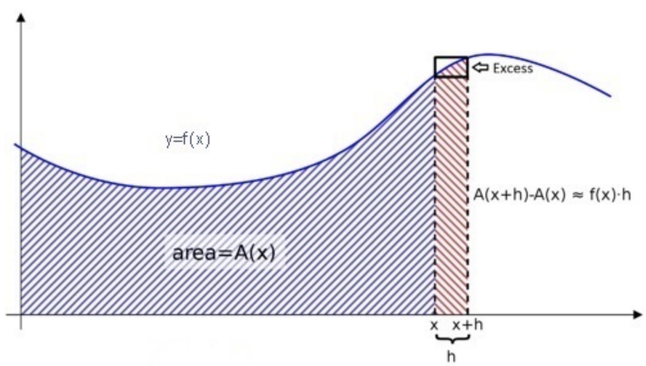
Nicholas Murray Butler [1]

Calculus has certainly been evolving over the past centuries. Even though the ability of the mathematicians to perform calculus was limited before the time of Newton, the concepts were not unfamiliar to them. Greek mathematicians already knew how to integrate and understood the concept, however their approach was much different. Differentiation was also not a strange concept to mathematicians centuries ago and they studied this notion in terms of motion. The invention of the Fundamental Theorem of Calculus was a breakthrough not because of the ability to perform these operations but because of the ability to recognise the link between them. Until the seventeenth century, people viewed the integration and differentiation as two unlinked operations and the realisation that in some aspect they were actually inverse of each other was astonishing and transcendental.

In this project I will firstly analyse the theorem itself followed by stating its formal definition and the proof. Then, I will look at the different interpretations made by some renowned mathematicians and try to understand their approach to the FTC. Meanwhile, I will see whether the seventeenth and early eighteenth century approach was rigorous enough and if not, I will try to explain why. Finally,I will provide an overall conclusion.

[1] Moritz, Robert Edouard. Memorabilia mathematica; or, The philomath's quotation-book. The Macmillan Company, 1914.

THE MODERN VERSION OF THE FUNDAMENTAL THEOREM OF CALCULUS.



[2]

Figure 1: Representation of the fundamental theorem of calculus.

The fundamental theorem of calculus reveals a close relation between two important concepts of calculus, the derivative and the integral; this was not recognised before the existence of the theorem even though the ancient mathematicians perfectly knew how to integrate and differentiate a long time before the discovery. This is why this connection is the real breakthrough, and not the ability to perform the operations themselves, hence being given the name 'Fundamental Theorem of Calculus'. The theorem can be interpreted in a geometrical and a physical manner. It is important to understand both and to be able to switch between them in order to grasp the full idea.

Geometrically speaking, taking any continuous function, say f, any value x will have a corresponding area under the function, $A(x)$ as represented in Figure 1. This is what we understand as the definite integral of a function, noting that the area under the curve is not a function but a set of functions and can be negative if it is located below the x axis. The area between x and $x + h$ can be calculated by subtracting the area between 0 and x from the area between 0 and $x+h$, therefore the needed area would be $A(x+h)-A(x)$.

Another way of looking at the problem is estimating the needed area to be found, here $A(x)$, using the method for finding the area of a rectangle and keeping in mind that the rate at which the area is changing is equal to the height of the function. Then $A(x + h) - A(x)$ is estimated to be$f(x) \times h$. If the presence of the excess is taken into account(as marked in Figure 1) then the estimation becomes an equality, $A(x+h) - A(x) = f(x) \times h -' excess'$. After rearranging the terms, so that $f(x)$ is the subject of the equation, we observe that the term including the excess of the area divided by h, tends to zero as h does, and therefore disappears from the equation. This can be explained as follows. If we take the height of the black rectangle to be a, then the area will be ah; dividing the area by h will simply yield in the height of the rectangle, $\frac{ah}{h} = a$, and it does go to zero when h does, so the excess does too. We get the following result:

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h},$$

that is $f(x) = A'(x)$ following the definition of the derivative. We can conclude that the derivative of the area function is the original function itself! It can be easily noticed from this implementation that the computation of the antiderivative of a function, that is the area function, and the derivative of the function, that is the rate of change of the area with respect to x, are inverse operations; and this is the epicentre of the Fundamental Theorem of Calculus.

The physical way of stating the theorem is to look at it in terms of distance and time. One could calculate any journey travelled by adding up its small intervals. To calculate the distance travelled in each interval we would use the basic theory of distance=speed $\times$ time, and by adding multiples of the velocity and time for the corresponding intervals we achieve the total distance. However, similarly as in the geometrical approach, the small intervals of time would disappear when dealing with such big sums and the overall distance travelled would become simply the summation of the velocities, which is the same as saying the integral of the velocity, since we know that summation corresponds to the integration process. Due to the fact that the derivative of the position function is the velocity, we clearly see that integrating velocity recovers the original position, hence the core idea of the Fundamental Theorem of Calculus.

Before further analysis of the theorem, I will state the following

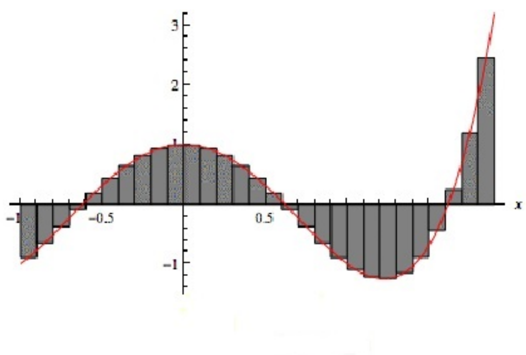
.

Definition 1
Let f be a function defined on a real, closed interval, $[a, b]$. Let P be the partition of the interval such that
$P = [x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$,
where
$a = x_0 < x_1 < x_2 < ... < x_n = b$.
The Riemann sum of f over the interval $[a, b]$ with partition P is defined as:
$S = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}), \quad x_{i-1} \leq x_i^* \leq x_i$.
The choice of x_i^* in the interval $[x_{i-1}, x_i]$ is arbitrary. [3]

Definition 2
We define the upper Riemann sum of f with respect to the partition P by

$$U(f; P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

where
$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$ relating to the interval $[x_{k-1}, x_k]$.



[4]

Figure 2: Upper and lower Riemann sums.

We define the lower Riemann sum of f with respect to the partition P by

$$L(f; P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

where $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ relating to the interval $[x_{k-1}, x_k]$.
For a particular partition P, we always have $U(f; P) \geq L(f; P)$. [5]

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[3]Thomas, George B. Jr.; Finney, Ross L. (1996), Calculus and Analytic Geometry (9th ed.), Addison Wesley, ISBN 0-201-53174-7,p.312

[5]Stephen Abbott, (2001), Understanding Analysis, Springer, ISBN 978-0-387-21506-8, p.186-187

The shaded areas in the graph above (Figure 2) show the lower and upper sums of the Riemann sum for a constant mesh size (i.e. the length of the longest subinterval).

Definition 3

Let R be the collection of all possible partitions of the closed interval $[a, b]$. The upper Riemann integral of f is defined to be

$$U(f) = \inf\{U(f; P) : P \in R\}$$

Similarly, we define the lower Riemann integral of f by

$$L(f) = \sup\{U(f; P) : P \in R\}$$

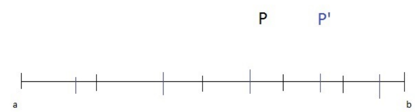


Figure 3: Partitions P and P' over the same interval

For any bounded function f on $[a, b]$
it is always the case that $U(f) \geq L(f)$. [6]
On an additional note, if we take two partitions, P and P', over the same closed interval $[a, b]$ (represented in Figure 3), where P' is a refinement of P,[7] then $U(f; P) \geq U(f; P')$.
By the same analogy we have $L(f; P) \leq L(f, P')$ and $U(f; P) \geq L(f; P)$.
Therefore it can be concluded that $U(f; P) - L(f; P) \geq 0$.
Moreover, $\inf_{\mathbf{P}}(U(f; P) - L(f; P)) = 0 \Rightarrow \inf_{\mathbf{P}} U(f; P) = \sup_{\mathbf{P}} L(f; P)$
and by the definition of $U(f)$ and $L(f)$ we have $U(f) = L(f)$.

Definition 4

Let f be a function defined over $\mathbb{R}$. Then we say that f is Riemann integrable everywhere if, for all $[a, b]$,

$$\inf\{U(f; P) - L(f; P)\} = 0.$$

where the infimum is taken over all partitions of $[a, b]$.

[6] Stephen Abbott, (2001), Understanding Analysis, Springer, ISBN 978-0-387-21506-8, p.188

[7] It means that P' contains at least all of the points of P; it could contain extra points. We say that P' is 'finer' than P.

Lemma 1
If f is a bounded, Riemann integrable function then its upper integral, $U(f)$, and its lower integral, $L(f)$, are equal. That is

$$U(f) = L(f)$$

Definition 5
We define the Riemann integral of f to be

$$U(f) = L(f) = \int_a^b f(x)dx.$$

It is the common value of $U(f)$ and $L(f)$. [8]
Proposition 1
If f is a Riemann integrable function then it is bounded. [9]

The Proposition implies that an unbounded function is never Riemann integrable (by applying the contrapositive of the statement).

Proof
We will prove this by contradiction. First assume that f is not bounded. Then by Definition 4 we see that

$$\inf \{U(f;P) - L(f;P)\} \neq 0$$

This is because if a function is unbounded then the upper integral will be infinite whilst the lower integral will be finite so the infimum will never be equal to 0. That means that the upper and the lower integrals are not the same, i.e.

$$U(f) \neq L(f)$$

This implies (by Lemma 1) that the function is not Riemann integrable. Therefore we get the result.
Q.E.D.

[8] Stephen Abbott, (2001), Understanding Analysis, Springer, ISBN 978-0-387-21506-8, p.188
[9] Max Jodeit,(2000),Introduction to Riemann integration, using Riemann sums, School of Mathematics, University of Minnesota

Theorem 1 *Lebesgue criterion for Riemann Integrability.*

Let f be a function defined on a closed interval, $[a, b]$, over $\mathbb{R}$. Then f is Riemann integrable if and only if it is bounded and continuous almost everywhere, i.e. the set of discontinuities of f has measure 0. [10]

On another note however, there exist functions that are integrable, but not Riemann integrable. An example of such a function would be a function f defined over a closed interval $[0, 1]$ by

$$f(x) = \begin{cases} 1, & \text{if} \quad x \notin \mathbb{Q} \\ 0, & \text{if} \quad x \in \mathbb{Q} \end{cases}$$

The function is bounded and can be regarded as continuous almost everywhere, as $f(x) = 1$, except on a set of measure zero; that is on the set of rational points in $[0, 1]$, where it is not continuous. Such functions are not Riemann integrable, but Lebesgue integrable. It is important to highlight that the FTC does not relate to functions of this type. We need to deal with functions that are Riemann integrable only and therefore satisfy the Lebesgue criterion for Riemann integrability.

Theorem 2. (*Integrability of Continuous Functions.*)
Suppose that f is a continuous function on a closed interval $[a, b]$. Then f is Riemann integrable over $[a, b]$. [11]

It is important to raise one point before moving further. The **indefinite integral** can be defined as the antiderivative of a function which means that we can apply it to the differentiated function in order to get back to the original function; it is simply understood to be the opposite operation of the derivative. Mathematically speaking, an antiderivative of a function $f(x)$ is a function $g(x)$ such that $g'(x) = f(x)$. The function $g(x)$ exists if $f(x)$

[10] R. Chen, *"Lebesgue Criterion for Riemann Integrability*, Advanced Calculus,Department of Mathematics, National Cheng Kung University, Taiwan, 4th May 2011,p.2

[11] This theorem is weaker than Theorem 1. Sourced from: Prof Bill Jackson,(2013),Calculus I,Thomas' Calculus,Sections 5.2 to 5.5, School of Mathematical Sciences,Queen Mary University of London

is continuous. The antiderivative of a function is not unique. If we take a function $f(x)$ then its antiderivative is $g(x)$ but, it is also $g(x) + C$ for some constant C (further details explained below). The **definite integral** is understood as the area function, that is the area under the curve as x varies. The fundamental theorem of calculus states that the two are equivalent (this is where the lack of uniqueness comes from).

Further explanation: If we let $f(x)$ be a function for which there exists an antiderivative $g(x)$, then we can say that this antiderivative could also be $g(x) + C$, where C is some constant. This is because if $A(x)$ and $g(x)$ are primitives of $f(x)$ then we know that $A(x) - g(x) = C$. After rearranging, we get $A(x) = g(x) + C$ where $A(x)$ is the area function of $f(x)$.

As discussed above, the theorem is understood that integration and differentiation are inverse operations, however this raises certain difficulties. The assumption leads us to think that a derivative of a differentiable function is always integrable, and we do know that derivatives of some functions, especially of ones that oscillate fast as they approach end points[12] and remain differentiable in such manners, are not integrable. An example of such a function would be

$$f(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ x^2 \sin(\frac{1}{x^2}), & \text{if} \quad x \neq 0 \end{cases}$$

where $x \in \mathbb{R} : x \neq 0$. The appearance of the function is presented in Figure 4, where the smaller figure is showing what is happening over a much smaller interval.

Firstly, I will differentiate for $x \neq 0$. The following is obtained:

$$f'(x) = 2x^2 \sin(\tfrac{1}{x^2}) - \tfrac{2}{x} \cos(\tfrac{1}{x^2}).$$

The function is presented in Figure 5.

For $x = 0$ the differentiation is a little bit more tricky, because I cannot simply differentiate the value 0. I will need to use the definition of a derivative. We know that a derivative of a function f at a value a is

[12] If we take a function $x^a \cos(\dfrac{1}{x^b})$, where $a, b \in \mathbb{Q}$ then the condition for a function that oscillates fast would be $a - 1 < b$. In this case essential singularity always occurs that gives raise to the oscillation. Obviously, the greater the difference in the inequality the more rapid the oscillation.

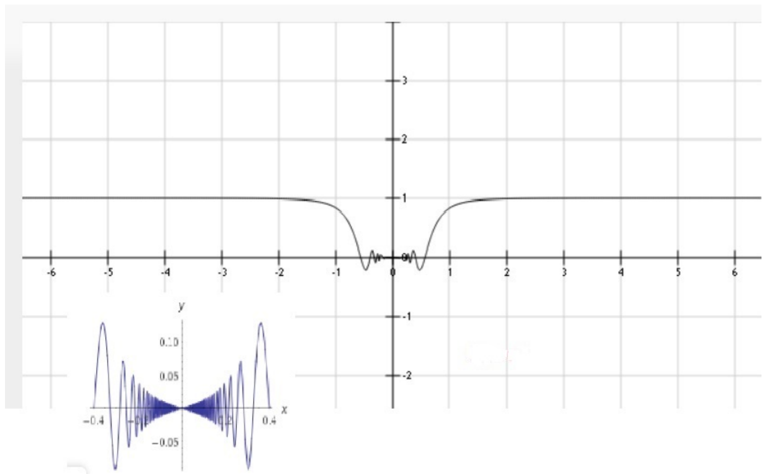


Figure 4: The representation of the original function $f(x)$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

We can easily substitute $a = 0$ to get the following:

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} =$$
$$= \lim_{h\to 0} \frac{h^2 sin(\frac{1}{h^2}) - 0}{h} =$$
$$= \lim_{h\to 0} \frac{h^2 sin(\frac{1}{h^2})}{h} =$$
$$= \lim_{h\to 0} h sin(\frac{1}{h^2}) = 0.$$

It is very clear that the above limit goes to zero as h does. Summarizing, we obtain the following function:

$$f'(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ 2x sin(\frac{1}{x^2}) - \frac{2}{x} cos(\frac{1}{x^2}), & \text{if} \quad x \neq 0 \end{cases}$$

The derivative of the function when $x \neq 0$ is represented in Figure 5. Something interesting happens when we take the derivative. Namely, at $x \neq 0$ the term $\dfrac{2}{x}$ implies that the function is not bounded, in fact it im-

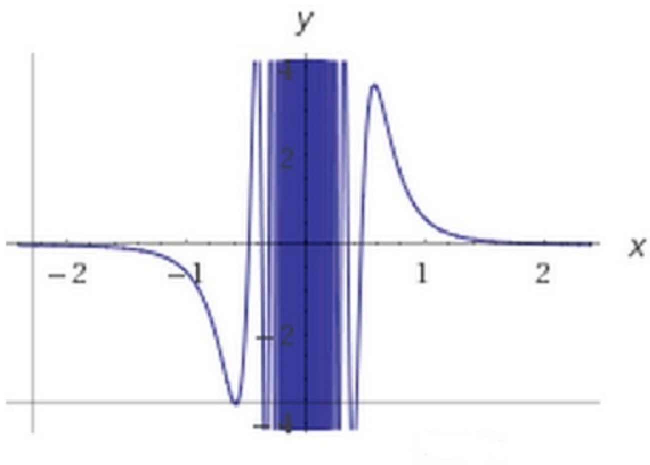


Figure 5: The representation of the derivative of $f(x)$

plies that it oscillates unboundedly as x tends to 0. This leads us to a quick conclusion that the unbounded function is not Riemann integrable by Proposition 1. This is a contradiction.

Another assumption that one might draw would be that if he takes a function and integrates it, its antiderivative will always be differentiable. However, this is impossible for some functions. As an example of this, let us take a function g defined below. The plot is represented in Figure 6.

$$g(x) = \begin{cases} 1, & \text{if} \quad x \geq 0 \\ -1, & \text{if} \quad x < 0 \end{cases}$$

This function is not continuous, however it is Riemann integrable. We can define its integral as

$$G(t) = \int_0^t g(x)dx = |t|.$$

By looking at Figure 7 we see that the antiderivative has a kink at the origin. This implies that the function cannot be differentiated at that point and hence is not differentiable everywhere. Summarizing, the integral cannot be differentiated back to the original Riemann integrable function; this is a contradiction.

The counterexamples stated above raise an important point. One should be careful whilst defining the types of the functions for which the theorem works. We should not state the theorem without making very specific assumptions about the functions, because, as shown in the aforementioned, it does not work in absolutely all cases and therefore we cannot define it as such. The examples have shown us that in order for the Fundamental Theorem of Calculus to work, the functions need to be continuous and Riemann integrable. I will talk about the needed assumptions and the mistakes that have risen due to the lack of carefully defining the functions further in the next Chapter, whilst analysing proofs of different mathematicians over the past centuries.

Before officially stating the theorem and its proof, there are additional definitions and properties that need to be stated, such as the definition of continuity or the properties of Riemann Integrals. They can be seen below. I will refer to these later on.

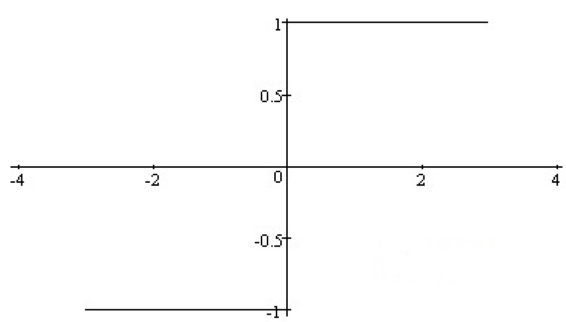


Figure 6: The representation of the function $g(x)$

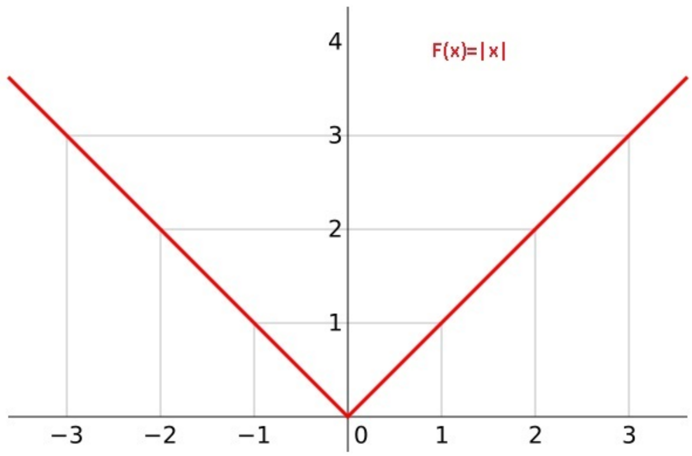


Figure 7: The representation of the antiderivative of the function $g(x)$

Definition 6

f is continuous at c if and only if $\forall \epsilon > 0, \exists \delta > 0$ such that,

$$\mid x - c \mid < \delta \Rightarrow \mid f(x) - f(c) \mid < \epsilon.$$

Theorem 3. (*Properties of Riemann Integrals*)

Suppose that f and g are Riemann integrable functions defined on $[a, b]$ and that $a < c < b$. Then the following rules apply:

Rule 1. $\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b (f(x) + g(x))dx$.

Rule 2. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.

Rule 3. $\int_a^b (cf(x) + dg(x))dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$.

Rule 4. $\mid \int_a^b f(x)dx \mid \leq \int_a^b \mid f(x) \mid dx$.

Rule 5. If $g(x) < f(x)$ then $\int_a^b g(x)dx \leq \int_a^b f(x)dx$.

Rule 6. With the conditions assumed above $f(x)g(x)$ is integrable on $[a, b]$.

Rule 7. $\int_a^b f(x)dx - \int_a^c f(x)dx = \int_c^b f(x)dx$. [13]

[13] Prof Bill Jackson,(2013),Calculus I,Thomas' Calculus,Sections 5.2 to 5.5, School of Mathematical Sciences,Queen Mary University of London

FUNDAMENTAL THEOREM OF CALCULUS, PART I

Suppose that f is a bounded, Riemann integrable function defined on a closed, bounded interval $[a,b]$. Define $F(x) = \int_a^x f(t)dt$. Then F is continuous in $[a,b]$. If f is also continuous, then F is differentiable in (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$.

This part shows how to construct an anti-derivative from a definite integral and therefore is called the anti-derivative part of the theorem.

FUNDAMENTAL THEOREM OF CALCULUS, PART II

Let f be continuous on $[a,b]$ and let G be any anti-derivative of f, then:

$$\int_a^b f(t)dt = G(b) - G(a).$$

This is the opposite to the first part, that is a construction of the definite integral using the anti-derivative, called the evaluation part of the theorem.

PROOF, PART I

We know that f is Riemann integrable on $[a,b]$, so it is bounded on $[a,b]$, by Proposition 1. Therefore there must exist some M such that
$\mid f(t) \mid \leq M \quad \forall t \in [a,b]$.

Now, let's take an arbitrary $c \in [a,b]$ and prove that F is continuous at c.
For any $x \in [a,b]$ we have

$\mid F(x) - F(c) \mid =$

$= \mid \int_a^x f(t)dt - \int_a^c f(t)dt \mid =$

$= \mid \int_c^x f(t)dt \mid \leq$

$\leq \mid \int_c^x \mid f(t) \mid dt \mid \leq$

$\leq \mid \int_c^x M dt \mid =$

$= M \mid x - c \mid.$

Some explanations: The first equality holds by definition of F. The second by Rule 7. The third inequality by Rule 4. The last inequality holds because $\mid f(t) \mid \leq M$ as mentioned earlier.

Now given $\epsilon > 0$, we choose $\delta = \dfrac{\epsilon}{M}$.
Then if $\mid x - c \mid < \delta$ then $\mid F(x) - F(c) \mid \leq M \mid x - c \mid < M\delta = M\dfrac{\epsilon}{M} = \epsilon$.

Summarising, if $\mid x - c \mid < \delta$ then $\mid F(x) - F(c) \mid < \epsilon$, which is the definition of continuity, so we can say that F is continuous at c. Since c is arbitrary (as assumed at the beginning), we deduce that F is continuous anywhere on the interval $[a, b]$.
Now assume that f is continuous and let $c \in (a, b)$ and $x \in [a, b]$. Then

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| =$$

$$= \left| \frac{\int_a^x f(t)dt - \int_a^c f(t)dt}{x - c} - f(c) \right| =$$

$$= \left| \frac{\int_c^x f(t)dt}{x - c} - f(c) \right| =$$

$$= \left| \frac{\int_c^x (f(x) - f(c))dt}{x - c} \right| \leq$$

$$\leq \left| \frac{\int_c^x \mid f(t) - f(c) \mid dt}{x - c} \right|.$$

The first equality holds by definition of F. The second by Rule 7. The third equality comes from the fact that $f(c)$ is constant. Finally, the last inequality holds by Rule 4.

We assumed that f is continuous and this, by the definition of continuity, means that given
$\epsilon > 0, \;\; \exists\delta > 0$ such that $\mid f(t) - f(c) \mid < \epsilon$ for $\mid t - c \mid < \delta$.
Then for $\quad 0 < \mid x - c \mid < \delta$ we have

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \;\; \leq \;\; \left| \frac{\int_c^x \mid f(t) - f(c) \mid dt}{x - c} \right| \;\; \leq \;\; \left| \frac{\int_c^x \epsilon dt}{x - c} \right| = \;\; \epsilon.$$

This result means that $F'(c) = f(c)$ which is equivalent to saying that $F'(x) = f(x)$, hence f is differentiable in (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.

PROOF, PART II

Let G be an anti-derivative of f, as stated in the theorem.
Define a new function of F:

$$F(x) = \int_a^x f(t)dt$$

By the first part of the theorem, it can be seen that F is continuous on $[a, b]$, differentiable on (a, b) and $F'(x) = f(x)$ for $\forall x \in (a, b)$.
Define another function h:

$$h(x) = F(x) - G(x)$$

Then h is continuous on $[a,b]$ and differentiable on (a,b), as it is a difference of two differentiable functions.[14] Moreover, if $x \in (a,b)$ then $h'(x) = F'(x) - G'(x)$, but we know that $F'(x) = f(x)$ by the first part of the theorem, and $G'(x) = f(x)$ by definition of anti-derivative.
Therefore, $h'(x) = f(x) - f(x) = 0$ for every $x \in (a,b)$ and, because in addition h is continuous at a and b, h is constant on [a,b], and hence $h(a) = h(b)$. In particular:

$h(b) = h(a)$

$F(b) - G(b) = F(a) - G(a)$ (By definition of h)

$F(b) = F(a) + (G(b) - G(a))$

$\int_a^b f(t)dt = \int_a^a f(t)dt + (G(b) - G(a))$ (By definition of F)

$\int_a^b f(t)dt = 0 + G(b) - G(a)$

$\int_a^b f(t)dt = G(b) - G(a)$.

And therefore the result is proved.

Q.E.D.

[14] The result that h is continuous is not immediately obvious unless we state the following conditions, consequently leading to the result:
1.1 If f is a continuous function on $[a,b]$ and is differentiable on $[a,b]$ then f is bounded on $[a,b]$ and achieves its bounds.
1.2 If f is differentiable on (a,b) and if $c \in (a,b)$ satisfies $f(x)$ has maximum on (a,b) then $f'(c) = 0$.
1.3 *Rolle's Theorem*
If f is differentiable on (a,b) and continuous on $[a,b]$ and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.
1.4 *(Mean Value Theorem)* If f is differentiable on (a,b) and continuous on $[a,b]$ then $\exists c \in (a,b)$ such that $f'(c) = \dfrac{(f(b) - f(a)}{b-a}$
1.5. This leads us to the final conclusion:
If $f(d) \neq f(a)$ for some $d \in [a,b]$, then $\exists c \in [a,d]$ such that $f'(c) = \dfrac{f(d) - f(a)}{d-a} \neq 0$.

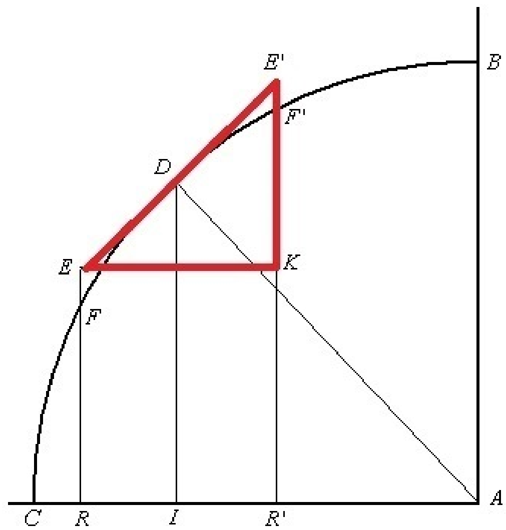
ANALYSIS OF THE HISTORICAL PROOFS

Mathematics has been evolving and changing constantly throughout many centuries. Every year we gain more knowledge and understanding in many different areas. We improve the already existent theorems making them more rigorous in order to fit the standards. However, certain problems arise when it comes to analysis of some theorems with a perfect example being the Fundamental Theorem of Calculus. The differences in the approach to mathematics between the 17th century mathematicians and the current one is rather significant, and many times misleading. In this chapter I will point out the main mistakes and the hidden assumptions made by great mathematicians such as Isaac Newton or his teacher Isaac Barrow, looking at it from a modern perspective.

The connection between differentiation and integration only started to be recognised in the 17th century. Back then, concepts such as integrals, derivatives or even functions did not exist in the modern sense. Newton thought of an integral as a way of calculating the area and this led him to a rushed assumption that the integral was well defined because the area was. From our modern approach, one has to prove that the integral is well defined in order to prove that the concept of the area is. On the other side, the derivative was thought of as the ratio of the sides of the characteristic triangle, which is presented in Figure 8.

Today we know that these definitions are not entirely correct. This leads us onto another important issue. In the 17th century, calculus was not applied to functions but to curves. If a function is defined in a modern sense, then it can have many properties that would have not been thought of a few centuries ago. For example, a function could be continuous but not differentiable anywhere. Moreover, the distance between any two points on the curve representing such a function, measured along the curve, could be infinite. All this would make it impossible to draw such a function. On the other side, a curve was understood as something that could have been easily drawn, without these surprising properties mentioned above. This is why most of the time, mathematicians assumed that all functions on a closed interval, $[a, b]$, were continuous and Riemann integrability was assumed rather than proved. When it comes to open intervals (a, b), we can have a function that is continuous but not Riemann integrable on $(0, 1)$, such as

$$f(x) = \frac{1}{x}$$



[15]

Figure 8: The characteristic tirangle

However, they imagined functions to be only well behaved and did not take these instances into the account. Now we define curves differently than in the 17th century.

Therefore one can say that these problems appeared due to the nature of the definition of a function back in those days. For example, Newton defined it as the position of a particle at some given time. The derivative of the function was therefore the speed of the particle. He made these assumptions without stating them beforehand. He also assumed that these functions were continuous and piecewise monotonic, again without mentioning it beforehand. Such misleading definitions imply that all functions fulfil these conditions and lead up to cases presented in Figures 4 and 6, where the Fundamental Theorem of Calculus cannot be applied.

While proving the FTC, we encounter some problems that were simply beyond the reach of 17th century mathematicians. For instance, the theorem states that for any function f that is continuous, $\int_a^b f'(x)dx = f(b) - f(a)$. This implies that if the speed remains zero in an interval of time, then there is no movement. This is relatively easy to prove if it is assumed that the result holds when $f'(x) = 0$ for all x in the interval. However, Newton attempted to tackle this in the Principia, which often led to problems when calculating the areas because he gave the proof without assuming the result or even without assuming that if $f'(x)$ is given and continuous then the function is given up to a constant. Whilst this approach is taken, the proof is basically out of reach.

The distinctive ability, that both Newton and Leibniz had, was the skill to switch between both the dynamical and the geometrical approach in understanding the FTC. We combine these two aspects of calculus in the 21st century, as the only difference between them is that in one we view the independent variable as the marking distance and in the other as the elapsed time. However, the difference between the two took a while to be fully understood in history and therefore it was not obvious to the mathematicians in the past.

The importance of understanding the meaning of quantifiers is also great. Nowadays, we are aware of the logical use of quantifiers and apply them comfortably as opposed to the past, when they were not such familiar concepts and many times were used vaguely and led to difficulties.

Another issue that is worth a thought, is whether the assumption that $f'(x)$ is continuous is needed for the theorem to work. It very much depends on the version of the integral used. Even though this is not going to be analysed in this project, one should not ignore the problem and assume that $f'(x)$ is integrable.

Different mathematicians approached the Fundamental Theorem of Calculus differently. To begin with, Leibniz sought to find the area under a curve. He stated that if there exists a function F for which the original curve is described as $y = F'(x)$ then the area underneath the function is expressed in terms of F. This yields part II of the FTC. One issue to point out is the manner that Leibniz expressed the idea. The values of what would now respectively be understood as the horizontal axis are positive both below and above the axis. This resulted in problems around the fact that some values appeared to be increasing where they were not. Another issue to point out is the understanding of ratios. In the past, a ratio of lengths could only equal to another ratio of lengths. This is how the distances were referred to in mathematics. Nowadays, the concept is still the same, since if we think of a string being 2 meters long, then the ratio of its length to the length of a standard meter is 2:1. The significant difference is in the notation. While understanding was the same as currently, this notation can cause some confusion being analysed from the modern perspective.

Isaac Barrow had the sufficient knowledge in order to differentiate any curve, by drawing its tangent. In addition, he was aware of the differential triangle method (that is very similar to the characteristic triangle mentioned before), therefore he was also able to compute the derivative numerically. He was the first person to notice the distinct connection between the derivatives and the integrals and represented it as a picture that we can see in Figure 9. He constructed a curve $VIFI$ so that if we made a rectangle with sides DF and R, its area would be the same as the area enclosed by $VZDE$. We identify this curve with the derivative. Next, he reflected this curve across the horizontal axis, i.e.$ZGEG$, constructing what we would call an antiderivative. It is constructed in a way that the ordinates to the horizontal axis decrease in length as we go from the left to the right. Additionally, he drew the figure so that the ratios $DT : R = DF : DE$ and $LF : LK = DE : R$ agreed. Due to this, we have $LK * DE = LF * R$. By the hypothesis, $LF * R$ was the area enclosed by $PGED$. We know that this area is greater than the area enclosed by the ordinates to the horizontal axis which is located to the left of the preceding area (in this case also $PDGE$), and so on. This is because the curve $ZGEG$ was constructed in this particular way. It leads us to conclude that KFK touches the upper curve at a point, and in particular this point is F as represented.

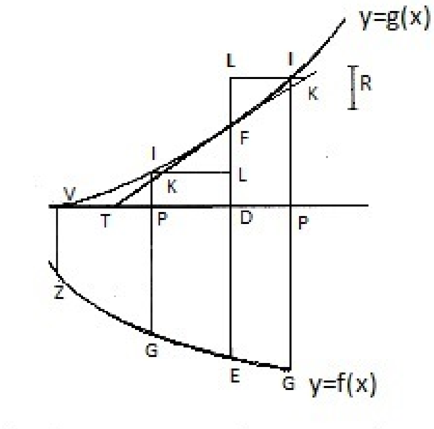
If we define the general ordinate of the curve $ZGEG$ to be y, and the general ordinate of the curve $VIFI$ to be y_1 then the theorem becomes such that

$$\int y = \text{area VDEZ} = \text{R.DF=R.}y_1;$$

It follows that dy_1/dx=FL/LK=(area PGED/R)LK=DE/R,

$$\text{i.e. R.d}y_1\text{/dx=y.}$$

In addition, he shows that the sides of the characteristic triangle of the antiderivative are equal to the ratio of the height of the original curve to some constant. The contribution of Isaac Barrow into the mathematical world was great, taking into account the little time that he had spent working in the field. However, even though it can be concluded that Barrow did recognize the strong connection between integration and differentiation (for example, he did recognise the necessity to prove not only the theorem but also its converse), he never used the theorems. Besides, he never realised the importance of his discoveries, nor did he complete his work. Furthermore, he took things as fact, even though he omitted their proofs; it was very intuitive (for example in Lecture XI, Chapter 27)[16]. We should therefore realise that it is the combination of the algorithmic technique and the full understanding of the theorem when it becomes the most useful.



[17].

Figure 9: Geometric representation of the fundamental theorem of calculus by Issac Barrow

As mentioned beforehand, Newton understood a function as a rate of change and the definite integral of this function as the accumulation of the change (which can be understood in the modern terms as the Riemann Sum). Furthermore, he states that the motion by which the curve increases is the ordinate of the curve, that is the height of the curve, leading to the first part of the theorem: 'the rate of change of the area is given by the ordinate of the bounding curve'.[18] If the antiderivative (the rate of change) is known, then the area under the curve can be calculated, which essentially is the second part of the theorem. In order to understand this approach, we do need to realise the meaning of the ordinate of the curve as the rate of change of the area.

"While calculus as understood in the late 17th and early 18th century was recognized to have broad applications to variable phenomena, it was always presented as a tool for analysing curves. Euler was the person chiefly responsible for the shift from an analysis of curves to an analysis of functions."[19].

[16]J.M.Child,"Criticisms and discussions.",Derby, England,1914,p.256-261

[18]David M. Bressoud. *Historical Reflections on Teaching the Fundamental Theorem of Integral Calculus.* (The American Association Monthly, Vol. 118, No. 2 (February 2011), p.104

[19]David M. Bressoud. *Historical Reflections on Teaching the Fundamental Theorem of*

This can be argued. Firstly, what does "broad applications to variable phenomena" mean? For example, Newton took a curve and parametrised it by some parameter that, in his opinion, corresponded to time. It does not mean that such parameter did correspond to time. Therefore, the speed with which a point moved along the curve defined by Newton did not correspond to the actual speed of a physically defined moving point. The question of this parameter, differently defined, due to the lack of rigorousness, is the distinction between the 'curves' and 'variable phenomena'. A tangent of a curve or an area under a curve are independent of parametrization of the curve therefore, it is rather meaningless to state that calculus was a tool for analysing curves (instead of stating that calculus was being concerned with 'variable phenomena').It is important to highlight that the main concern here is not the distinction between the functions, curves or 'variable phenomena' but the type of functions that were considered. For example, it is not clear to which extend Newton considered negative exponents in the definition of the functions that he considered, even though he did consider power series and fractional exponents. In this manner, there is no evidence that he considered functions such as $f(x) = \dfrac{1}{x}$ for $x \neq 0$, that has a negative exponent for x^{-n} for $\forall n > 0$. Finally, one needs to keep in mind that functions were not defined in the same manner in the 17th or 18th century as today. Functions were defined to be well-behaved, continuous, geometrical objects that can be easily drawn (that is very far from the recent definition). If we consider this, then it seems that from a modern point of view, there is not a big difference between a curve (that represents such easily drawn geometrical object) and a function, so the importance of Euler's impact vanishes.

Meanwhile, Cauchy introduced the definition of the definite integral in the 19th century. He defined it in accordance with the modern approach, that is

$$\textstyle\int_a^b f(x)dx = f(b) - f(a)$$

The indefinite integral was defined as

$$F(x) = \textstyle\int f(x)dx.$$

Integral Calculus, p. 106

In order to prove that $F'(x) = f(x)$ he used the mean value theorem (stated in a footnote explaining a result in the proof of FTC) and the fact that f is continuous, that is the first part of the theorem. Then he proves that any function whose derivative is zero must be constant and so that any two derivatives of the same function must differ by a constant; putting this in a mathematical context
$\int f(x)dx = F(x) + C$ where C is some constant. Hence, the second part of the theorem follows.
Cauchy made a clear difference between the definite integral and the concept of integral understood as the antiderivative.

The notion of a function only became sufficiently rigorous in the 19th century. Mathematicians have realised that functions that are nowhere continuous exist and therefore the statement of the Fundamental Theorem of Calculus had to become more rigorous as well, namely the type of functions that it considers. We should not remove these assumptions and make sure that we state them very clearly before proceeding any further. One needs to realise that FTC has been developed over many years and looked at from many points of view, but on the way, there have been disagreements and problems encountered by the mathematicians due to the lack of the rigorous approach that was caused partially by the lack of knowledge that we have in 21st century.

CONCLUSION

During the extensive research over the past months, I have realised what a long path it has been towards the reformation of the discovery discussed, so that it could be studied and understood in the twenty first century in terms of general functions. One should not be surprised about the amount of research that is still carried out around this topic, taking into account the time taken by mathematicians just to define some concepts, such as the integral or the derivative.

In this project I have first analysed the theorem, looking at the geometrical and the dynamical approaches that deepen its understanding, and then I have discussed how challenging it was to notice the connection by the seventeenth century mathematicians. These days we know that to make the switching between the two approaches easier, one could focus on the dynamic approach first and then visualise it via the geometrical approach.

I have talked about certain mathematicians, such as Newton, Barrow or Leibniz, who lacked in rigorous approach when stating definitions or made hidden assumptions; later, I have given examples of problems that could occur when this happens. For example, we have seen that it is not enough to understand integration as simply the opposite process of differentiation but also as the limit of Riemann sums. One can use this limit in order to construct an antiderivative for any function that is continuous and, if such antiderivative is known, one can evaluate back the limit from the antiderivative. [20]

It is now clear that the Fundamental Theorem of Calculus cannot be applied to all functions, as it was believed a few centuries ago. One should keep this in mind when applying the theorem or when reading the literature from that period, not only because of the different notation used but also due to the gaps in the interpretation of some important mathematical concepts.

Today, we have sufficient knowledge to avoid this approach and fully understand the theorem. We should not mechanically apply the methods in order to find areas under curves, but stop and deeply think about what it means that we are doing. Moreover, we should see functions for what they are, that is expressions describing relations between varying quantities[21]. We

[20] David M. Bressoud, *Historical Reflections on Teaching the Fundamental Theorem of Integral Calculus, The American Association Monthly, Vol. 118, No. 2 (February 2011), p.112*

[21] ibid.

should appreciate how privileged we are because of all the knowledge that is available to us at this time in the history. If we want to understand the Fundamental Theorem of Calculus fully, we should also make sure that we know, up to the recent rigour, not only the concept of a function, but also notions of accumulation, rate of change, antiderivative, or derivative. Calculus emerges from these ideas, therefore one should not move forward without fully grasping what is beforehand.