An Introduction to the Cellular Automaton Model of Spontaneous, Single Lane, Traffic Jams

A Masters Thesis by Andy Drizen

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1 Introduction

Despite the fact that in 2000, the British government developed a 10-year plan to stabilise congestion levels, the volume of cars is increasing and average traffic speeds are falling. After just three years the government had to admit their plan had failed [4] - in fact, congestion is more likely to increase by 25% [7]. Cities like London, (amongst the worst sufferers of congestion in Europe) are reporting loses between £2-4 million per week in terms of lost time, with drivers spending 50% of this time in queues [5]. Nationally, it is reported that industry and commerce is losing an estimated £3 billion a year [4]. Tony Blair, (former Prime Minster of England) released a statement in which he said that increasing the capacity of roads would not be an option as ”one mile of motorway costs as much as £30 million” [7]. Traffic congestion has many guises and is not just limited to the road, for example you may experience queues when you buy your shopping, wait for a car park space, browse the Internet or have a mobile telephone call disrupted due to the network being ”busy” - a problem that always occurs immediately after a disaster as the public try to contact their loved ones and overload the network. As well as being time consuming, bad for business and the environment, a German study [6] suggests that you are three times more likely to suffer a heart attack from traffic jam related stress.

It is clear that defeating the congestion problem is in the best interests of our society. I will study a basic system, which expresses the characteristics of traffic jams. To avoid obscurities I will investigate traffic jams that occur on a single lane without the influence of external causes (e.g. road works, collisions, breakdowns). To do this I use the model described by Nagel and Schreckenberg [1], which provides a realistic model of traffic flow. I will begin by explaining the rules of model and then I will analyse and discuss the jam lifetime.

2 The Model

To begin, consider your favourite stretch of single carriage-way road. Now divide it in to L equal parts called *cells* and only let one car in each cell at a time, labelled with its instantaneous velocity (figure 1). We define the *lane length*, L, to be the number of cells in the lane. Note that the velocity means how many cells the car will move in one time step. To prevent the car from being between cells, we enforce a rule, which states that the positions of the cars are only updated after one complete time step (i.e. this is a discrete system, not a continuous one) and all velocities are integers (at least zero) and less than the speed limit of the lane; call this v_{max}. The final notable point about this model is that there are two types of boundary conditions, i.e. what happens to the cars as they reach the end of the lane. The first lane type is *periodic*. This means that if a car passes the L^{th} cell, it will reappear at the beginning of the lane in cell 1; this is a loop system e.g. a race course circuit. The second type is *open*. In an open boundary system the cars enter and leave the lane with certain probabilities.

We model the lane in the following way. Denote each cell in the lane by a hyphen. If a car is in a particular cell, replace the hyphen with the velocity of the car. As this *cellular automaton model*, (which means it is a discrete, regular grid of cells and each one is in a finite number of states) is of a single lane, cars may not overtake each other. Therefore each car is in exactly one cell and each cell can have at most one car in it. For example, An empty lane would be written as ”- - - - - - - - - - - - - - -”, if the lane had a car with velocity 5 on it in cell three, the representation of the lane would look like this ”- - 5 - - - - - - - - - - - -”.

As cars are not permitted to overtake other cars, if the velocity of some car drops to zero for a sufficiently long time (on an open boundary system or on a periodic boundary system with at least 2 cars), a traffic jam will occur. We can formalise what we have stated in the following four rules, which were originally formed by [1]. Note that time and space are one-dimensional variables and the cars move along a one-dimensional chain.

1. **Acceleration:** if the velocity v of a vehicle is lower than v_{max} and if the distance to the

Figure 1: Here is a lane with two cars on it. I have put their velocities in red on each car. This can represented as "- - 2 - - - 3 - - - - -".

next car ahead is larger than $v+1$, the speed is advanced by one, i.e. $f_1 : v \to v+1$.

2. **Slowing Down (due to congestion):** if a vehicle at site i sees the next vehicle at site $i+j$ (with j≤ v), it reduces speed to $j-1$, i.e. $f_2 : v \to j-1$.
3. **Randomisation:** with probability p, the velocity of each vehicle (if greater than zero) is decreased by one, i.e. $f_3 : v \to v-1$.
4. **Car motion:** each vehicle advances v sites.

The first rule ensures that cars drive at the speed limit if they are able to. The second rule looks ahead to see if any two cars are getting too close, meaning the rear car must decelerate (as overtaking is not permitted). The third rule prevents the system from being deterministic, which accounts for the varying behaviour of drivers, e.g. slow reacting drivers, slow moving vehicles drivers. Without rule 3, the system could be described by a recurrence relation and possibly a closed formula for any time t. The fourth rule controls our discrete time steps - the system is updated, in parallel, every time rule four is applied. One complete time step consists of applying these four rules consecutively. Figure 2 depicts the system for some initial conditions.

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Figure 2: The first line may be thought of as the time-step $t = 0$, the second line would be $t = 1$, etc. Every hyphen represents an empty cell, and every number implies that a car occupies the cell where the number is the instantaneous velocity of the car.The traffic jam is the set of cars with zero velocity. The vertical length of the traffic jam represents the jam lifetime, T. In this case, T=14. The number of zeroes is known as the traffic jam mass, M. In this scenario, M = 25. This screen capture shows part of the traffic system which has periodic boundary condiditons.

A common feature of many traffic analysis work is the aptly named "Fundamental Diagram", which illustrates theoretical and experimental traffic flow data, where the flow rate is shown against traffic density. I shall now investigate a small system to demonstrate how to create such a diagram.

I have generated some data (see appendix 1) using a computer program which I created. This data is based on a system with lane size (or number of cells) L=10, maximum velocity, $v_{max} = 4$, number of time steps (or iterations) I=10 and $p = 0.5$ (this is the probability mentioned in rule 3) and the number of cars, N varies. The raw data appears to show that congestion starts to become a significant issue by at least 30% density. We will use the fundamental diagram to inspect this property momentarily. First, lets look at how we construct the fundamental diagram for this data.

We begin by defining the *flow*, which is the average number of cars that pass a given point per time step. To calculate the flow we place a marker between cells i and $i+1$ at time t. We define a variable $k(t)_{i,i+1} = 0$ and increment it by 1 for every car that passes site i in one time step. In the illustration in figure 3, we see that in one time step, one car would have passed through the marker, so $k(0)_{6,7} = 1$.

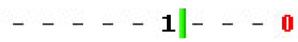

Figure 3: After one time step, one car passes through the marker, so $k(0)_{6,7} = 1$

Now sum over all values of t in your system and then divide by the number of time steps to obtain the *average flow*. I have shown a system (figure 4) with periodic boundary conditions, lane size $L = 10$ and number of time steps $I = 10$ (taken from appendix 1, (2 cars, trial 2)).

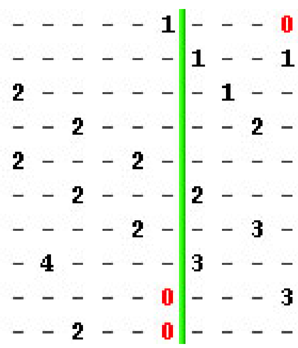

Figure 4: After ten time steps, three cars will pass the marker. As there are 10 time steps, the average flow is $3/10 = 0.3$

To further improve this average we sum over $i = 1, ..., L$ (i.e. we place the marker between every pair of adjacent cells) and obtain the *total average flow*, j, for the system by dividing by L.

$$j = \frac{\sum_{i=1}^{L} \sum_{t=1}^{T} k(t)_{i,i+1}}{LT} \tag{1}$$

Equivalently, we observe that if a car in position u has velocity v then it will contribute to the total average flow if we place the marker between cells $(u, u+1), (u+1, u+2), ..., (u+v-1, u+v)$ e.g. a car with velocity 4 increments 4 markers (see figure 5). In general, each car b at time-step t with velocity $v_{b,t}$ increments exactly $v_{b,t}$ markers. Therefore,

$$\sum_{i=1}^{L} k(t)_{i,i+1} = \sum_{b=1}^{N} v_{b,t} \tag{2}$$

for given t.

So we can calculate the total average flow by summing all velocities of all N cars and then dividing by the product of the lane length (L) and number of iterations (I). Hence, we can rewrite the total average flow as:

$$j = \frac{\sum_{t=1}^{I} \sum_{b=1}^{N} v_{b,t}}{LT} \tag{3}$$

We note that as this is a periodic system, the density is

$$\rho = \frac{N}{L} \tag{4}$$

where N = Number of cars in the loop and L = Total number of sites in the lane and

$$\begin{aligned} j &= \frac{\sum_{t=1}^{I} \sum_{b=1}^{N} v_{b,t}}{LT} \\ &= \frac{\sum_{t=1}^{I} \sum_{b=1}^{N} v_{b,t}}{NT} \frac{N}{L} \\ &= \bar{v}\rho \end{aligned}$$

which states that the total average flow, j, is the average velocity multiplied by the density.

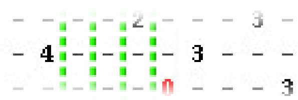

Figure 5: The dotted green line represents the four markers that will count the car with velocity 4.

Using (3) to calculate the total average flow for figure 4, which has density 2/10 = 0.2, we find that j=0.35. We now have the first point we can plot on our fundamental diagram, which is (0.2, 0.35). We now repeat the test numerous times with the same density to improve the accuracy of the reproducibility of the test. Now we want to compare this result with varying density values, (which I have done for the collected data) and create a scatter plot with a line of best fit - this is the completed fundamental diagram (figure 6). We can see from the plot that when the density is 0.1, (i.e. one car in our 10-site system) the car is able to move freely. However, the flow is not optimal as only one car is contributing to it. The flow increases to a peak very quickly at the density 0.2 (where there are 2 cars on the road), at which point cars are able to accelerate to their maximum speed easily. This is the optimum number of cars for these initial conditions. Increasing the density further results in reducing the flow implying that congestion occurs.

This example has been useful to explain how the Fundamental Diagram is created and to demonstrate what it can show. However, with only 10 cells and 10 iterations, it does not provide a very big sample. I will now use the same computer program to create a simulation that has lane size L=500, number of iterations I=500 and p=0.5.

From figure 7, we can see a similar curve to what we previously obtained. The flow starts low but rapidly increases to a peak at 15% density. The flow then proceeds to decrease as the density increases. The general shape of the curve is satisfying as it is fairly similar to the fundamental diagram found in [1], who used a much bigger time-step average (figure 8).

A particular use of the fundamental diagram is to show how enforcing speed restrictions can help beat congestion. In [8] there are six superimposed fundamental diagrams which demonstrate this. Keeping the same value of p, lane size and number of iterations as seen in figure 7, I ran three more simulations with varying v_{max} (figure 9). As [8] observes, "the density of maximum throughput decreases" as v_{max} increases. After this maximum has passed, congestion follows. Therefore, by reducing v_{max}, the maximum throughput is shifted further and further to right, meaning the road can get denser and denser before congestion sets in. This technique is advised for certain roads in the UK, and is compulsory for 30KM of the M25 - the motorway orbiting London. It is an automatic system that works by having sensors buried in the road which monitor the traffic flow. These sensor then relay what the speed limit should be to the overhead gantries. Although this does not increase the number of cars through the road, it does reduce accidents by removing the ability for drivers to break heavily [9].

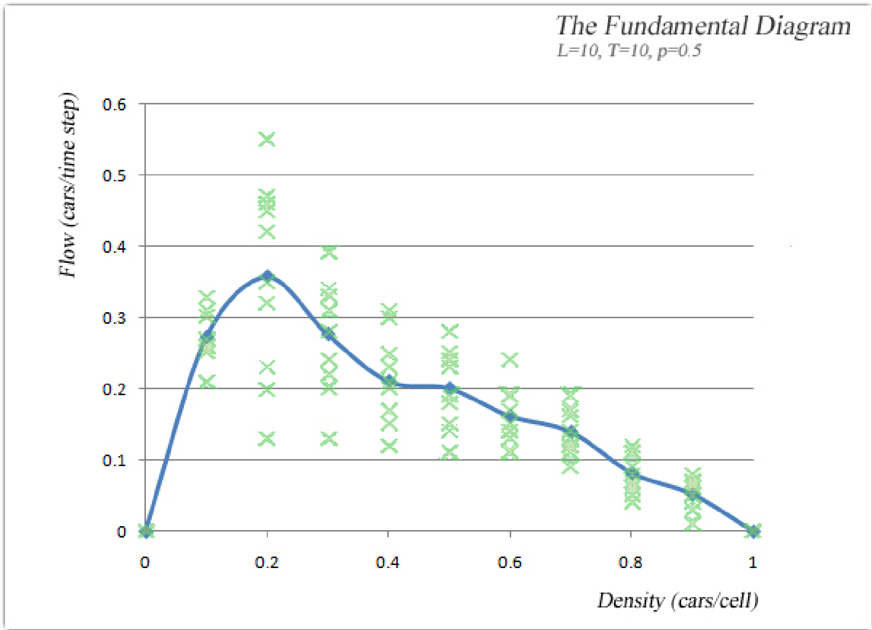

Figure 6: The green crosses represent the average flow for 10 time steps. The blue line is the average of 10^2 time steps. In this simulation, $v_{max} = 4$

The fundamental diagram provides an intuitive understanding of the traffic flow. However, the details contained in such a diagram and the underlying model are in fact still too complicated. So in order to give a more detailed analytical account of traffic flow, I will follow the presentation in [2]. I am now going to focus on properties of a single traffic jam which may occur in the system. The general set up is still to complicated to be handled analytically so I will consider a simplified type of model. With this in mind, we consider a system with open boundary conditions (i.e. cars enter the system with probability p') and induce a traffic jam by setting exactly one of the velocities to zero, and all of the other velocities to one.

Note that by doing this we can combine rules 1 and 3 to obtain 3′:

3′. Randomisation: If $v = 0$ and the adjacent site to the right is empty, then the velocity increases to 1 with probability p.

Although in [2], a detailed analysis of traffic flow required a restriction of $v_{max} = 1$, I see no reason for doing this for the subsection of the analysis I am focusing on and will therefore omit this restriction.

The two factors which keep this system from being deterministic are:

1. The stopped car will accelerate away with probability p (by rule 3′).
2. The probability that a car joins the back of the queue is p'

Finally, we say the traffic jam is resolved when all the vehicles have velocity of at least one.

3 Analysis of Jam Lifetime

I will consider the first traffic jam on the road (from the left) and claim it originated due to a single car stopping for non-perminant reasons (e.g. traffic accident). The evolution of the queue is probabilistic and depends on the following three outcomes, called the *Queue Transition Probabilities*:

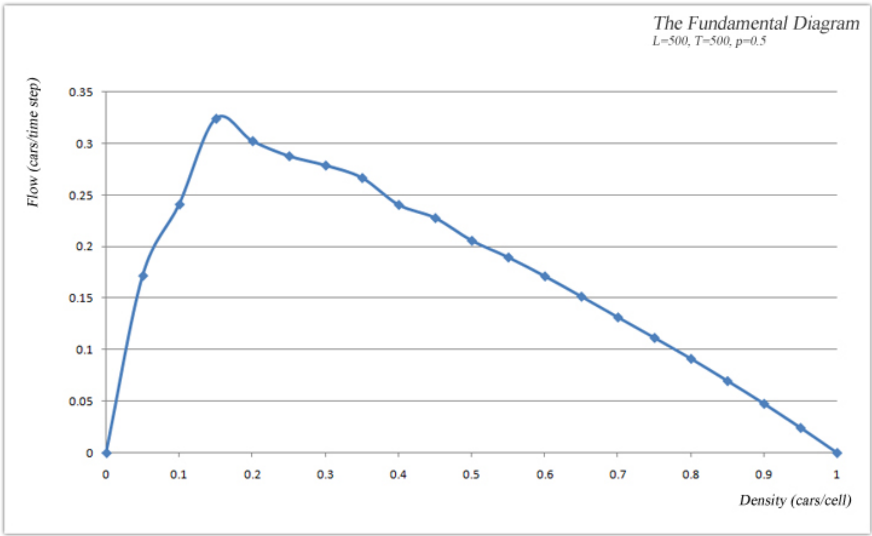

Figure 7: The points on the blue line represents averages of 500 time steps. In this simulation, $v_{max} = 4$. We can see that the curve is wavy - I would suggest this is the fingerprint of an inadequate sample size and not a true reflection of the realistic situation.

1. The queue grows: a car joins the back of the queue and the front car of the jam does not leave - this occurs with probability $p'(1-p) =: P_+$;

2. The queue remains the same length: either a car joins the back of the queue and the front car leaves, or no car joins the back and the front car doesn't leave. This occurs with probability $p'p + (1-p')(1-p) =: P_0$;

3. Or finally, the queue shrinks: no car joins the back of the queue and the front car accelerates away. This occurs with probability $(1-p')p =: P_-$.

It was observed in [3] that the queueing model was little more than a lattice path $w = (s_0, s_1, ..., s_n) \in \mathbb{N}\times\mathbb{N}$ such that the starting point is $s_0 = (0,0)$, the ending point is $s_n = (n,0)$ moving across the lattice using three moves: north-east, east or south-east; this is known as a *Motzkin path*.

We can easily convert our model to a Motzkin path by the following method: instead of finishing at $s_n = (n,0)$, finish at $s_n = (n-1,0)$, which would mean there is exactly 1 car in the queue with the understanding that at time n the final car in the queue accelerates, ending the traffic jam. Figure 10 illustrates how the Motzkin path might look for jam that lasts for 8 time-steps.

As previously defined, the queue transition probabilities are as follows:

$P_+ := p'q$, $P_- := pq'$, $P_0 := pp' + qq'$, where $q = 1-p$ and $q' = 1-p'$.

Let us now begin to look at the probabilities that a jam lasts for n time-steps. We do this by asking such questions as, "What is the probability that the taffic jam lasts for n time-steps?" and we denote this by $P(T = n)$. As we begin the process by setting the velocity of one of the cars to zero, the probability that a jam lasts for zero time-steps is 0, i.e., $P(T = 0) = 0$. If the jam only lasts for 1 time-step, (a lifetime-1 jam), this means at time 1, the stopped car accelerated away without another car joining the back of the queue, so we have $P(T = 1) = P_-$. A lifetime-2 jam can only occur if the stopped car remains stopped after one time-step, and then it accelerates away in the next, i.e. $P(T = 2) = P_0P_-$.

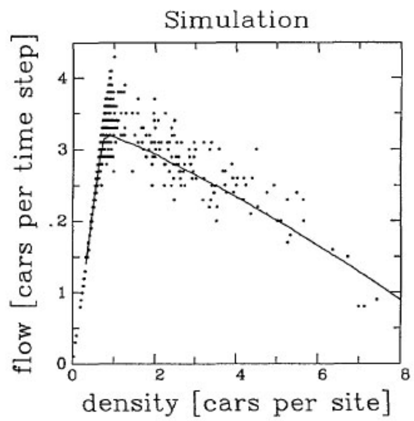

Figure 8: This is the fundamental diagram produced in [1]. We observe that it is basically the same shape as the diagram we produced. This version is based on a loop system where the lane size, $L = 10^4$, dots represent averages over 100 time-steps whereas the line represents averages over 10^6 time-steps.

Now lets consider how we can create a recurrence relation for t time-steps. We know that $T \geq 1$ as the model involves deliberately stopping a car to induce the jam, so we know the first point of the Motzkin path will be at $(0, 0)$. After one time-step (and not ending the jam) there are two possible moves on the Motzkin path, namely east or north-east.

If we take the east path, then there are still $t - 1$ time-steps left, and we know that the probability of the jam lasting this long is exactly $P(T = t - 1)$. So this contributes $P_0 P(T = t - 1)$ to the recurrence relation. This is depicted in figure 11.

The alternative to going east is to move north-east. As the end of the path must occur at the x-axis there must be a point k where the path meets the axis again - however, not necessarily the end of the path. So by going north-east initially, we contribute P_+, then looking at the probability of a queue lasting $k - 1$ time-steps contributes $P(T = k - 1)$ and finally we look at the remaining $t - k$ time-steps, which contributes $P(T = t - k)$. Summing over all possible values of k yields the probability of a queue lasting t time-steps given that the first move was north-east. Note that k starts from 2 as $P(T = 0) = 0$ (which occurs when $k = 1$). Also, as we define what happens at time-step 1, the sum will go up to $t - 1$. Figure 12 illustrates this.

So this contributes:

$$P_+ \sum_{k=2}^{t-1} P(T = k - 1) P(T = t - k), \tag{5}$$

and by renumbering the summation,

$$P_+ \sum_{k=1}^{t-2} P(T = k) P(T = t - k - 1), \tag{6}$$

So for $t \geq 3$ we have:

$$P(T = t) = P_0 P(T = t - 1) + P_+ \sum_{k=1}^{t-2} P(T = k) P(T = t - k - 1). \tag{7}$$

Now I will show how to get a generating function from eq.(7). The generating function is useful because one can sometimes find very interesting results such as a closed formula instead of

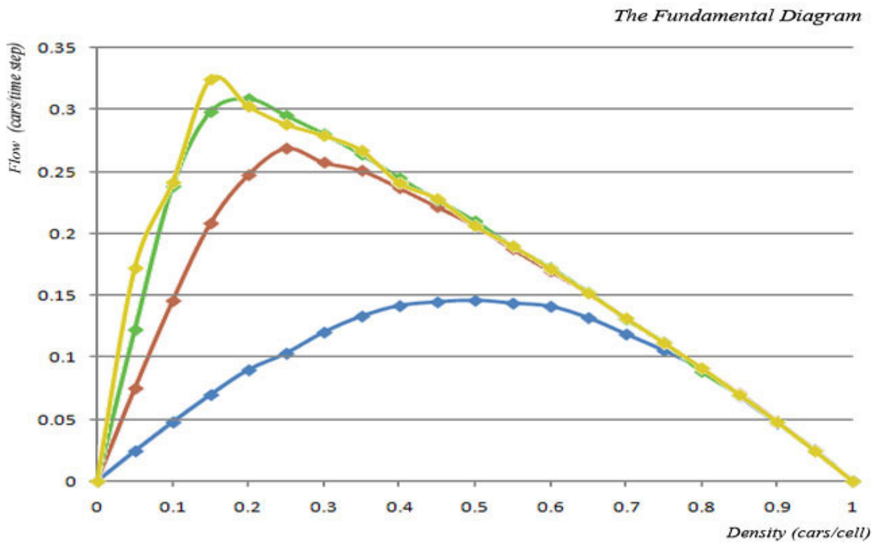

Figure 9: The yellow line is the fundamental diagram that we saw before (with $v_{max} = 4$). The other lines represent the same time averaged results over the same lane size, but with different values of v_{max}, namely: green has $v_{max} = 3$, red has $v_{max} = 2$ and blue has $v_{max} = 1$. It is also important to note that although the yellow line dips below the green line here, this is merely the artifacting created from a low sample size. In reality we do not expect lowing v_{max} to produce an increased flow rate.

a recurrence, or in our case, the expected value of the jam lifetime. We use the ordinary generating function where $|x| < 1$ (as we will want to analyse it later):

$$\begin{aligned} G(x) \equiv \sum_{t=0}^{\infty} P(T=t)x^t &= (P_-x + P_-P_ox^2) + \sum_{t=3}^{\infty}[P_0P(T=t-1) + P_+\sum_{k=1}^{t-2} P(T=t-1-k)P(T=k)]x^t. \\ &= (P_-x + P_-P_0x^2) + P_0\sum_{t\geq 3} P(T=t-1)x^t + P_+\sum_{t\geq 3}\sum_{k=1}^{t-2} P(T=t-1-k)P(T=k)x^t \end{aligned}$$

Let

$$\begin{aligned} A &= P_-x + P_-P_0x^2 \\ B &= P_0\sum_{t\geq 3} P(T=t-1)x^t \\ C &= P_+\sum_{t\geq 3}\sum_{k=1}^{t-2} P(T=t-1-k)P(T=k)x^t \end{aligned}$$

So $G(x) \equiv \sum_{t=0}^{\infty} P(T=t)x^t = A + B + C$

Lets begin by studying B:

$$\begin{aligned} B &= P_0\sum_{t\geq 3} P(T=t-1)x^t \\ &= xP_0\sum_{t\geq 2} P(T=t)x^t \\ &= xP_0(G(x) - P_-x) \\ &= xP_0G(x) - P_-P_0x^2 \end{aligned}$$

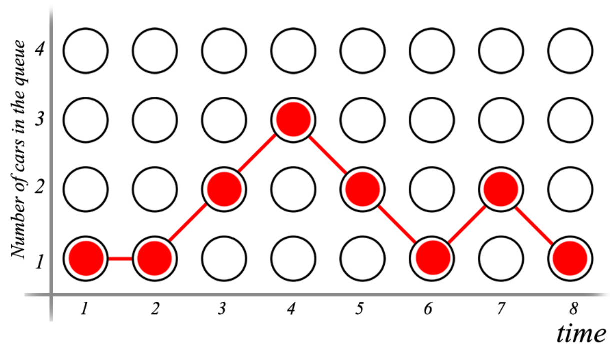

Figure 10: This shows how the Motzkin path can resemble the traffic jam. Using the queue transition properties, the probability for the occurence of such a jam would be $P_0P_+P_+P_-P_-P_+P_-P_-$ where the last P_- is not pictured, but it is added for the reason previously stated.

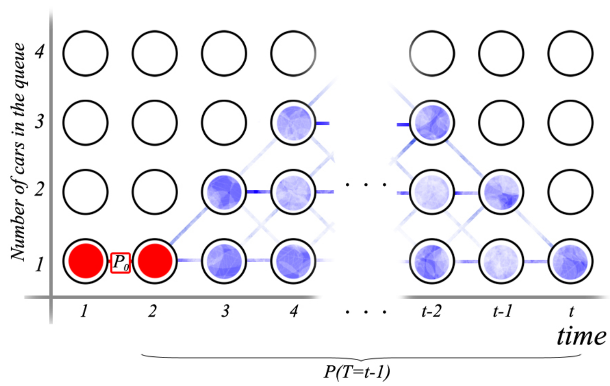

Figure 11: The red lines and circles represent actual parts of the Motzkin path and the blue lines and circles represent possible paths.

Now lets study C:

$$\begin{aligned} C &= P_+ \sum_{t\geq 3} \sum_{k=1}^{t-2} P(T = t-1-k)P(T = k)x^t \\ &= P_+ \sum_{t\geq 3} \sum_{k=0}^{t-3} P(T = t-1-(k+1))P(T = k+1)x^t \\ &= P_+ \sum_{t\geq 0} \sum_{k=0}^{t} P(T = t-2-k)P(T = k+1)x^t \end{aligned} \tag{8}$$

Setting $a_t = P(T = t+1)$ and $b_t = P(T = t-2)$ we can transform (8) to

$$= P_+ \sum_{t\geq 0} \sum_{k=0}^{t} b_{t-k}a_k x^t$$

Which is the Cauchy product of two power series, so

$$C = P_+ (\sum_{t\geq 0} a_t x^t)(\sum_{t\geq 0} b_t x^t)$$

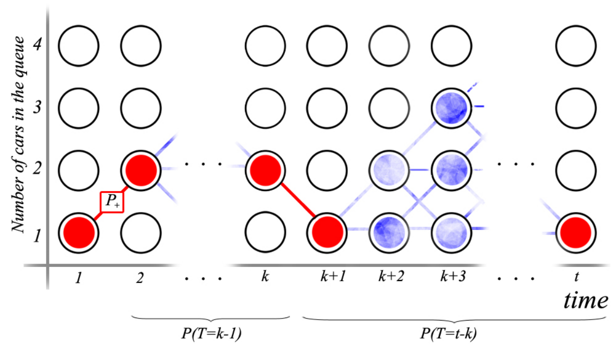

Figure 12: Red and blue represent the same as they did in figure 11. Notice that at time $k+1$ the path hits the axis again for the first time since the start of the path.

$$\begin{aligned}
&= P_+(\sum_{t\geq 0} P(T=t+1)x^t)(\sum_{t\geq 0} P(T=t-2)x^t) \\
&= xP_+(\sum_{t\geq 0} P(T=t+1)x^{t+1})(\sum_{t\geq 0} P(T=t-2)x^{t-2}) \\
&= xP_+(\sum_{t\geq 1} P(T=t)x^t)(\sum_{t\geq -2} P(T=t)x^t)
\end{aligned}$$

As $P(T=i)=0$ if $i \leq 0$,

$$\begin{aligned}
C &= xP_+(\sum_{t\geq 0} P(T=t)x^t)(\sum_{t\geq 0} P(T=t)x^t) \\
&= xP_+(G(x))(G(x)) \\
&= xP_+G(x)^2
\end{aligned}$$

So

$$\begin{aligned}
G(x) \equiv \sum_{t=0}^{\infty} P(T=t)x^t &= P_-x + P_-P_0x^2 + xP_0G(x) - P_-P_0x^2 + xP_+G(x)^2 \\
&= P_-x + xP_0G(x) + xP_+G(x)^2 \qquad (9)
\end{aligned}$$

By using the quadratic formula it is straight forward to solve (9) for $G(x)$ to get

$$G(x) = \frac{1 - P_0x \pm \sqrt{(1-P_0x)^2 - 4P_+P_-x^2}}{2P_+x} \qquad (10)$$

Note that there are two solutions to this equation. Using the initial conditions we can conclude that the positive solution is not appropriate. Remember that $G(x) \equiv \sum_{t=0}^{\infty} P(T=t)x^t$. Observe that when $x=0$, $G(x) = G(0) \equiv \sum_{t=0}^{\infty} P(T=t)0^t = P(T=0) = 0$. Now lets use that initial condition to see what happens when we use the two given solutions.

Taking the positive root:

$$\lim_{x\to 0} G(x) = \lim_{x\to 0} \frac{1 - P_0x + \sqrt{(1-P_0x)^2 - 4P_+P - x^2}}{2P_+x}$$

As the numerator tends to a finite number > 0 and the denominator tends to 0, the limit is undefined and thus cannot not fulfil the initial condition.

Lets look at the negative root:

$$\lim_{x\to 0} G(x) = \lim_{x\to 0} \frac{1 - P_0x - \sqrt{(1-P_0x)^2 - 4P_+P - x^2}}{2P_+x}$$

As both the numerator and denominator limit to 0; we are in a position to apply L'Hôpital's rule, which states that if $f(x)$ and $g(x)$ are two functions that both tend to 0 or $\pm\infty$ as $x \to c$ then the $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ (where here $'$ denotes the first derivative) as long as the limit of the quotient exists. Applying this theorem,

$$\begin{aligned}
\lim_{x\to 0} G(x) &= \lim_{x\to 0} \frac{1 - P_0x - \sqrt{(1-P_0x)^2 - 4P_+P - x^2}}{2P_+x} \\
&= \lim_{x\to 0} \frac{-P_0 - \sqrt{1 - 2P_0x + P_0^2x^2 - 4P_+P_-x^2}(-P_0 + xP_0^2 - 4P_+P_-x)}{2P_+} \\
&= \frac{0}{2P_+} \\
&= 0
\end{aligned}$$

which fulfils the initial condition. So,

$$G(x) = \frac{1 - P_0x - \sqrt{(1-P_0x)^2 - 4P_+P_-x^2}}{2P_+x} \tag{11}$$

To reiterate, in this calculation we have not mentioned that $v_{max} = 1$. Furthermore, in the event that a car is forced to stop for non-perminant reseasons, as rule 3′ is equivalent to rules 1 and 3 $G(x)$, this analysis is valid for all $v_{max} = n, n > 0, n \in \mathbb{N}$ in the model defined by [1] for a particular jam.

Using the generating function we are able to recover the probability mass function (pmf), $P(T = k)$, by evaluating $\frac{G^{(k)}(0)}{k!}$. Using Maple I have calculated the pmf for $k = 1, ..., 20$ for three different sets of probabilities and plotted the probability distribution for those values (figure 13). We observe that the probability that the jam resolves quickly is much higher when $p > p'$, i.e. when the probability of cars leaving is higher than the probability of cars joining the jam. It also appears that when $p' > p$ the probability that the jam resolves takes much longer - perhaps forever? We shall investigate these situations now.

We want to probe the case where $P(T = \infty)$. Recall that the probability of a finite jam can be expressed as:

$$\sum_{n=0}^{\infty} P(T = n). \tag{12}$$

Also, we know that the sum of all the possible probabilities is one, so the probability of an infinite-lifetime queue is:

$$P(T = \infty) = 1 - \sum_{n=0}^{\infty} P(T = n). \tag{13}$$

So $P(T = \infty) = 0 \Leftrightarrow \sum_{n=0}^{\infty} P(T = n) = 1$. If we now compare this to how we defined $G(x)$, we can see that

$$P(T = \infty) = 0 \Leftrightarrow \lim_{x\to 1} G(x) = 1 \tag{14}$$

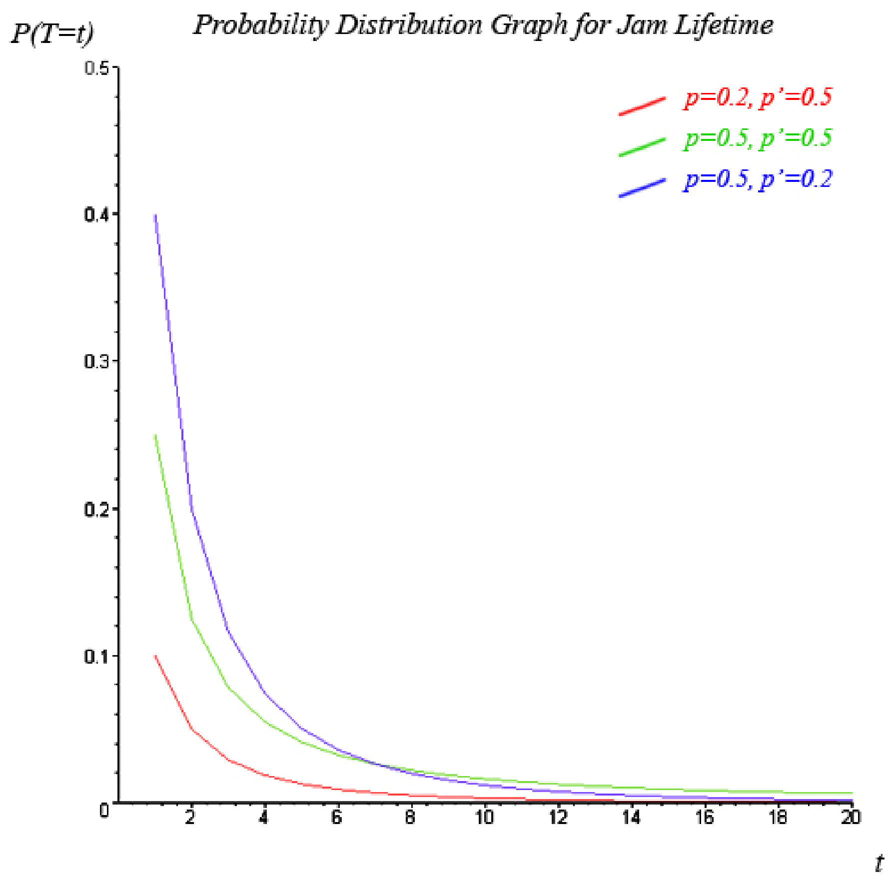

Figure 13: Probability distribution for the jam lifetime, for three sets of initial values.

As $G(x)$ is continuous at 1, we can replace the limit by the value of the function at $x = 1$, i.e. $G(1)$.

$$\frac{1 - P_0 - \sqrt{(1-P_0)^2 - 4P_+P_-}}{2P_+} = 1 \quad \Leftrightarrow \quad 1 - P_0 - 2P_+ = \sqrt{(1-P_0)^2 - 4P_+P_-}$$
$$\Rightarrow \quad 1 - P_0 - 2P_+ \geq 0$$

Now we substitute $P_0 = pp' + qq'$, $P_- = pq'$ and $P_+ = p'q$ to get

$$\begin{aligned} 1 - pp' - qq' - 2p'q \geq 0 \quad &\Rightarrow \quad 1 \geq pp' + qq' + 2p'q \\ &\Rightarrow \quad 1 \geq pp' + (1-p)(1-p') + 2p'(1-p) \\ &\Rightarrow \quad 0 \geq -p' - p + 2p' \\ &\Rightarrow \quad 0 \geq p' - p \\ &\Rightarrow \quad p \geq p' \end{aligned}$$

So $P(T = \infty) = 0 \Rightarrow p \geq p'$. This means that if cars leave the system more frequently than they enter it, the queue will disperse. This agrees with the probability distribution graph shown in figure 13.

Now lets turn our attention to the rather more worrying case, namely when $P(T = \infty) > 0$. Before we analyse this, we note two things:

Firstly, we note that

$$\begin{aligned} P_+ > P_- &\Leftrightarrow p'q > pq' \\ &\Leftrightarrow p'(1-p) > p(1-p') \\ &\Leftrightarrow p' - pp' > p - pp' \\ &\Leftrightarrow p' > p \end{aligned}$$

Secondly, we note that queue length, as we previously discussed, can decrease, stay the same or increase. So $1 = P_0 + P_- + P_-$, or analogously, $(1 - P_0) = P_+ + P_-$.

Now lets look at the radical part of $G(1)$.

$$\begin{aligned} (1-P_0)^2 - 4P_+P_- &= (P_+ + P_-)^2 - 4P_+P_- \\ &= (P_+ + P_-)^2 - 4P_+P_- \\ &= P_+^2 - 2P_+P_- + P_-^2 \\ &= (P_+ - P_-)^2 \end{aligned}$$

So now we can rewrite $G(1)$ as

$$\begin{aligned} G(1) &= \frac{1 - P_0 - |P_+ - P_-|}{2P_+} \\ &= \frac{P_+ + P_- - |P_+ - P_-|}{2P_+} \end{aligned}$$

And when $P_+ \geq P_-$, (i.e. $p' > p$), $G(1) = \frac{P_-}{P_+}$, which is not equal to 1, which means $P(T = \infty) = 1 - G(1) = 1 - \frac{P_-}{P_+} > 0$. We can also reshow that when $p \geq p' (\Leftrightarrow P_- \geq P_+)$, $G(1) = 1$, which reiterates the probaility of an infinite length jam is 0 when $p \geq p'$.

We have shown that $P(T = \infty) > 0$ when $p' > p$. This means that when the probability cars leave the system (denoted by p) is lower than the probability cars enter it (denoted by p') there is a non-zero probability that the jam will never end.

This is a satisfying conclusion as the probability distribution graph in figure 13 appeared to show the same result.

We now have enough information to calculate a very useful quantity, namely the expected value of the jam lifetime. Recall that expectation is defined to be

$$< T >= \sum_{n=0}^{\infty} nP(T = n) \tag{15}$$

which is precisely what we obtain if we evaluate $G'(x)$ at 1, i.e.

$$G'(x) = \sum_{n=0}^{\infty} nP(T = n)x^{n-1} \tag{16}$$

and hence

$$G'(1) = \sum_{n=0}^{\infty} nP(T = n) \tag{17}$$

So now lets look at the expected lifetime, $< T >$, of the traffic jam:

In the case $P(T = \infty) > 0$ (when $p' > p$), $< T >= \infty$. Lets look at the case where $P(T = \infty) = 0$. We know that for this to be true, $G(1) = 1$. So the expected jam lifetime is:

$$
\begin{aligned}
< T >= \frac{d(G(x))}{dx}|_{x=1} &= -\frac{P_0 + \sqrt{(1-P_0)^2 - 4P_+P_-} - 1}{2P_+\sqrt{(1-P_0)^2 - 4P_+P_-}} \\
&= \frac{G(1)}{\sqrt{(1-P_0)^2 - 4P_+P_-}} \\
&= \frac{G(1)}{\sqrt{(1-P_0)^2 - 4P_+P_-}} \\
&= \frac{1}{\sqrt{1 - 2P_0 + P_0^2 - 4P_+P_-}} \\
&= \frac{1}{\sqrt{1 - 2(pp' + (1-p)(1-p')) + (pp' + (1-p)(1-p'))^2 - 4p'(1-p)p(1-p')}} \\
&= \frac{1}{\sqrt{1 - 2(2pp' + 1 - p - p') + (pp')^2 + 2pp'(1-p)(1-p') + [(1-p)(1-p')]^2 - 4pp'(1-p)(1-p')}} \\
&= \frac{1}{\sqrt{-2pp' + p^2 + p'^2}} \\
&= \frac{1}{\sqrt{(p-p')^2}}
\end{aligned}
$$

and as $p > p'$,

$$< T >= \frac{1}{p-p'}$$

The first thing to notice is that even though $P(T = \infty) = 0$ when $p = p'$, the expected jam lifetime diverges in this case. The reason for this is that distribution decays too slowly. Notice than in figure 13 the case where the $p = p'$ had the greatest distance from the t-axis as $t \to \infty$. In fact, for this critical case, $P(T = t) \sim t^{-3/2}$ as $t \to \infty$ [2].

4 Conclusion

We have seen from the fundamental diagram that the maximum throughput of a road occurs relatively early (between 10% - 20% density). After this peak, congestion follows. Moving the peak to a higher density, (meaning that the congestion is postponed) is an attractive prospect. This is exactly what happens when the speed limit of the road is reduced, albeit at the cost of lowering the flow when compared to the normal speed limit. It would be interesting to investigate other changes that one could make to the system to see if it is possible to shift the maximum throughput, without sacrificing the flow. One way that this could be shown is if the line representing $v_{max} = 1$ (from figure 9) rose above any of the lines where $v_{max} \geq 2$. It may be the case that in a real system the speed-restricted flow does rise above the flow associated with the normal speed limit. The concept of reducing speed limits to battle congestion feels very counter-intuitive when one is forced to drive at 30mph on the motorway when it appears to be freely flowing. Many drivers do not appreciate that the lower speed limit has actually prevented them from being caught in a traffic jam and prevented the jam-related stress mentioned earlier. It is my opinion that the public should be further exposed to the reasons behind speed limits and the fundamental diagram is a great tool to achieve this.

For a realistic open boundary road traffic, an infinite jam is not possible due to the fact that there is not infinite space, time or cars. However, in say, a computer network, where the computer is stuck in a loop requesting information at the same rate or more quickly than the server can process the requests, the possibility of an infinite jam could exist, as expected.

We have also confirmed that if the probability of cars entering the jam is greater than the probability of cars leaving the traffic jam, then there is a non-zero probability that the jam is not going to resolve. If however the probability of leaving the jam is greater than that of joining, then we have a probabilistic expected time for how long the traffic jam will last before it is totally dispersed.

In a large system with several jams, we note that cars leave a particular jam with probability p, and join another jam with probabilty p'. In this case $p = p'$ (as cars move from one jam to the next, flowing freely between them and no cars are removed from the road) and as we observed earlier, such jams then obey a power law distributuion with infinite expectation value.

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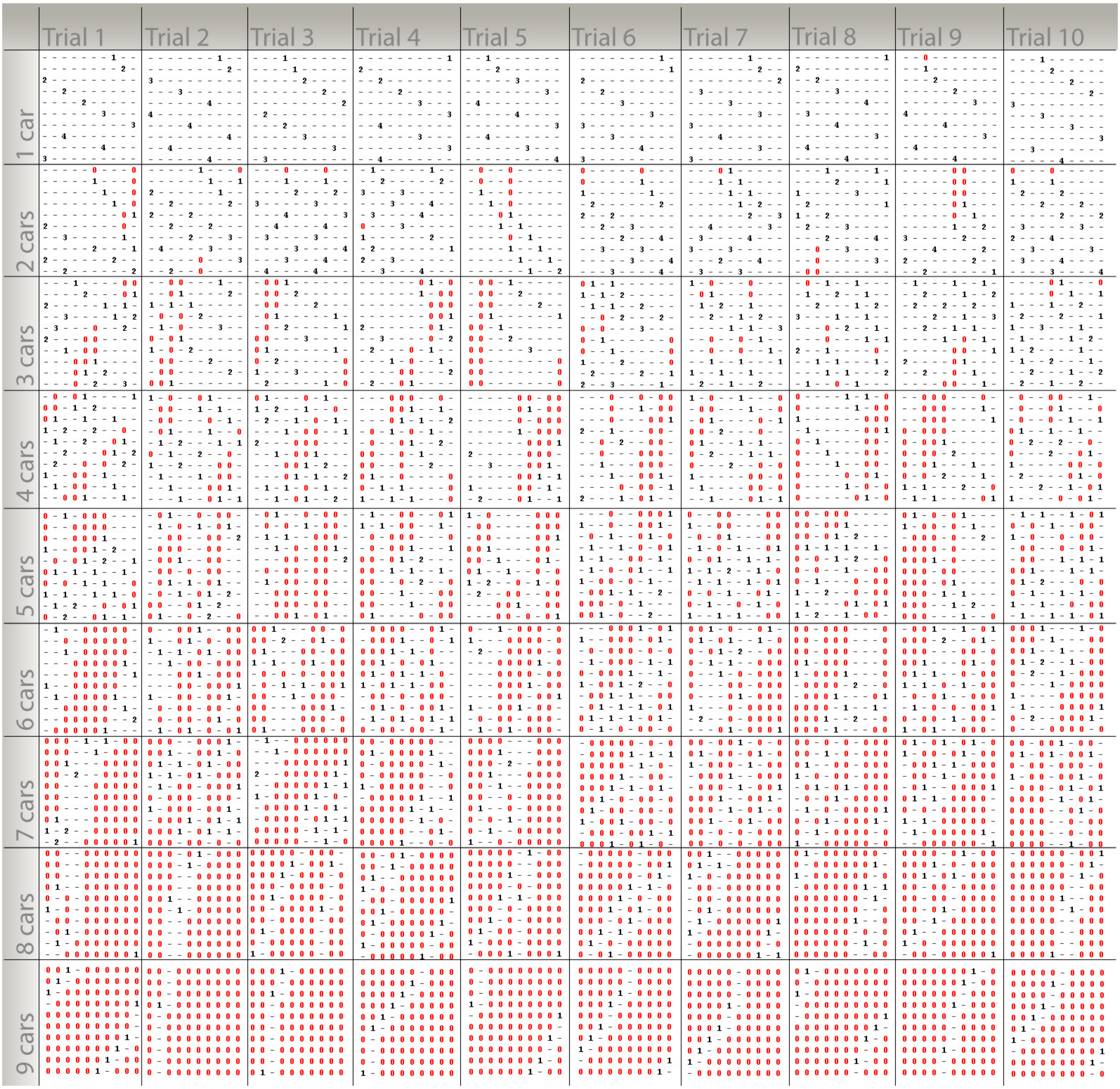
Appendix 1


Trial 1
Trial 2
Trial 3
Trial 4
Trial 5
Trial 6
Trial 7
Trial 8
Trial 9
Trial 10
1 car
2 cars
3 cars
4 cars
5 cars
6 cars
7 cars
8 cars
9 cars

Appendix 2

Fundamental Diagram for L=500, T=500, p=0.5, v_max=4

Sum of Velocities	Number of cars	L	T	Density	Flow
0	0	500	500	0	0
43000	25	500	500	0.05	0.172
60304	50	500	500	0.1	0.241216
81089	75	500	500	0.15	0.324356
75665	100	500	500	0.2	0.30266
72006	125	500	500	0.25	0.288024
69743	150	500	500	0.3	0.278972
66724	175	500	500	0.35	0.266896
60195	200	500	500	0.4	0.24078
56997	225	500	500	0.45	0.227988
51505	250	500	500	0.5	0.20602
47448	275	500	500	0.55	0.189792
42864	300	500	500	0.6	0.171456
37941	325	500	500	0.65	0.151764
32901	350	500	500	0.7	0.131604
27924	375	500	500	0.75	0.111696
22851	400	500	500	0.8	0.091404
17458	425	500	500	0.85	0.069832
11941	450	500	500	0.9	0.047764
6098	475	500	500	0.95	0.024392
0	500	500	500	1	0