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A Study of The Korteweg-De Vries Equation and its Soliton Solutions

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Abstract

This thesis discusses properties of waves, in particular, reference is made to ocean waves in order to help one's mathematical understanding of how tsunami waves differ from normal ocean waves.
We do this by finding solutions to various partial differential equations, using a variety of mathematical methods. We will then illustrate these graphically and observe the important characteristics of the waves and their interactions with one another.

This builds up to our study of a specific type of partial differential equation; the Korteweg de-Vries Equation, in which we observe particular solutions which hold some very special properties, differing to some of our initial observations.

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1 An Introduction to Waves

Waves are an extremely crucial yet fascinating natural phenomenon which arises in many subject areas today. It is the travelling of waves over millions of kilometres by which light reaches the Earth and by which sound is able to be heard. Built upon this phenomenon is the latest technology such as that used to search for oil on the Earth's subsurface using seismic waves. Wave energy is a relatively new form of renewable energy that can potentially be utilised in the Energy Sector.

The importance of the different wave types is due to their ability to transfer large amounts of energy whilst travelling through a wide range of media such as through vacuums, solid materials or our atmosphere.

A precise definition for waves cannot be produced but we can most certainly observe the properties and characteristics of waves. A wave is a set of oscillations which propagate with time through a medium. The propagation of a wave is associated with a transfer of energy through a medium from one position to another. The travelling of a wave involves the displacement of particles within a medium as they vibrate, and these vibrations are passed on to neighbouring particles and then to further neighbouring particles, which in turn carry the energy in the form of waves. An uninterrupted propagating wave consists of a pattern of consecutive crests and troughs, which form a sinusoidal wave [2]. Such waves are known as travelling waves. These waves consist of an amplitude which is the distance between the top of the crest or the bottom of the trough and the stationary position of the wave. It represents how much energy is being carried by the wave, for example a loud sound has a large amplitude sound wave and a bright light has a large amplitude light wave as it is of a large intensity. The frequency of a wave is defined as the number of oscillations produced per second. The wavelength is the width of a wave which is the distance between one crest and the next neighbouring crest. These characteristics of a wave can be illustrated in Figure 1 below.

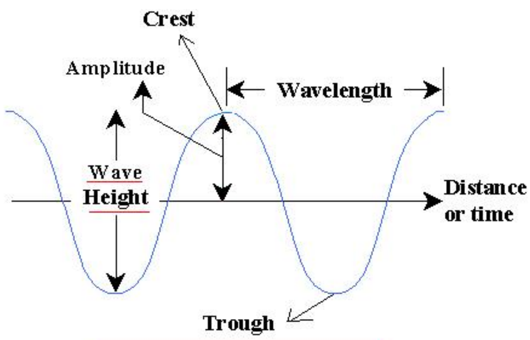


Figure 1: This figure illustrates the characteristics of a wave structure [1]

A particular type of travelling wave is a water waves (also known as a surface wave and a gravity wave). In particular ocean waves are the cause of much natural destruction through tsunamis, triggered by earthquakes in oceans. The reason behind this natural disaster, is the larger than normal amplitude of the waves which form as a result of a disturbance on the seabed. Tsunamis are difficult to detect as the waves are spanned over a very large area horizontally such that wavelengths are very large. Consequently these are not easily visible in an ocean of depth approximately 14000ft deep. It is only when the waves approach the shore that the depth of the water, relative to the wave's amplitude decreases making the large amplitude waves more prominent at the shore. Water waves which cause tsunamis have constant amplitude as they travel through the ocean in comparison to normal water waves whose amplitude decrease relative to the water depth as they reach the shore. What determines whether a wave's amplitude remains constant or decreases is the relationship between the wavelength k, and frequency ω, of a wave. This relation is better known as the Dispersion Relation, which we will later discuss in further detail.

1.0.1 Boundary Conditions

When studying waves we must note that there exists wave boundaries and initial conditions at time, $t = 0$. The knowledge of these are required in order to obtain a wave solution. The boundaries can come in different forms depending on the waves we are studying.

On studying waves produced from plucking a finite string on a guitar, the boundary condition is the finite length of the string being plucked, and the initial conditions refer to how much we displace the string by from its initial shape to when it is plucked.

Whereas, when studying water waves we consider boundaries in a slightly different manner. There is no limit as to how long the wave is horizontally thus we could potentially have waves with very large wavelengths, but we are limited to finite amplitudes, which is the height of the wave from the crest. Hence we can deduce that boundaries in water waves lie in the direction of the amplitude; the vertical direction, as opposed to the horizontal direction with length of a string as on a guitar. As a result of this we become slightly limited to the types of equations which can be solutions to wave equations as they must be bounded in the vertical direction such as those we will study in this thesis. We can rule out all linear and some quadratic functions existing as wave equation solutions, such as the exponential function and multiples of this as these functions tend to infinity. Whereas, functions such as e^{-x^2} and its translations are more likely solutions.
Although this may appear to be a very subtle idea it is important to know this when recognising wave equation solutions.

In the subsequent chapters we will study three partial differential equations. We will begin with introducing the linear Wave Equation and Schrödinger's Equation and finding their solutions through the use of Fourier Transformations. Illustrating these wave solutions graphically, we will observe their motion over time and make some important observations through introducing concepts such as the Superposition Principle and the dispersion relation.

Furthermore, we will study the Korteweg-de Vries (KdV) equation which models shallow water waves together with some of its properties. In addition to this, we will discuss its soliton solutions and properties of such solutions. Using a direct integration method; separation of variables, we will obtain a one-soliton solution of this equation. In studying this equation further we will adopt Hirota's Method to help us calculate a two-soliton solution. We will then interpret both these solutions graphically and analyse the illustrations produced. In doing this we will be able to see how the Korteweg-de Vries equation differs significantly from the wave equation and the Schrödinger's Equation, and finally, discuss the limitations in the methods we used to find solutions to the KdV equation.

All the illustrations presented in this thesis have been produced using a mathematical software Matlab, unless otherwise stated.

2 The Linear Wave Equation

2.1 The Wave Equation Explained

Definition 1. *The **Wave Equation** is a linear, second-order partial differential equation*

$$\frac{\partial^2 u}{\partial t^2}\frac{1}{c^2} = \bigtriangledown^2 u \tag{2.1}$$

where $\bigtriangledown^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ with respect to the function $u(x, y, z, t)$, where x, y, z are the three-dimensional space variables and t is the time variable.
For the remainder of this thesis we will be studying the wave equation in one-dimensional form thus making use of the following equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{2.2}$$

Partial differential equations crop up in a branch of applied mathematics called mathematical analysis. A partial differential equation is a mathematical equation which consists of partial derivatives of a number of variables. The order of the equation refers to the highest power contained in the equation. In equation (2.2) both partial derivatives are to the power 2 hence this equation is 2^{nd} order.

The wave equation models waves such as water waves which arise in a variety of fields such as fluid dynamics and electrodynamics.

2.1.1 The Superposition Principle

The wave equation is linear and so we can take a linear combination of wave equation solutions and this would also form a solution. A solution to the wave equation will be of the form $u(x,t)$=$f(x \pm ct)$ where c is the velocity of the wave and $f(x,t)$ is a function $f : \mathbf{R}^2 \to \mathbf{R}$ which determines the shape of the wave. In order to see what a solution $u(x,t)$ to the equation looks like, we can first see what the equation represents at initial time $t_0 = 0$, $u(x, 0)$. Seeing the initial shape of the wave represented by the solution, helps us to gather an intuition of the solution graphically. To be able to do this we can plot the solution at a time $t_{0+\delta} = 0 + \delta$, $u(x, 0 + \delta)$ where δ is very small and $t_{0+\delta}$ is very close to t_0 to enable us to see the shape produced by the wave solution [3].
Once we have obtained one solution to the linear wave equation we can obtain other solutions using the very important superposition principle.

Definition 2. *The **Superposition Principle** is an important property within linear equations such as the wave equation. This principle states that once one has obtained a solution $u(x,t)$ to a linear equation, one can obtain a finite number of solutions to the equation from this in the following way*

$$\sum_{i=1}^{n} a_i u_i(x,t) \tag{2.3}$$

where a_i is an arbitrary constant and $u_i(x,t)$ is the $i - th$ solution [19].

It is important to clarify the difference between the superposition principle in partial differential equations (PDEs) such as the wave equation, where the general solutions consist of arbitrary functions, and the superposition principle in ordinary differential equations (ODEs) where the general solutions contain arbitrary constants.

2.2 Solutions to the Wave Equation

2.2.1 The d'Alembert Solution

The following derivation has been taken from [19].

Now that we understand the benefit of the superposition principle being applied to linear equations, we will use a method known as d'Alembert's solution to obtain general and particular solutions to the one-dimensional wave equation.

In order to carry out d'Alembert's method it is necessary to introduce two new variables, of which both are dependent on a single spacial coordinate x and time t;

$$r = r(x,t) = x - ct \tag{2.4}$$

$$s = s(x,t) = x + ct \tag{2.5}$$

r and s can be re-arranged such that x and t can be written as follows

$$x = \frac{1}{2}(r+s) \tag{2.6}$$

$$t = \frac{1}{2c}(s-r) \tag{2.7}$$

giving two implicit functions of r and s.

We will make use of the chain rule for differentiation in order to find the partial derivatives of a solution to the wave equation, u with respect to r and s. Having written x and t in terms of r and s, we can use (2.6) and (2.7) to differentiate as follows

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial r} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2c}\frac{\partial u}{\partial t} \tag{2.8}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2c}\frac{\partial u}{\partial t} \tag{2.9}$$

Multiplying $\frac{\partial u}{\partial r}$ with $\frac{\partial u}{\partial s}$ from (2.8) and (2.9) gives us the following expression

$$\frac{\partial^2 u}{\partial r \partial s} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2} + \frac{1}{4c}\frac{\partial^2 u}{\partial x \partial t} - \frac{1}{4c}\frac{\partial^2 u}{\partial x \partial t} - \frac{1}{4c^2}\frac{\partial^2 u}{\partial t^2}. \tag{2.10}$$

In order to proceed we need to remember that second order partial derivatives with respect to two distinct variables are commutative, thus $\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x}$.

This simplifies our expression to

$$\frac{\partial^2 u}{\partial r \partial s} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\right) \tag{2.11}$$

within the brackets in (2.11) we have the wave equation, which when re-arranged from (2.2), is equal to zero. Thus leaves us with the following, so called **Canonical Form** of the wave equation

$$\frac{\partial^2 u}{\partial r \partial s} = 0. \tag{2.12}$$

This partial differential equation has a general solution of the form

$$u = u(r, s) = u_1(r) + u_2(s) \tag{2.13}$$

where u_1, u_2 are arbitrary functions of r and s respectively. Equivalently, using (2.4) and (2.5), (2.13) can be written as

$$u(x, t) = u_1(x - ct) + u_2(x + ct) \tag{2.14}$$

which is a general solution of a wave travelling over infinite time.

Now we need to find a solution satisfying a set of initial conditions of initial time ($t_0 = 0$) and initial wave velocity so we can study waves travelling finitely. These initial conditions of time and wave velocity are

$$u(x, 0) = u_0(x)$$
$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = v_0(x).$$

We can apply these initial conditions to our general solution in (2.14), giving us

$$u_0(x) = u_1(x) + u_2(x) \tag{2.15}$$
$$v_0(x) = cu_2'(x) - cu_1'(x) \tag{2.16}$$

integrating (2.16) gives us[1]

$$\frac{1}{c}\int_a^x v_0(\alpha)d\alpha = u_2(x) - u_1(x) \tag{2.17}$$

adding (2.15) and (2.17) gives us

$$\begin{aligned} u_0(x) + \frac{1}{c}\int_a^x v_0(\alpha)d\alpha &= u_1(x) + u_2(x) + u_2(x) - u_1(x) && (2.18)\\ &= 2u_2(x) && (2.19)\\ \frac{1}{2}u_0(x) + \frac{1}{2c}\int_a^x v_0(\alpha)d\alpha &= u_2(x) && (2.20) \end{aligned}$$

subtracting (2.15) and (2.17) gives us

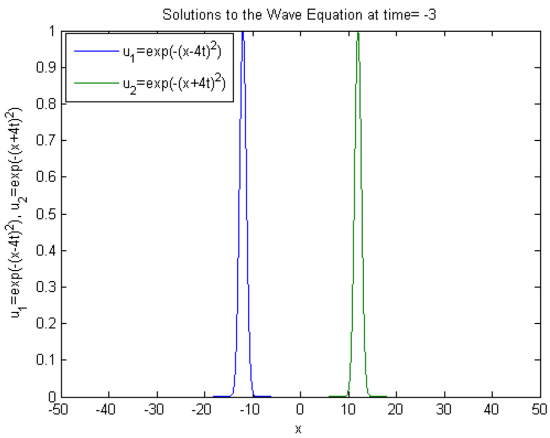
$$\begin{aligned} u_0(x) - \frac{1}{c}\int_a^x v_0(\alpha)d\alpha &= u_1(x) + u_2(x) - u_2(x) + u_1(x) && (2.21)\\ &= 2u_1(x) && (2.22)\\ \frac{1}{2}u_0(x) - \frac{1}{2c}\int_a^x v_0(\alpha)d\alpha &= u_1(x). && (2.23) \end{aligned}$$

Now we can make the substitutions for $u_1(x)$ and $u_2(x)$ into the general solution in (2.14), resulting in

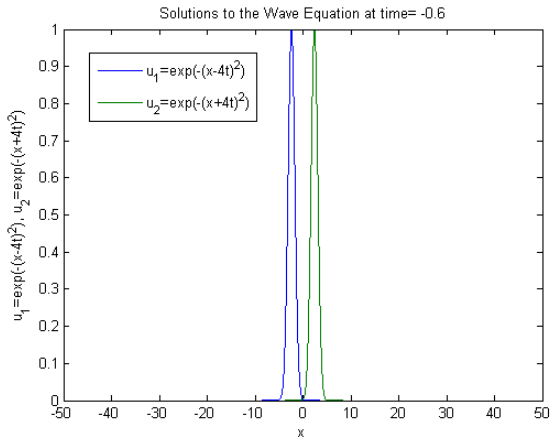
$$u(x, t) = \frac{1}{2}u_0(x - ct) - \frac{1}{2c}\int_a^{x-ct} v_0(\alpha)d\alpha + \frac{1}{2}u_0(x + ct) + \frac{1}{2c}\int_a^{x+ct} v_0(\alpha)d\alpha \tag{2.24}$$

[1] In our integral, the lower limit 'a' is a constant which we can drop when carrying out the integration, as this is chosen such that it is very large.

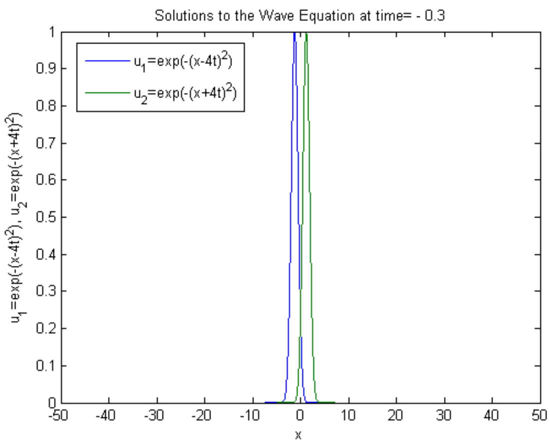
which is the d'Alembert solution to the wave equation subject to initial conditions. Here we have replaced x by $x - ct$ and $x + ct$ as we require the solution for all time and not just initial time. From this solution, one can see that the solution to the one-dimensional wave equation is a sum of two functions u_1 and u_2, where u_1 travels in the right direction along the positive $x - axis$ and u_2 travels in the left direction along the negative $x - axis$. Note that the overall shape of the wave solution remains the same as it travels at a velocity c which is illustrated in Figure 2 on the following page.



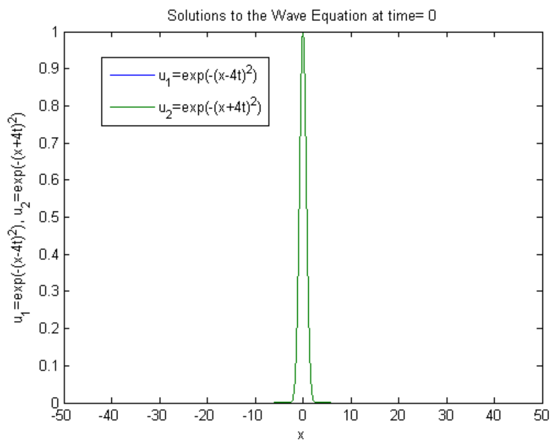
(a) Time= -3



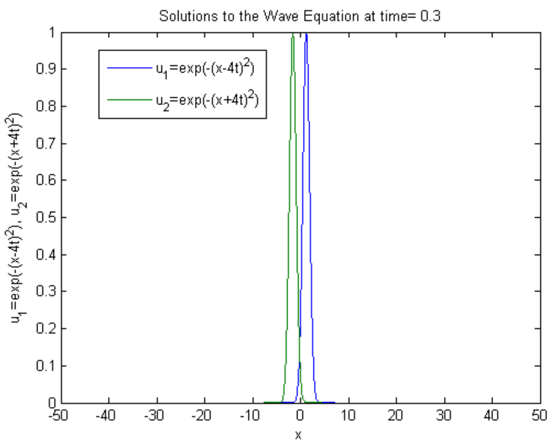
(b) Time= -0.6



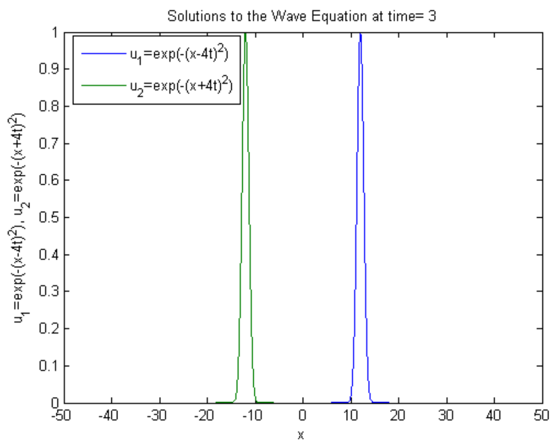
(c) Time= -0.3



(d) Time= 0



(e) Time= 0.3



(f) Time= 3

Figure 2: This figure neatly demonstrates the superposition principle in d'Alembert's solution to the wave equation at progressive times

In Figure 2 we have plotted $u_1 = e^{-(x-4t)^2}$ and $u_2 = e^{-(x+4t)^2}$ where $c = 4$, as possible solutions to the wave equation. We can confirm that the waves produced by u_1 and u_2 do indeed travel towards the right and left respectively as time progresses.

Another characteristic of the wave equation solutions which was mentioned earlier as the superposition principle, is also illustrated in Figure 2. As the waves approach each other at the same speed, they merge together and at time $t = 0$ we have obtained another solution to the wave equation as a result of the current two solutions being added together in a linear form. Once they have passed each other the shape of the wave has not been altered and they both continue travelling at the same speed and same amplitude as they did before they passed each other. This is a result of the superposition principle.

2.3 The Dispersion Relation and other relations: The Wave Equation

Another possible solution of the wave equation is in the form of sine and cosine trigonometric functions. A solution of such a form is called a plane wave. In order to be able to define the plane wave we must first define the dispersion relation.

Definition 3. *The* ***Dispersion Relation*** *of a linear, partial differential equation is an equation which relates the wavenumber $k = \frac{2\pi}{\lambda}$ and the frequency $\omega(k) = \frac{2\pi}{\lambda}$ of a wave, such that the plane wave solves the equation, where λ is the wavelength.*

Definition 4. *A* ***Plane Wave*** *also a type of travelling wave, is a possible solution to a linear wave equation if it satisfies one condition; the dispersion relation. The plane wave takes the form*

$$u(\overrightarrow{x}, t) = Ae^{i(\overrightarrow{k}\,\overrightarrow{x} - \omega t)}$$

where A is an arbitrary amplitude, i is a complex number, $i = \sqrt{-1}$, ω is the wave vector, x is a point along the x-axis, k is the angular frequency and t is the time. As we are only studying the wave equation in one dimension, k and x will be scalar quantities thus

$$u(x, t) = e^{i(kx - \omega t)} \tag{2.25}$$

In terms of sine and cosine (2.25) is equivalent to

$$u(x, t) = \cos(kx - \omega t) + i\sin(kx - \omega t)$$

In order to show $u(x, t)$ is a solution to the wave equation by which we also obtain the dispersion relation, we need to substitute the required derivatives of $u(x, t)$ into the wave equation.

$$\frac{\partial^2 u}{\partial x^2} = -k^2 e^{i(kx - \omega t)} \tag{2.26}$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 e^{i(kx - \omega t)}. \tag{2.27}$$

Now we can substitute these derivatives into the wave equation (2.2)

$$-c^2k^2 e^{i(kx - \omega t)} = -\omega^2 e^{i(kx - \omega t)} \tag{2.28}$$

we can cancel out the common term $e^{i(kx - \omega t)}$ on both sides and multiply throughout by -1, to obtain

$$\sqrt{c^2k^2} = \sqrt{\omega^2} \tag{2.29}$$

$$\omega = \pm ck \tag{2.30}$$

which is the linear dispersion relation for the wave equation.

For the remainder of this section, we will take the positive value of the frequency, although the negative wavelength would also work in the negative time direction.

We can see that in order for the plane wave to satisfy the wave equation, the linear dispersion relation $\omega = ck$ (where c is the wave velocity) must be satisfied. In general the linearity of the dispersion relation for an equation determines whether a wave's amplitude decreases. This implies the wave is carrying a decreasing amount of energy as time progresses; such waves are called dispersive waves. Conversely, a wave's amplitude can remain constant hence carrying a constant amount of energy, such waves are called non-dispersive waves. A linear dispersion relation as for the wave equation implies the latter of the two as we will illustrate later.

Using the wave number and frequency, we can also make some other observations regarding the motion of waves.

One observation we can make is explained by the wave's *group velocity*. This is the rate of change of the frequency with respect to the wave number. It is the velocity at which energy flows in a wave packet through a medium [12]. Using the dispersion relation we can write the group velocity for the wave equation as [10]

$$\frac{d\omega}{dk} = c. \tag{2.31}$$

We can also define another type of velocity which tells us the rate at which the phase (a period) of the wave travels through a medium. This is known as the *phase velocity* which is [10]

$$c_{ph} = \frac{\omega}{k} \tag{2.32}$$

In the case for a non-dispersive wave, the phase and group velocity are the same as we can show below [10]

$$c_{ph} = \frac{\omega}{k} = \frac{ck}{k} = c = \frac{\partial \omega}{\partial k}. \tag{2.33}$$

The above tells us that the wave does not disperse as it travels with time [13].
An example of where this relation can be demonstrated in the natural world is in a tsunami. Disruptions in the ocean such as by earthquakes and landslides are major causes of a tsunami. A disturbance in the ocean caused by an earthquake occurs when the tectonic plates under the ocean rub against each other causing friction as one plate is pushed up above the other. This results in hundreds of cubic kilometres of water being disturbed which in turn triggers very large waves to travel in the ocean away from the epicentre. In the ocean where the depth of the water is large, the waves travel at very high speeds but as the water approaches the shore, the depth decreases and the speed of the wave slows down as wave amplitude becomes larger [4]. The occurrence of a tsunami is dependent on the strength of the earthquake. A tsunami can be triggered if the jolting of the tectonic plate by an earthquake is large enough. This in turn brings about the effect of the linear dispersion relation, causing the wave's to maintain their shape and conserve their energy as they travel through the ocean towards the shore. To put this into perspective, a large enough magnitude earthquake, say 9.0 such as that which occurred on the coast of Sumatra, Indonesia in 2004, triggered a heavily destructive tsunami of waves with height 50 metres that travelled 5 kilometres inland [5]. This helps one understand the existence of the dispersion relation in he real world.

2.4 Solutions to the Wave Equation continued

We have already come across some solutions to the wave equation. Our aim now is to be able to find a way where we can easily obtain as many solutions to the equation as we require. We can make this possible with the use of the Fourier Transformation.

This Fourier transformation is a useful method which can be used on a function such as a waveform which is composed of sine and cosines. It allows the waveform function to be written as a continuous sum of sinusoidal functions.

Definition 5. *The Fourier Transform can be defined as follows [6]*

$$F\{g(k)\} = G(f) = \int_{-\infty}^{\infty} g(k)e^{-2\pi ifk}dk \tag{2.34}$$

where we can write the function $G(f)$ as an integral of $g(t)e^{-2\pi ifk}$, and so we have a continuous sum of $g(k)$.

In relation to what we are dealing with, we need to find $g(k)$ which we will rename as $A(k)$ using the Fourier Transformation in Definition 5. $A(k)$ contains the coefficients of the plane wave solutions which will aid us in using the linear superposition principle.

In the example that follows, we will work to calculate a wave-packet solution. A wave-packet also known as a wave group is a group of two or more waves travelling simultaneously through a medium. Wave-packets exist as a result of the linear superposition principle [18].

The following is a general form of a solution of a wave-packet

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}dk \tag{2.35}$$

where $u(x,t)$ is a function of the one-dimensional spacial coordinate x and time t of a wave.

With this general form we can find a continuous sum of the solutions of the wave equation with the help of $A(k)$. We will obtain a wave-packet solution of the form $f(x-ct)$ where our wave-packet will propagate in the positive x-direction and we can let $c = 1$ for simplicity.

To begin with, we need to calculate the amplitude, $A(k)$ which can be done by inverting the general form of a wave packet in (2.35) at $t = 0$

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx. \tag{2.36}$$

We can illustrate the use of the Fourier Transform in the following example. This will eventually lead us to a solution of the wave equation.

For the sake of this example we can allow $u(x,0) = e^{-x^2}$, which is an equation of a wave at initial time $t = 0$. Substituting this into (2.36) gives us

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2}e^{-ikx}dx & (2.37) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2-ikx}dx. & (2.38) \end{aligned}$$

We are able to write the exponent from the integral in (2.38) as a sum of two squares as follows

$$-(x^2 + ikx) = -\left(x^2 + ikx + \left(\frac{ik}{2}\right)^2 - \left(\frac{ik}{2}\right)^2\right) \quad (2.39)$$

$$= -\left(x + \frac{ik}{2}\right)^2 + \left(\frac{ik}{2}\right)^2. \quad (2.40)$$

By writing the exponent in this form, it allows us to use the method of substitution to solve the integral in (2.38). Thus to use this method we make the following substitution

$$y = x + \frac{ik}{2} \quad (2.41)$$

$$\text{thus } dy = dx \quad (2.42)$$

and our integration limits will remain as they are.

The integral we are now required to calculate, having made the substitution for y and dy in (2.38) is

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2 + (\frac{ik}{2})^2} dy \quad (2.43)$$

$$= \frac{1}{\sqrt{2\pi}} e^{(\frac{ik}{2})^2} \int_{-\infty}^{\infty} e^{-y^2} dy. \quad (2.44)$$

Using our knowledge, we know the integral $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$ which we can make use of in (2.44) to cancel out $\sqrt{\pi}$, giving us

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{4}} \sqrt{\pi} \quad (2.45)$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{k^2}{4}}. \quad (2.46)$$

Now substituting this amplitude, $A(k)$ into the original general form of the wave-packet in (2.35) gives us the following

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-\frac{k^2}{4}} e^{i(kx - \omega(k)t)} dk. \quad (2.47)$$

We can now consider the linear dispersion relation which we obtained earlier. As we mentioned before, only when the dispersion relation is satisfied, does there exist a solution to the wave equation in terms of the trigonometric functions sine and cosine.

Again, we will make use of the substitution method to integrate (2.47).

Using the dispersion relation $\omega(k) = k$ from earlier calculations, we can substitute for

$\omega(k)$ into the exponent in (2.47)

$$\begin{aligned}
-\frac{k^2}{4} + i(kx - kt) &= -\frac{k^2}{4} + ik(x-t) && (2.48)\\
&= -\frac{1}{4}(k^2 - 4ik(x-t)) && (2.49)\\
&= -\frac{1}{4}\left[k^2 - 4ik(x-t) + \left(\frac{-4i(x-t)}{2}\right)^2 - \left(\frac{-4i(x-t)}{2}\right)^2\right] && (2.50)\\
&= -\frac{1}{4}\left[(k - 2i(x-t))^2 - (-2i(x-t))^2\right] && (2.51)\\
&= -\frac{1}{4}\left[(k - 2i(x-t))^2 + 4(x-t)^2\right] && (2.52)\\
&= -\frac{1}{4}\left[k - 2i(x-t)\right]^2 - (x-t)^2. && (2.53)
\end{aligned}$$

In steps (2.48) – (2.53) we have taken the exponent within the integral in (2.47) and rewritten it, such that we can take the sum of two squares and write this in such a way as to make a helpful substitution back into (2.47). We can see that the exponent can eventually be written as shown in (2.53). Using this form we can make the following substitution

$$\begin{aligned}
y &= \frac{1}{2}(k - 2i(x-t)) && (2.54)\\
dy &= \frac{1}{2}dk. && (2.55)
\end{aligned}$$

Substituting (2.54) and (2.55) into (2.47) we can write the integral as follows

$$u(x,t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2-(x-t)^2} 2dy. \qquad (2.56)$$

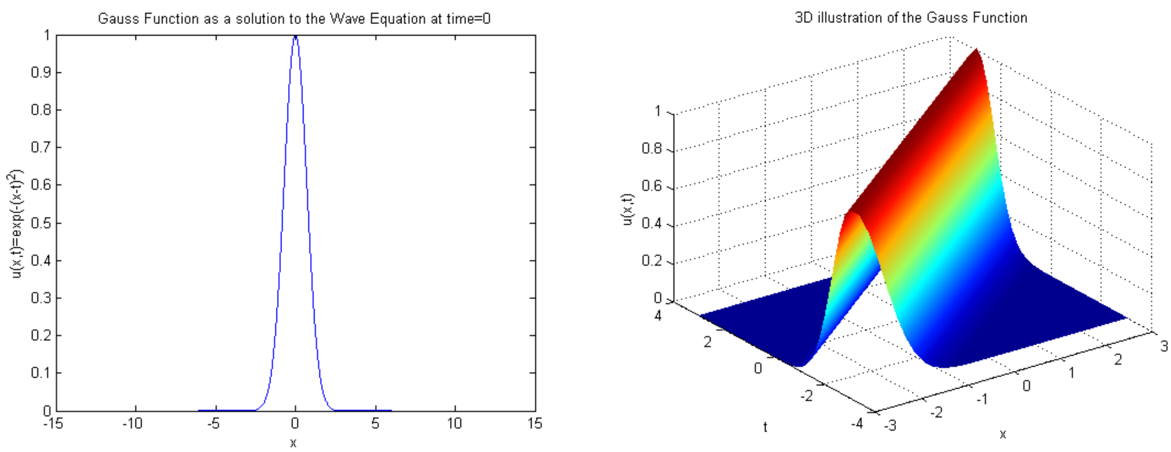
We can now simplify this integral and use the result $\int_{\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}$ to obtain a solution

$$\begin{aligned}
u(x,t) &= \frac{2e^{-(x-t)^2}}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}dy && (2.57)\\
&= \frac{1}{\sqrt{\pi}}e^{-(x-t)^2}\sqrt{\pi} && (2.58)\\
&= e^{-(x-t)^2}. && (2.59)
\end{aligned}$$

We can conclude that

$$u(x,t) = e^{-(x-t)^2} \qquad (2.60)$$

is a wave-packet solution to the wave equation which takes the form of a Gauss function as illustrated on the next page.



(a) 2D Plot

(b) 3D Plot

Figure 3: This figure illustrates the solution we obtained from the linear wave equation both in two-dimensional and three-dimensional form

In Figure 3b we can see that as time progresses, the shape of the wave remains consistent; a property resulting from the linear dispersion relation.

From our solution in (2.60) we can see it takes the form $u(x - ct)$ where $c = 1$, thus the wave propagates in the positive x-direction. Our solution is a localised wave which is in fact a Gauss function that is bounded in the direction of the amplitude, as is required of the wave solutions that we discussed earlier.

A wave of this kind is one which a tsunami is formed from, since it does not reduce in amplitude as it arrives to the shore.

3 The Schrödinger Equation

3.1 The Schrödinger Equation explained

In the previous chapter we studied waves of constant amplitude over time, such as those which form tsunamis. There also exists waves which do not have constant amplitude; they are the complete opposite due to the fact that they spread as they travel over time.

The Schrödinger Equation is an example of an equation from which we can obtain solutions of such waves.

This equation has its uses in Quantum Mechanics, which is the study of the behaviour of particles in terms of matter and energy on a nanoscopic scale including atoms such as those in the elements in the Periodic Table. Comparatively, Classical Mechanics differs from this in terms of the relatively larger objects studied in this branch. Due to the differences in behaviour at the two scales we require two sets of rules, where Quantum Mechanics observations involve the Planck's constant, $h = 6.62606957 \times 10^{-34} m^2 kg/s$.
Within Quantum Mechanics, the same particle can behave very differently at different times. This interpretation is commonly known as the *Copenhagen Interpretation*, as most of the research in this field was carried out by Niels Bohr; a Danish physicist working in Copenhagen [7].

Definition 6. *[8] The* ***Schrödinger Equation***

$$i\hbar\frac{\partial}{\partial t}\psi(\overrightarrow{r},t) = -\frac{\hbar^2}{2m}\bigtriangledown^2\psi(\overrightarrow{r},t) + V(\overrightarrow{r},t)\psi(\overrightarrow{r},t) \tag{3.1}$$

- i *is the complex number* $\sqrt{-1}$
- $\hbar$ *is the Planck's Constant* $h = 6.62606957 \times 10^{-34} m^2 kg/s$ *divided by* 2π
- $\psi(\overrightarrow{r},t)$ *is the wave function with respect to space and time*
- m *is the mass of the particle*
- $\bigtriangledown^2$ *is the Laplacian operator* $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- $V(\overrightarrow{r},t)$ *is the potential energy of the particle*

This is the Schrödinger equation inclusive of all variables for the reader to appreciate the full equation. However, for the calculations carried out in this section we will use the following equation, a simplified one-dimensional version of the Schrödinger equation.

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} \tag{3.2}$$

3.2 The Dispersion Relation: The Schrödinger Equation

Previously we have shown how the wave equation illustrated a linear dispersion relation. Now we will calculate the non-linear dispersion relation for Schrödinger's equation using the plane wave as before. We have chosen this equation as it is the simplest wave equation with which we can show the non-linear dispersion relation.

We will calculate the 1^{st} and 2^{nd} derivatives of the plane wave with respect to t and x respectively

$$\psi = e^{i(kx-\omega(k)t)} \tag{3.3}$$

$$\frac{\partial\psi}{\partial t} = -i\omega(k)e^{i(kx-\omega(k)t)} \tag{3.4}$$

$$\frac{\partial^2\psi}{\partial x^2} = -k^2 e^{i(kx-\omega(k)t)}. \tag{3.5}$$

We can substitute this into the Schrödinger equation in (3.2) giving us

$$i(-i\omega(k)e^{i(kx-\omega(k)t)}) = -\frac{1}{2}(-k^2 e^{i(kx-\omega(k)t)}). \tag{3.6}$$

We can cancel out the common term $e^{i(kx-\omega(k)t)}$ from both sides and multiply the complex numbers $i \cdot -i = 1$. This results in the following non-linear dispersion relation for the Schrödinger equation

$$\omega(k) = \frac{1}{2}k^2. \tag{3.7}$$

A wave equation which insists on a non-linear dispersion relation is satisfied by wave-packet solutions which consist of waves of different speeds and amplitudes. As time progresses the wave-packet spreads and flattens out, no longer keeping its shape. This will be illustrated in the next subsection where we will be able to obtain a solution with the help of this relation.

3.3 Solutions to the Schrödinger Equation

As we did for the wave equation in the previous section, we are able to use the Fourier Transformation to obtain the amplitude of a wave and calculate a solution to the Schrödinger equation. With this solution we will be able to find more solutions to the equation using a linear combination of current solutions through the superposition principle. It is here where the amplitude $A(k)$ is of importance as this contains the coefficients necessary to construct linear combinations of solutions.

The following example will lead us to a solution of the Schrödinger equation. In order to achieve this we must first calculate the amplitude as we did for the wave equation. We then use the non-linear dispersion relation we obtained in the previous subsection to calculate a solution to the Schrödinger equation, from which we can obtain more solutions.

To begin with we can use the plane wave to obtain the amplitude. Steps (2.35) – (2.46) in which we calculated an amplitude for the wave equation solution will be repeated exactly giving us

$$A(k) = \frac{1}{\sqrt{2}} e^{-\frac{k^2}{4}}. \tag{3.8}$$

Substituting this into the general form of a solution of a wave-packet in (2.35), and also making the substitution of the non-linear dispersion relation $\omega = \frac{1}{2}k^2$ from (3.7) results in

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} e^{-\frac{k^2}{4} + i(kx - \frac{1}{2}k^2 t)} dk. \tag{3.9}$$

Taking the exponent $-\frac{k^2}{4} + i(kx - \frac{1}{2}k^2 t)$ from (3.9), notice we can re-write this as the sum of two squares. This notation will later help us make a substitution such that the integral is easier to solve. Re-arranging this exponent and writing it as a sum of two squares gives us

$$\begin{aligned} -\frac{k^2}{4} + i\left(kx - \frac{1}{2}k^2 t\right) &= \frac{-1-2it}{4}\left[k^2 + \left(\frac{4ix}{-1-2it}\right)k\right] && (3.10) \\ &= \frac{-1-2it}{4}\left[\left(k + \frac{2ix}{-1-2it}\right)^2 - \left(\frac{2ix}{-1-2it}\right)^2\right]. && (3.11) \end{aligned}$$

Writing the exponent as in the form (3.11) allows us to make a suitable substitution and differentiate this as follows

$$\begin{aligned} \text{let } y &= \sqrt{\frac{1+2it}{4}}\left(k + \frac{2ix}{-1-2it}\right) && (3.12) \\ \frac{2dy}{\sqrt{1+2it}} &= dk. && (3.13) \end{aligned}$$

For the convenience of the reader, intermediate calculations have been omitted. For the full length of calculations between (3.10) – (3.13) please refer to Appendix 1.

Using the substitution for y in (3.12), and the expression for dk in (3.13), we can substitute these into the expression for $\psi(x,t)$ in (3.9) to give us

$$\psi(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{1+2it}} e^{-y^2 - \left(\frac{-1-2it}{4}\right)\left(\frac{2ix}{-1-2it}\right)^2} dy \tag{3.14}$$

rearranging this gives us

$$\frac{1}{\sqrt{\pi(1+2it)}}e^{-\left(\frac{-1-2it}{4}\right)\left(\frac{2ix}{-1-2it}\right)^2}\int_{-\infty}^{\infty} e^{-y^2}dy$$

using the result $\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}$, we obtain

$$\psi(x,t) \quad = \quad \frac{1}{\sqrt{(1+2it)}}e^{-\left(\frac{-1-2it}{4}\right)\left(\frac{2ix}{-1-2it}\right)^2}. \tag{3.15}$$

Now we can simply rewrite the exponent $\left(\frac{-1-2it}{4}\right)\left(\frac{2ix}{-1-2it}\right)^2$ to $\frac{x^2}{1+2it}$ by cancelling out common factors in the numerator and denominator. This can now be rewritten by multiplying numerator and denominator by $(2it-1)$, giving us $\frac{2itx^2}{4t^2+1} - \frac{x^2}{4t^2+1}$. Thus writing $\psi(x,t)$ as

$$\psi(x,t) = \frac{1}{\sqrt{(1+2it)}}e^{\left(\frac{2itx^2}{4t^2+1}-\frac{x^2}{4t^2+1}\right)}. \tag{3.16}$$

In order to be able to plot this, we are required to take the absolute value of $\psi(x,t)$ in (3.16) with the aim to get rid of the complex numbers, i.

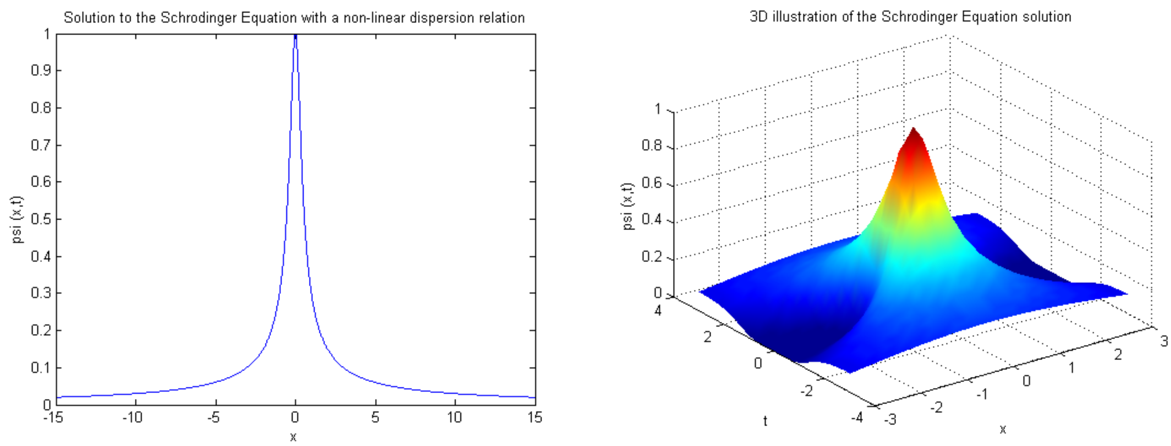
$$|\psi(x,t)| = |\frac{1}{\sqrt{(1+2it)}}||e^{\frac{2itx^2}{4t^2+1}}||e^{-\frac{x^2}{4t^2+1}}|. \tag{3.17}$$

We arrive at a solution of the Schrödinger equation

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{1+4t^2}}e^{-\frac{2x^2}{1+4t^2}} \tag{3.18}$$

For the convenience of the reader, intermediate calculations have been omitted here. To see the full length of calculations between (3.17) – (3.18) please refer to Appendix 2.

The following two-dimensional and three-dimensional graphs are a plot of the above solution of the Schrödinger Equation.

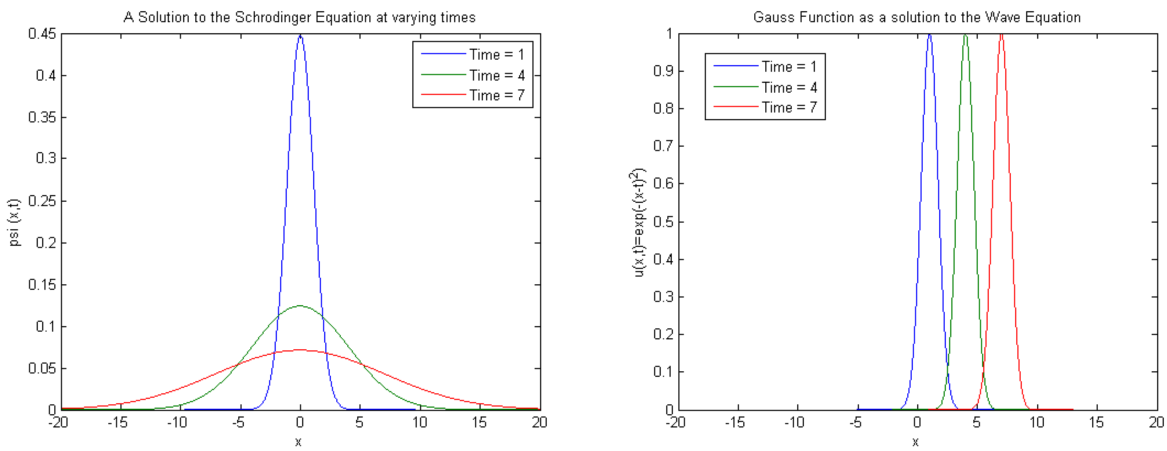


(a) 2D plot (b) 3D plot

Figure 4: This figure shows a two-dimensional and three-dimensional illustration of the Schrödinger equation solution which we obtained in our calculations

From Figure 4 we can see that the Schrödinger equation exhibits a non-linear dispersion relation. In particular, in Figure 4b we can see that as time progresses, the wave-packet decreases in amplitude as the wave flattens and spreads whilst it travels. This is due to the loss of energy carried by the wave. As described before, this characterises a dissipative wave [10]. We have plotted the solution in forwards and backwards time which displays symmetry at time $t = 0$.

Looking at the solution itself in (3.18) we can see that the amplitude $\frac{1}{\sqrt{1+4t^2}}$, which is the coefficient in front of the exponential function is dependent on time. The time variable, t is presented within the square-root of the denominator which tells us that as time increases the amplitude becomes increasingly smaller which is illustrated in Figure 4.



(a) Illustration of the non-linear dispersion relation at time=1,4,7

(b) Illustration of the linear dispersion relation at time=1,4,7

Figure 5: This figure allows us to make better comparisons of the dispersion relations of the wave equation and the Schrödinger equation

Figure 5 allows us to see a clear comparison between the effect of the linear and non-linear dispersion relation on wave solutions obtained from linear partial differential equations such as those we studied in the last two chapters. This figure presents a neat summary to conclude what we have observed so far. The non-linear dispersion relation presented in the Schrödinger's equation has an effect of decreasing the amplitude as a wave progresses over time. This is due to the dependency of the amplitude on time in the solution we obtained for the Schrödinger's equation. It is possible to obtain other solutions to the Schrödinger equation, all of which will illustrate dissipative waves. These solutions can be obtained from the use of the linear superposition principle.

Comparatively, we have shown in Section 2 that the wave equation adopts a linear dispersion relation. This has the effect of the amplitude of a wave remaining constant as it travels through a medium, with respect to time. We can explain this by looking at the example of a solution we obtained to the wave equation. Equation (2.60) is our wave equation solution $e^{-(x-t)^2}$. The amplitude being the coefficient in front of the exponential, which in this case is 1, has no dependency on time. Thus linear partial differential equations which possess a linear dispersion relation are satisfied by wave-packet solutions of non-dissipative waves [10].

4 The Korteweg-de Vries Equation

4.1 The Korteweg-de Vries Equation explained

In this section we will study a relatively different type of differential equation known as the Korteweg-de Vries equation, which is a non-linear partial differential equation. The non-linearity in this equation differentiates it from the first two in a significant way. As a result of this, we are unable to use previous methods to obtain solutions; instead we will make use of more complicated methods as it will be explained later.

Definition 7. *The* ***Korteweg-de Vries Equation (KdV)*** *is a* 3^{rd} *order, non-linear, dispersive partial differential equation for* $\phi(x,t)$*, where* x *is the one-dimensional space variable,* t *is the time variable and* ϕ *is the amplitude of the wave in question [9].*

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + 6\phi\frac{\partial \phi}{\partial x} = 0 \tag{4.1}$$

where $-\infty < x < \infty, t > 0$.

The term $\frac{\partial^3 \phi}{\partial x^3}$ represents the dispersion of waves [13]. The order tells us that the largest derivative in this equation is the 3^{rd} derivative. The non-linearity aspect arises from the fact that the third term in the equation consists of a product of two terms $\phi\frac{\partial \phi}{\partial x}$. Studying the KdV equation in one-dimensional form means the equation only contains one space variable x and a time variable t. A potential solution to the KdV equation is denoted $\phi(x,t)$.

For some background knowledge of this equation we will look at how it was introduced and some of its properties. John Scott Russell a naval architect, made a significant observation in 1834 regarding waves along the Union Canal at Hermiston, Edinburgh. As a boat was in motion along the narrow channel, Russell noticed that the water around the front of the boat in motion piled up once it reached a halt, forming a wave one foot high and 30 feet long [11]. As the boat returned to motion, Russell followed it along the channel by horseback (for about an hour). He noticed that the shape of the wave formed from the accumulation of water remained unchanged as it propagated along the channel whilst also the speed remained constant. Russell described this phenomenon as the 'Wave of Translation' [10]. It is this phenomenon which is described by the KdV equation as explained in Definition 7. Solutions to the KdV equation are known as soliton solutions.

4.1.1 Soliton properties

The information provided in this section has been motivated by ideas from [13].

Long before soliton solutions and their respective equations were introduced, Russell had already seen a soliton in real-life, what he called it then was a 'Wave of Translation' [10]; a wave which did not break as it travelled [11]. Since then the theory behind solitons has been extensively researched by the likes of Zabusky, Kruskal, Korteweg and his student de Vries just to name a few [15]. Soliton solutions are governed by non-linear partial differential equations such as the Sine-Gordon Equation, Korteweg-de Vries Equation and Kadomtsev-Petviashvili equation just to name a few.

Below is an image of a soliton and we will explain some of the properties of this structure which make them worth studying and highlight why they are so important in the natural world.

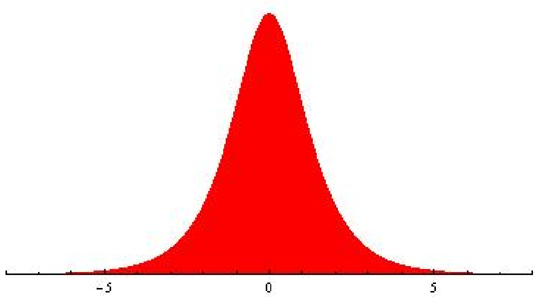


Figure 6: This figure illustrates the shape of a Soliton [20]

1) ***Symmetry in Solitons*:** Solitons are symmetric about the point of highest amplitude which is evident when viewing the soliton in its cross sectional form. Furthermore, there also exist symmetrical properties in the equations from which the soliton solutions are obtained from. In the following subsection we will show a type of symmetry which exists in the KdV equation.

2) ***Solitons retain their shape and velocity*:** Whilst a soliton travels and also what Russell observed, is that a solitary wave does not disperse as it travels. The shape of the wave remains unchanged as it passes through another solitary wave, hence we can say a solitary wave is shape invariant.
Solitons *scatter elastically*, this means that whilst they are shape invariant, the velocity at which the wave travels is also constant upon approaching another solitary wave. In our example of a two-soliton solution using Hirota's Method we will be able to illustrate these properties.

3) ***Solitons exhibit a kind of "non-linear" superposition* principle:** When two solitons are travelling and pass each other, although the speed and shape of the two waves (we will see that one will always be taller than the other) remain constant, the waves actually experience a phase shift. This means that the waves are slightly displaced

compared to if they had not passed through each other. We will show this in our analysis of a two soliton solution later on. It is this phase shift that allows us to disregard the linear superposition principle.

4) Solitary waves only appear locally at a point in time: Solitons are continuous and smooth objects. They are isolated and appear locally at a point in time such that if $u(x,t)$ represents a soliton solution and $x \to \pm\infty$, then $u(x,t) \to 0$.

The shape of a soliton as it travels and the velocity at which it travels is determined by the equation of the wave-packet.

The properties explained above relate back to the observation made by Russell where he noticed the accumulation of water at the front of the canal boat. His observation is what we describe now as a Solitary Wave. As mentioned above $\phi(x,t)$ represents a solution to the KdV equation which we can now refer to as a soliton solution.

The KdV equation is unidirectional, thus the solution $\phi(x,t)$ obtained will describe solitons collectively travelling in one direction only. This eliminates one's need to observe the idea of how solitons would interact had they approached each other whilst travelling in opposite directions, with a head-on collision [22].

Methods to obtain soliton solutions differ to the methods used to obtain solutions to the wave equation and Schrodinger's equation. This is due to the difference being in the non-linearity of the KdV equation. A non-linear partial differential equation is comparatively difficult to solve as the linear superposition principle does not apply to such an equation. This is due to the presence of the non-linear $6\phi\frac{\partial\phi}{\partial x}$ term in the KdV equation. As we are not able to obtain a general solution and thus take advantage of the linear superposition principle we will need to obtain a particular solution to the KdV equation.

4.2 Some properties of the Korteweg-de Vries Equation

4.2.1 Symmetric Properties

The following ideas have been used from [15].

For this section we will write the KdV equation as

$$\phi_t + \gamma\phi\phi_x + \lambda\phi_x^3 = 0 \tag{4.2}$$

where $-\infty < x < \infty, t > 0$.

Here the subscript indicates the variable at which ϕ is partially differentiated with respect to. The constants γ and λ are arbitrary and they can be changed, or all be made to equal 1 depending on how we choose to rescale the equation.

The aim of this section is to show that the KdV equation is invariant, such that if $\phi(x,t)$ is a solution then so is $\phi(-x,-t)$. We will demonstrate this using an example of rescaling the variables in such a way that we return back to the original form.

We can begin with rescaling the space variable x, and t such that

$$x = XY \tag{4.3}$$
$$t = TZ \tag{4.4}$$

where Y and Z are arbitrary constants.

We can now use the chain rule to write the partial derivative with respect to x and t as

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{dX}{dx}\frac{\partial}{\partial X} \\ &= \frac{1}{Y}\frac{\partial}{\partial X} \qquad (4.5)\\ \frac{\partial}{\partial t} &= \frac{dT}{dt}\frac{\partial}{\partial T} \\ &= \frac{1}{Z}\frac{\partial}{\partial T} \qquad (4.6)\end{aligned}$$

We can now use the expressions for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ and apply these to (4.2). This gives us

$$\frac{\phi_T}{Z} + \frac{\gamma\phi\phi_X}{Y} + \frac{\lambda\phi_{XXX}}{Y^3} = 0 \qquad (4.7)$$

multiplying each component by Z gives

$$\phi_T + \frac{\gamma Z\phi\phi_X}{Y} + \frac{\lambda Z\phi_{XXX}}{Y^3} = 0. \qquad (4.8)$$

As we mentioned above, rescaling the variables should leave our equation unchanged, so we need to undertake another rescaling to demonstrate the invariance. To do this we can introduce a rescaling for ϕ;

$$\phi = CU \qquad (4.9)$$

where C is an arbitrary constant and the solution to the KdV equation is also renamed $\phi(x,t) = U(X,T)$. We can now use this rescaled ϕ and apply it to (4.8)

$$U_T + \frac{\gamma CZUU_X}{Y} + \frac{\lambda ZU_{XXX}}{Y^3} = 0. \qquad (4.10)$$

We mentioned before that γ and λ are arbitrary and so we can choose what to make them and solve for C, Y, Z accordingly. For this example we will choose for them both to equal 1.

As we are now working with the rescaled KdV equation (4.10) the coefficient γ from the original KdV equation (4.2) is now $\frac{\gamma CZ}{Y}$ and λ is now $\frac{\lambda Z}{Y^3}$. We equate these to 1 as this is what we require the coefficient to be for this example.

$$\begin{aligned}\frac{\gamma CZ}{Y} &= 1 \qquad (4.11)\\ \frac{\lambda Z}{Y^3} &= 1. \qquad (4.12)\end{aligned}$$

We now have to solve two equations for three unknowns. This indicates the KdV equation lies in a *symmetry group* which explains why our solution to (4.2) is $\phi(x,t) = \phi(-x,-t)$. To solve (4.11) and (4.12) we can let any of C, Y, Z represent an arbitrary constant and then find the other two variables in terms of this constant.

We can let $Y = \alpha$ and rearrange using equation (4.12) and we obtain $Z = \frac{\alpha^3}{\lambda}$. This implies $C = \frac{\lambda}{\alpha^2\gamma}$ by rearranging (4.11). As α is some arbitrary constant, we can show the invariant property for a particular value of α, for example we can let $\alpha = 1$, then C, Y, Z can be written as follows

$$\begin{aligned}C &= \frac{\lambda}{\gamma} \qquad (4.13)\\ Y &= 1 \qquad (4.14)\\ Z &= \frac{1}{\lambda}. \qquad (4.15)\end{aligned}$$

Now that we have values for C, Y, Z, substituting this into our rescaled equation from (4.10) and simplifying gives us

$$U_T + UU_X + U_{XXX} = 0 \tag{4.16}$$

which brings us back to our original equation with the coefficients we required.

This is a particular example of how we could go about changing coefficients in the KdV equation. If we wanted coefficients γ and λ to be 2 and 3 respectively then we could equate (4.11) and (4.12) to 2 and 3 and solve in the same way.

The type of symmetry we have shown here is also known as a *scaling symmetry group* or *similarity transformation group.*

4.2.2 Conservation law

The following ideas have been taken from [16] and [23].

The KdV equation satisfies infinitely many conservation laws. The local conservation law is

$$A_t + B_x = 0 \tag{4.17}$$

where A and B are the conserved local density and flux respectively, where the flux is the flow of a fluid. For an equation to satisfy the conservation law it must be possible to write it in the form in (4.17). Here A and B represent a finite number of functions of ϕ which are partially differentiated with respect to t and x respectively.

For the remainder of this subsection we will write the equation using slightly different notation for simplicity as shown below

$$\phi_t - 6\phi\phi_x + \phi_{xxx} = 0. \tag{4.18}$$

This can be re-written in conservation form in (4.17) as follows

$$\frac{\partial}{\partial_t}\phi + \frac{\partial}{\partial_x}(\phi_{xx} - 3\phi^2) = 0 \tag{4.19}$$

where $A = \phi$ and $B = \phi_{xx} - 3\phi^2$.

Thus we can see that $\int_{-\infty}^{\infty} \phi dx$ is a conserved quantity, where

$$\int_{-\infty}^{\infty} \phi dx = \text{constant} \tag{4.20}$$

is the conservation of mass of the water waves.

The KdV equation also satisfies another conservation law. In multiplying (4.18) by ϕ we obtain the following

$$\phi\phi_t - 6\phi^2\phi_x + \phi\phi_{xxx} = 0. \tag{4.21}$$

We can now write this in conservation form as follows

$$\frac{\partial}{\partial_t}\left(\frac{1}{2}\phi^2\right) + \frac{\partial}{\partial_x}\left(\phi\phi_{xx} - \frac{1}{2}\phi_x^2 - 2\phi^3\right) = 0 \tag{4.22}$$

where $A = \frac{1}{2}\phi^2$ and $B = \phi\phi_{xx} - \frac{1}{2}\phi_x^2 - 2\phi^3$.

Thus, we can see that $\int_{-\infty}^{\infty} \phi^2 dx$ is also a conserved quantity, where

$$\int_{-\infty}^{\infty} \phi^2 dx = \text{constant} \tag{4.23}$$

is the conservation of momentum of water waves.

We have shown that the conserved densities ϕ and ϕ^2 which describe the conservation of mass and momentum respectively, are satisfied by the KdV equation. This implies that there exists a corresponding conserved quantity; $\int_{-\infty}^{\infty} 3\phi^2\phi_t + \phi_x\phi_{xt} dx$. We can obtain this by calculating the following

$$3\phi^2(\phi_t - 6\phi\phi_x + u_{xxx}) + \phi_x(\phi_{xt} - 6\phi_x^2 - 6\phi\phi_{xx} + \phi_{xxxx}) = 0 \tag{4.24}$$

where the first bracket contains the KdV equation as in (4.18) and the second bracket contains the partial derivative of the KdV equation with respect to x. Writing (4.24) in conservation form, it can be written as

$$\frac{\partial}{\partial_t}\left(\phi^3 + \frac{1}{2}\phi_x^2\right) + \frac{\partial}{\partial_x}\left(-\frac{9}{2}\phi^4 + 3\phi^2\phi_{xx} - 6\phi\phi_x^2 + \phi_x\phi_{xxx} - \frac{1}{2}\phi_{xx}^2\right) = 0. \tag{4.25}$$

Referring back to (4.17), $A = \phi^3 + \frac{1}{2}\phi_x^2$ and $B = -\frac{9}{2}\phi^4 + 3\phi^2\phi_{xx} - 6\phi\phi_x^2 + \phi_x\phi_{xxx} - \frac{1}{2}\phi_{xx}^2$.

Thus we can see that $\int_{-\infty}^{\infty} 3\phi^2\phi_t + \phi_x\phi_{xt} dx$ is a conserved quantity, where

$$\int_{-\infty}^{\infty} 3\phi^2\phi_t + \phi_x\phi_{xt} dx = \text{constant} \tag{4.26}$$

is the conservation of energy.

We have shown the existence of the three main conserved quantities; the conservation of mass, momentum and energy, together they describe a physical system in one-dimension, which in our case is the KdV equation.

After further tedious calculations, Gardner, Kruskal and Miura eventually obtained 8 more conserved quantities for the KdV equation. They eventually deduced that in fact the KdV equation satisfies infinitely many conservation quantities this in turn corresponds to the KdV equation being associated with an infinite Hamiltonian system. This is still an active form of research. For further reading on this, please refer to Chapter 5 of [23].

4.2.3 Hirota's bilinear D-operator

Hirota discovered a method to find soliton solutions of the KdV equation by introducing a bilinear D-operator as defined below.

Definition 8. *The* ***Hirota bilinear D-operator*** *can be applied to two functions say, f and g. If we were to differentiate these functions m,n-times respectively, then the Hirota D-operator applied to these is defined with respect to two variables as the following*

$$D_t^m D_x^n (f \cdot g) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')|_{x'=x,t'=t}$$

where $m, n \in \mathbb{Z}^+$ [23].

In a later section we will obtain a two-soliton solution using Hirota's method. Alternatively, we could also use the D-operator to obtain the same answer and also derive n-soliton solutions to the KdV equation. If you would like to learn more of this method, please refer to Chapter 5 of [23] where the two-soliton solution is constructed using the D-operator.

4.3 The Dispersion Relation: The Korteweg-De Vries Equation

As before, we will obtain the dispersion relation, this time for the KdV equation to help us understand the type of wave solution this equation produces. However we do not explicitly use it in our one and two-soliton solutions which follow.
We have stated in Definition 3 that the dispersion relation only exists for a linear partial differential equation. Therefore we must linearise the KdV equation and to do this we consider when the amplitude ϕ is very small thus $|\phi| \ll 1$. It is only then that the term $6\phi\frac{\partial \phi}{\partial x}$ in Definition 7 can be ignored, since the product of two small values results in a smaller value and so $6\phi\frac{\partial \phi}{\partial x} \approx 0$, hence we are able to disregard this. We can now write the linearised form of the KdV equation as:

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} = 0. \tag{4.27}$$

where $-\infty < x < \infty, t > 0$.

In order to deduce the dispersion relation from (4.27) we use the plane wave $\phi(x,t) = e^{i(kx-\omega(k)t)}$ from Definition 4, Section 2.3 as we did to calculate dispersion relation previously.

We need to calculate the 1^{st} and 3^{rd} derivatives with respect to t and x respectively and substitute into the linearised KdV equation. This gives

$$-iwe^{i(kx-\omega t)} - ik^3 e^{i(kx-wt)} = 0. \tag{4.28}$$

We can cancel and simplify like terms on both sides of the equation in (4.28) which results in the following non-linear dispersion relation

$$w(k) = k^3. \tag{4.29}$$

In order for the plane wave to be a solution to the linear KdV equation, the non-linear relationship between the wave number and the angular frequency must be satisfied. We have seen from the Schrödinger equation that this has a significant impact on the shape of the wave as it propagates. Relating to ocean waves, as time progresses, the wave amplitude decreases progressively, by the time the wave approaches the shore, its amplitude is comparatively smaller and no longer large enough to be destructive.

We have just solved the linear KdV equation where dispersive waves exist. However, now we will study the non-linear KdV Equation where $|\phi| \gg 1$, and we will see that such dispersive waves do not exist and we in fact obtain soliton solutions.

4.4 A One-Soliton solution to the Korteweg-de Vries Equation

Now we will carry out an example, in which we find a solution to the KdV equation that consists of the propagation of one soliton as time progresses. We will do this by direct integration of the 3^{rd} order KdV equation (4.1). The way in which we will approach this calculation is by aiming to reduce the 3^{rd} order KdV equation into a 1^{st} order equation as this is much simpler to solve. We will integrate via the so called separation of variables method. The type of solution we are aiming to achieve will take either of the form $\phi(x \pm ct)$. This is a general form of a wave-packet solution which propagates to the right if the sign in between is negative, or propagates to the left if the sign in between is positive. The speed at which the corresponding wave travels at, is represented by c.

As we mentioned above, $\phi(x,t)$ represents a solution to the KdV equation so we can let $\phi(x,t) = u(x-ct)$. In this particular example, which has been motivated by [11], we will obtain a single soliton solution that represents a solitary wave which propagates to the right as time progresses.

We can differentiate $u(x-ct)$ according to the derivatives required in the non-linear KdV equation. This gives us

$$\frac{\partial \phi}{\partial t} = -c\frac{du}{dt}, \quad \frac{\partial \phi}{\partial x} = \frac{du}{dx}, \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{d^2u}{dx^2}, \quad \frac{\partial^3 \phi}{\partial x^3} = \frac{d^3u}{dx^3}. \tag{4.30}$$

Equivalently, $u(x-ct) \equiv u(\xi)$. Using this change of variables we can substitute the derivatives from (4.30) into the KdV equation obtaining

$$-c\frac{du}{d\xi} + 6u\frac{du}{d\xi} + \frac{d^3u}{d\xi^3} = 0. \tag{4.31}$$

We can integrate each of the above terms once with respect to $d\xi$, and as a result of this we obtain an arbitrary constant c_1 on the right-hand side as shown in (4.34) below.

$$\int \left(-c\frac{du}{d\xi} + 6u\frac{du}{d\xi} + \frac{d^3u}{d\xi^3}\right) d\xi = \int 0 d\xi \tag{4.32}$$

$$-cu + \frac{6u^2}{2}u + \frac{d^2u}{d\xi^2} = c_1 \tag{4.33}$$

$$-cu + 3u^2 + \frac{d^2u}{d\xi^2} = c_1 \tag{4.34}$$

We have now arrived at a 2^{nd} order normal differential equation.

In order to obtain the 1^{st} order equation we are aiming for, we need to reduce (4.34) further by multiplying it by $\frac{du}{d\xi}$ and then by $d\xi$ as follows

$$-cu\frac{du}{d\xi} + 3u^2\frac{du}{d\xi} + \frac{d^2u}{d\xi^2}\frac{du}{d\xi} = c_1\frac{du}{d\xi} \tag{4.35}$$

$$-cudu + 3u^2du + \frac{d^2u}{d\xi^2}du = c_1du. \tag{4.36}$$

We can then integrate each term in (4.36) separately with respect to du. In particular, to integrate the 3^{rd} term on the left hand side we can write it as follows

$$\int \frac{d^2u}{d\xi^2}du = \int \frac{d^2u}{d\xi^2}\frac{du}{d\xi}d\xi \tag{4.37}$$

$$= \int \frac{d}{d\xi}\frac{1}{2}\left(\frac{du}{d\xi}\right)^2 d\xi. \tag{4.38}$$

In (4.37) we have multiplied by $\frac{d\xi}{d\xi}$ and then re-written the integral such that the integral and $\frac{d}{d\xi}$ cancel each other out. Using the result in (4.38) we can integrate (4.36) and we obtain

$$-c\int udu + \int 3u^2du + \int \frac{d}{d\xi}\frac{1}{2}\left(\frac{du}{d\xi}\right)^2 d\xi = \int c_1du \tag{4.39}$$

$$-c\frac{u^2}{2} + u^3 + \frac{1}{2}\left(\frac{du}{d\xi}\right)^2 = c_1u + c_2 \tag{4.40}$$

where c_1 and c_2 are arbitrary constants.

The equation (4.40) is a 1^{st} order normal differential equation. This is what we were aiming for, as it makes it slightly simpler, although not trivial, to solve in order to find a particular solution to the KdV equation.

From our knowledge of soliton properties from Section 4.1.1, we know that solitons are localised waves. We saw in property 4 when $x \to \pm\infty$ then $u(\xi) \to 0$ which implies that $\frac{du}{d\xi} \to 0$, and $\frac{d^2u}{d\xi^2} \to 0$. Thus it follows that $c_1 = c_2 = 0$ whilst also after multiplying (4.40) throughout by 2 we eventually obtain

$$-cu^2 + 2u^3 + \left(\frac{du}{d\xi}\right)^2 = 0. \tag{4.41}$$

.

From (4.41) we are required to find the solution $u(\xi)$. Rearranging this to make $(\frac{du}{d\xi})^2$ the subject and factorising it gives:

$$\left(\frac{du}{d\xi}\right)^2 = u^2(c - 2u) \tag{4.42}$$

$$\frac{du}{d\xi} = u\sqrt{c - 2u}. \tag{4.43}$$

To summarise so far, we have obtained a 1^{st} order normal differential equation in (4.43) from our 3^{rd} order non-linear KdV equation which we can now integrate more easily. To do so we can use the method of separation of variables for which the following form rearranged from (4.43) is required to integrate

$$\int \frac{du}{u\sqrt{c-2u}} = \int d\xi. \tag{4.44}$$

Introducing integration limits requires us to change the variables with which we are integrating with respect to;

$$\int_0^u \frac{d\zeta}{\zeta\sqrt{c-2\zeta}} = \int_0^\xi d\eta. \tag{4.45}$$

To carry out the integral in (4.45), we can use the following substitution

$$\zeta = \frac{1}{2}c\mathrm{sech}^2 w. \tag{4.46}$$

Using this substitution for ζ we can rearrange the terms within the square root in the denominator in (4.45) to give

$$\begin{aligned} c-2\zeta &= c-2\left(\frac{1}{2}c\mathrm{sech}^2 w\right) && (4.47)\\ c-2\zeta &= c-c\mathrm{sech}^2 w && (4.48)\\ c-2\zeta &= c(1-\mathrm{sech}^2 w) && (4.49) \end{aligned}$$

using the hyperbolic trigonometric identity $\tanh^2(w)+\mathrm{sech}^2(w)=1$ we obtain

$$c-2\zeta = c\tanh^2 w. \tag{4.50}$$

We can rearrange for ζ we arrive at

$$\zeta = \frac{1}{2}c - \frac{1}{2}c\mathrm{sech}^2 w. \tag{4.51}$$

We can then differentiate (4.51) as this will be needed further on. The chain rule for differentiation can be used with respect to w and we get the following derivative

$$\frac{d\zeta}{dw} = -c\frac{\mathrm{sinh}w}{\cosh^3 w}. \tag{4.52}$$

The upper limit of the integral on the left hand side in (4.45) can be transformed using the substitution in (4.46), rearranging for w gives

$$w = \mathrm{sech}^{-1}\left(\sqrt{\frac{2\zeta}{c}}\right). \tag{4.53}$$

Going back to the initial integral we needed to solve in (4.45), we can now use the substitution in (4.46) together with (4.50) and (4.52) and substitute these into (4.45). We can then simplify this further which gives the following calculations

$$\begin{aligned} \int_0^\xi d\eta &= \int_0^u \frac{d\zeta}{\zeta\sqrt{c-2\zeta}} && (4.54)\\ \xi &= \int_0^w -c\frac{\sinh w}{\cosh^3 w}\frac{1}{\frac{1}{2}c\mathrm{sech}^2 w}\frac{1}{\sqrt{c\tanh^2 w}}dw && (4.55)\\ &= \frac{-2}{\sqrt{c}}\int_0^w dw && (4.56)\\ &= \frac{-2}{\sqrt{c}}w. && (4.57) \end{aligned}$$

Now we can substitute (4.53) into the right hand side of (4.57) which gives us

$$\xi = -\frac{2}{\sqrt{c}}\text{sech}^{-1}\left(\sqrt{\frac{2u}{c}}\right). \tag{4.58}$$

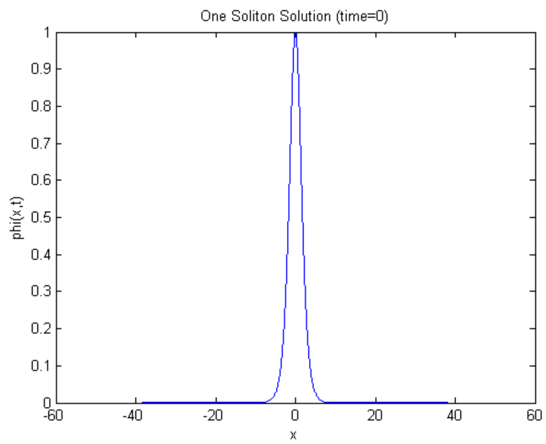
We can then rearrange (4.58) as follows in order to obtain an expression for u in terms of ξ;

$$\begin{aligned} -\frac{\sqrt{c}}{2}\xi &= \text{sech}^{-1}\left(\sqrt{\frac{2u}{c}}\right) && (4.59)\\ \text{sech}^2\left(-\frac{\sqrt{c}}{2}\xi\right) &= \left(\sqrt{\frac{2u}{c}}\right)^2 && (4.60)\\ \text{sech}^2\left(-\frac{\sqrt{c}}{2}\xi\right) &= \frac{2u}{c} && (4.61)\\ u(\xi) &= \frac{c}{2}\text{sech}^2\left(-\frac{\sqrt{c}}{2}\xi\right). && (4.62) \end{aligned}$$

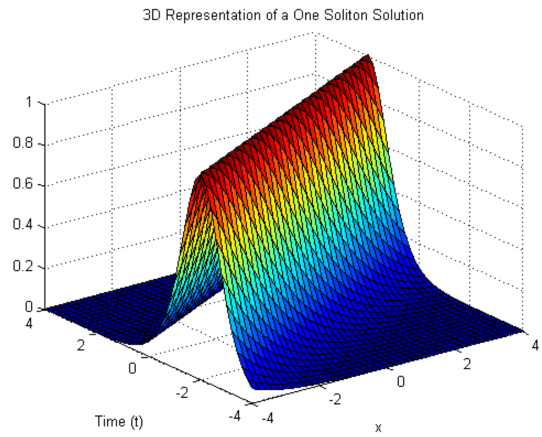
Remembering that we initially made a change of variables $\phi(x,t) = u(x-ct) \equiv u(\xi)$, we can now return to our original variables t and x to obtain the following

$$\phi(x,t) = \frac{c}{2}\text{sech}^2\left(-\frac{\sqrt{c}}{2}(x-ct)\right). \tag{4.63}$$

We have arrived at a one-soliton solution to the KdV equation which is illustrated in Figure 7 below.



(a) 2D Plot



(b) 3D Plot

Figure 7: This figure presents our one-soliton solution graphically in two-dimensional and three-dimensional form

From our equation for a one-soliton solution in (4.63) we can make an important observation. The amplitude of the wave is $\frac{c}{2}$ and the wavelength, in other words the width of the wave is $-\frac{\sqrt{c}}{2}$. This tells us that they are both dependent on the velocity c at which the wave propagates. Hence, as the velocity increases then so does the wave amplitude and wavelength. However, although the amplitude and wavelength may change, the actual shape of the soliton is invariant whilst it travels, maintaining its hump-like shape as is evident from Figure 7. Also evident from this figure is the localised wave structure which we explained previously. We can see that as $x \to \pm\infty$ the solitary wave flattens out and $\phi(x,t) \to 0$.

Our one-soliton solution differs in comparison to the linear wave equation solution we obtained in a previous example. In this previous example we were able to find one solution to the wave equation, and by varying the amplitude we were able to obtain multiple solutions, where the amplitude was independent on the velocity of the wave.

The dependency of both the wavelength and amplitude on the velocity in soliton solutions obtained from the KdV equation, is due to the presence of the the non-linear dispersion relation. It is this non-linear dispersion relation, together with the non-linearity of the KdV equation, that form the basis upon which this dependency occurs.

What we have just studied is a one solitary wave. One may now wonder what happens if we have two solitary waves? We will illustrate this in the next subsection where we will derive a particular two-soliton solution (more explicitly) to the KdV equation. In this solution, we will see that the linear superposition principle will only be satisfied to a certain extent.

4.5 A Two-Soliton solution to the Korteweg-de Vries Equation

A two-soliton solution represents two solitary waves travelling together. As mentioned previously, we cannot use exactly the linear superposition principle, instead we need to find a *particular* two-soliton solution rather than a general one. In this example we will approach this calculation using a different method called Hirota's Method.

The solution we will obtain will have some characteristic properties which will explain some of our observations. We will notice that our solution will illustrate two solitary waves where the taller and narrower wave will travel faster compared to the shorter wider wave. This is due to the dependence of the amplitude on the speed [22].

The following example has been motivated by [17].

For convenience, we can re-write the non-linear KdV equation with slightly different notation in line with [17].

$$u_t = -u_{xxx} + 6uu_x \tag{4.64}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$ and $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$.

Remark. *The variable 'u' in this section has no correspondence with that of Section 2: The Wave Equation*

We will briefly explain the steps here to gather an overview of what we are aiming to achieve in this example. Using the substitution $u = \partial_x v$ to the KdV equation (4.64), we can obtain Hirota's form of the KdV equation. Using a general form of the solution denoted as $\eta(x,t)$, we can substitute this into the Hirota's form of the KdV equation and after a lengthy calculation, we eventually obtain a two-soliton solution.

The first step of the calculation involves making the substitution $u = \partial_x v$ into (4.64)

$$\partial_x(v_t + v_{xxx} - 3v_x^2) = 0. \tag{4.65}$$

We can then integrate this with respect to ∂_x so that we obtain the potential KdV (PKdV) equation below, assuming the arbitrary constant of integration, $C = 0$

$$v_t + v_{xxx} - 3v_x^2 = 0. \tag{4.66}$$

A solution to the PKdV equation in (4.66) is denoted $v(x,t)$. From this we can deduce a solution to the KdV equation from the relationship $u = \partial_x v$.

Hence a solution to the PKdV equation is

$$v(x,t) = -\sqrt{c}\left(\tanh\left(\frac{\sqrt{c}}{2}(x-ct)\right)+1\right) \tag{4.67}$$

which is in fact a one-soliton solution. This will aid us in generating Hirota's form of the KdV equation. We can re-write this in a different way so it is easier to work with.

$$\begin{aligned}
v(x,t) &= -\sqrt{c}\left(\frac{e^{\sqrt{c}(x-ct)}-1}{e^{\sqrt{c}(x-ct)}+1}+1\right) && (4.68)\\
&= -\sqrt{c}\left(\frac{e^{\sqrt{c}(x-ct)}-1}{e^{\sqrt{c}(x-ct)}+1}+\frac{e^{\sqrt{c}(x-ct)}+1}{e^{\sqrt{c}(x-ct)}+1}\right) && (4.69)\\
&= -\sqrt{c}\left(\frac{2e^{\sqrt{c}(x-ct)}}{e^{\sqrt{c}(x-ct)}+1}\right) && (4.70)
\end{aligned}$$

In (4.68) – (4.70) we have expressed tanh in its exponential form, $\tanh = \frac{e^{-2x}-1}{e^{2x}+1}$ and re-written it as a single fraction.

We denote (4.70) as

$$v = \frac{-2\eta_x}{\eta} \tag{4.71}$$

which is referred to a the Hopf-Cole transformation where $\eta = 1 + e^{\sqrt{c}(x-ct)}$ [23].

This motivates our use of making the *Hirota Substitution* $v = \frac{-2\eta_x}{\eta}$ into the PKdV equation. In order to do this we need to calculate the relevant derivatives using (4.71), with the use of the quotient rule for differentiation, stated as $\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$. Then we must substitute these derivatives into (4.66).

The terms we require to substitute into the PKdV equation are:

$$\begin{aligned}
v_t &= \frac{-2\eta_{xt}}{\eta} + \frac{2\eta_x\eta_t}{\eta^2} && (4.72)\\
v_{xx} &= \frac{6\eta^2\eta_x\eta_{xx} - 2\eta^3\eta_{xxx} - 4\eta\eta_x^3}{\eta^4} && (4.73)\\
v_{xxx} &= \frac{6\eta^6\eta_{xx}^2 - 2\eta^7\eta_{xxxx} - 4\eta^4\eta_x^4 - 24\eta^5\eta_x^2\eta_{xx} + 8\eta^6\eta_x\eta_{xxx} + 16\eta^4\eta_x^4}{\eta^8}. && (4.74)
\end{aligned}$$

Substituting these terms, we obtain the following lengthy equation

$$\begin{aligned}\frac{-2\eta_{xt}}{\eta}+\frac{2\eta_x\eta_t}{\eta^2} &= \frac{-6\eta_{xx}^2}{\eta^2}+\frac{2\eta_{xxxx}}{\eta}-\frac{12\eta_x^4}{\eta^4}+\frac{24\eta_x^2\eta_{xx}}{\eta^3}-\frac{8\eta_x\eta_{xxx}}{\eta^2}+\frac{12\eta_{xx}^2}{\eta^2}- \\ &\quad \frac{24\eta_x^2\eta_{xx}}{\eta^3}+\frac{12\eta_x^4}{\eta^4} \qquad (4.75)\end{aligned}$$

From (4.75) we can cancel out like terms which leaves us with

$$\frac{-2\eta_{xt}}{\eta}+\frac{2\eta_x\eta_t}{\eta^2}=\frac{2\eta_{xxxx}}{\eta}-\frac{8\eta_x\eta_{xxx}}{\eta^2}+\frac{6\eta_{xx}^2}{\eta^2}. \qquad (4.76)$$

To get rid of the denominator we can multiply each numerator by η^2 and simplify the coefficients, obtaining

$$\eta\eta_{xt}-\eta_x\eta_t+\eta\eta_{xxxx}-4\eta_x\eta_{xxx}+3\eta_{xx}^2=0. \qquad (4.77)$$

Now we have arrived at the Hirota form of the KdV equation. With this form it is now easier to find n-soliton solutions. Although equation (4.77) looks rather complicated, it can be written in a simpler form with the use of Hirota D-operator. Applying the D-operator, we can write the Hirota form of the KdV equation in the following short form as such

$$D_x(D_t+D_x^3)\eta\cdot\eta=0. \qquad (4.78)$$

Although we will not be using the D-operator method here, (4.78) allows us to see how we can write (4.77) using the D-operator from Definition 8. We can use this as a starting point in using this alternative method to obtain a two-soliton solution as further work.

A solution to the KdV equation illustrating two solitary waves takes the form

$$\eta=1+e^{\theta_1}+e^{\theta_2}+ae^{\theta_1+\theta_2} \qquad (4.79)$$

where $\theta_1=A_1x-A_1^3t+C_1$, $\theta_2=A_2x-A_2^3t+C_2$, A_1, A_2, C_1 and C_2 are $\in\mathbb{R}$, x and t are space and time variables respectively and $a=\left(\frac{A_1-A_2}{A_1+A_2}\right)^2$.

We can calculate all the required derivatives needed to substitute into Hirota's form of the KdV equation (4.77)

$$\begin{aligned}
\eta &= 1+e^{\theta_1}+e^{\theta_2}+ae^{\theta_1+\theta_2} && (4.80)\\
\eta_x &= A_1e^{\theta_1}+A_2e^{\theta_2}+a(A_1+A_2)e^{\theta_1+\theta_2} && (4.81)\\
\eta_t &= -A_1^3e^{\theta_1}-A_2^3e^{\theta_2}-(A_1^3+A_2^3)ae^{\theta_1+\theta_2} && (4.82)\\
\eta_{xx} &= A_1^2e^{\theta_1}+A_2^2e^{\theta_2}+(A_1+A_2)^2ae^{\theta_1+\theta_2} && (4.83)\\
\eta_{xxx} &= A_1^3e^{\theta_1}+A_2^3e^{\theta_2}+(A_1+A_2)^3ae^{\theta_1+\theta_2} && (4.84)\\
\eta_{xxxx} &= A_1^4e^{\theta_1}+A_2^4e^{\theta_2}+(A_1+A_2)^4ae^{\theta_1+\theta_2} && (4.85)\\
\eta_{xt} &= -A_1^4e^{\theta_1}-A_2^4e^{\theta_2}-(A_1^3+A_2^3)(A_1+A_2)ae^{\theta_1+\theta_2}. && (4.86)
\end{aligned}$$

We can multiply terms according to (4.77), and assuming we add the terms correctly, all the coefficients should cancel out. This should result in our equation to equal 0.

For the convenience of the reader the lengthy calculations have been omitted. Please refer to Appendix 3 if you would like to follow them through.

As a result of carrying out these calculations we can confirm that $\eta = 1+e^{\theta_1}+e^{\theta_2}+ae^{\theta_1+\theta_2}$ is a solution to the Hirota form of the KdV equation, only when $a = \left(\frac{A_1-A_2}{A_1+A_2}\right)^2$.

Now, in order to find a solution u to the KdV equation we use the following

$$u = \left(\frac{-2\eta_x}{\eta}\right)_x . \tag{4.87}$$

This requires us to use η and calculate its partial derivative with respect to x. The full form of the solution η can be written as

$$\eta(x,t) = 1 + e^{A_1x-A_1^3t+C_1} + e^{A_2x-A_2^3t+C_2} + \left(\frac{A_1-A_2}{A_1+A_2}\right)^2 e^{(A_1+A_2)x-(A_1^3+A_2^3)t+(C_1+C_2)} \tag{4.88}$$

For this example we can simply let $A_1 = 1$, $A_2 = 2$ and $C_1 = C_2 = 0$. We can re-write η with these constants as

$$\eta = 9 + 9e^{x-t} + 9e^{2x-8t} + e^{3x-9t}. \tag{4.89}$$

Calculating η_x, we can substitute this together with η into (4.87), and obtain the following

$$u = \frac{\partial}{\partial x}\left(\frac{-18e^{x-t} - 36e^{2x-8t} - 6e^{3x-9t}}{9 + 9e^{x-t} + 9e^{2x-8t} + e^{3x-9t}}\right). \tag{4.90}$$

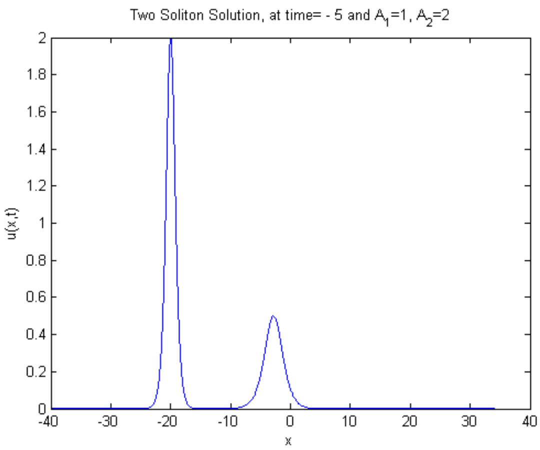
Intermediate steps to find this two-soliton solution have been provided in Appendix 4. We can use the quotient rule of differentiation to compute the derivative in (4.90). This calculates to

$$u(x,t) = \frac{-162e^{x-t} - 324e^{3x-9t} - 72e^{4x-10t} - 648e^{2x-8t} - 18e^{5x-17t}}{(9 + 9e^{x-t} + 9e^{2x-9t} + e^{3x-9t})^2}. \tag{4.91}$$

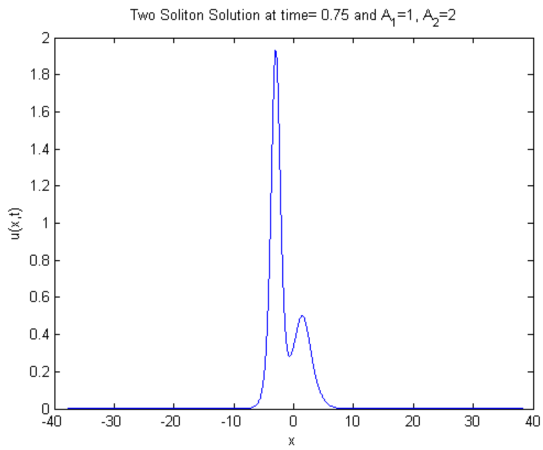
We can take out a factor of -18 and write $u(x,t)$ as

$$u(x,t) = -18\frac{9e^{x-t} + 18e^{3x-9t} + 4e^{4x-10t} + 36e^{2x-8t} + e^{5x-17t}}{(9 + 9e^{x-t} + 9e^{2x-9t} + e^{3x-9t})^2}. \tag{4.92}$$

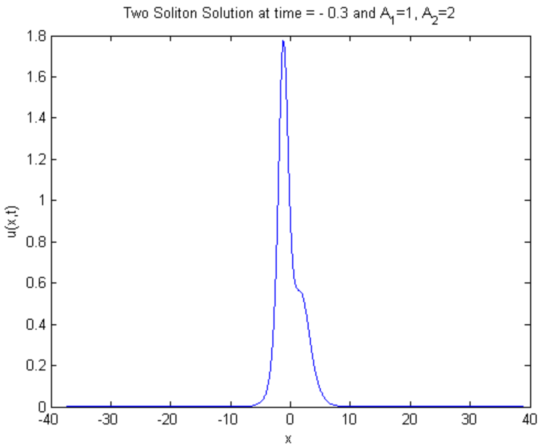
Finally, we arrive at a two-soliton solution to the Korteweg-de Vries equation. Illustrating (4.92) graphically we get the following graphs in Figure 8 at increasing times.



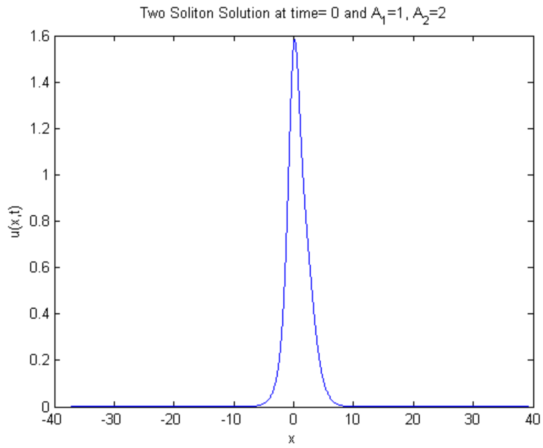
(a) Time= -5



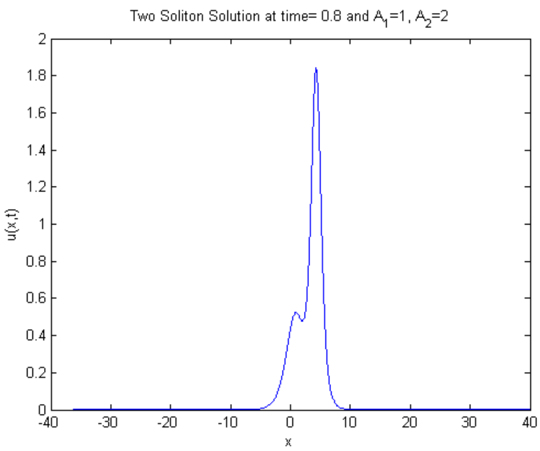
(b) Time= -0.75



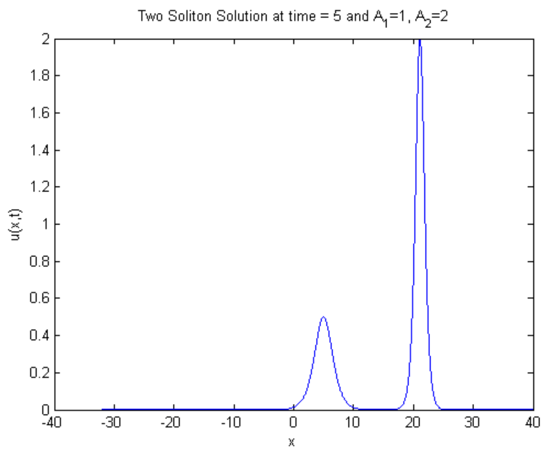
(c) Time= -0.3



(d) Time= 0



(e) Time= 0.8



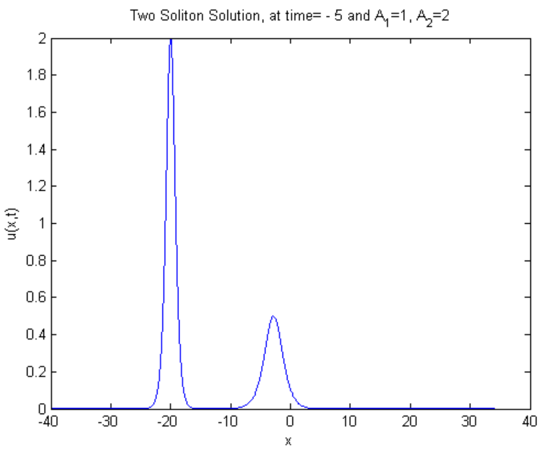
(f) Time= 5

Figure 8: This figure illustrates how two solitons travel in our two-soliton solution (4.92) at times $t = -5$, $t = -0.75$, $t = -0.3$, $t = 0$, $t = 0.8$ and $t = 5$. The constants in our solution are $A_1 = 1$, $A_2 = 2$ and $C_1 = C_2 = 0$.

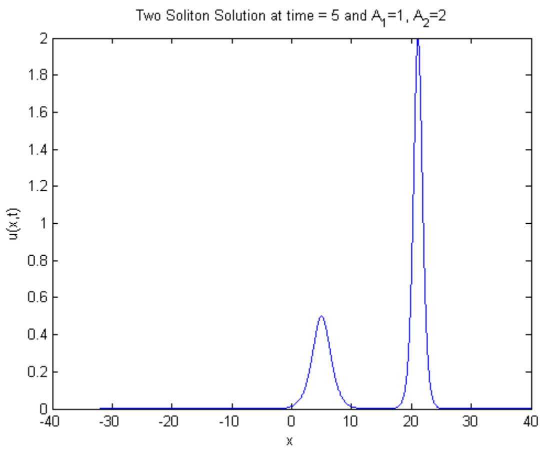
Figure 8 shows the illustrations of the function $-u(x,t)$ from (4.92). Producing graphs for $u(x,t)$ will plot the same graphs symmetrical in the x-axis. Figure 8 shows how two solitons interact both with each other and individually at progressive times. In Figure 8a at time, $t=-5$, it is clear that there are two solitary waves with distinct peaks, one significantly greater than the other, a characteristic common in solitons. As time progresses the waves travel in the positive x-direction as our solution takes the form $u(x-ct)$. In particular, the wave with the taller peak travels at a greater velocity than the smaller wave, eventually approaching the smaller wave, it travels right through.

At time, $t=0$ in Figure 8d, the two solitons have fully merged together forming one soliton. The phenomenon by which this occurs is known as *soliton resonance*. Also at this point, the solitons are defined as being stable, but very quickly become unstable as they separate from each other when the larger wave travels ahead of the smaller one [22]. The shape of both solitons remain unchanged through their motion.

According to Figure 8, the motion of the waves described so far is the same as that which would be observed in solutions for linear wave equations, where the linear superposition principle applies. However, we know that in a non-linear equation the same superposition principle would not apply, but rather a slight variation to this principle is required. This arises as the large amplitude wave travels through the smaller wave and a *phase-shift* becomes evident. This explains why the two waves are not in the same position as they would have been whilst travelling at uniform speed, had they not interacted with one another. We can explain this by looking at Figure 9, where we have taken two graphs from Figure 8 at time $t=-5$ and $t=5$ to better explain this phase-shift.



(a) Time= -5



(b) Time= 5

Figure 9: This figure allows us to see a phase-shift more clearly, from our two soliton solution

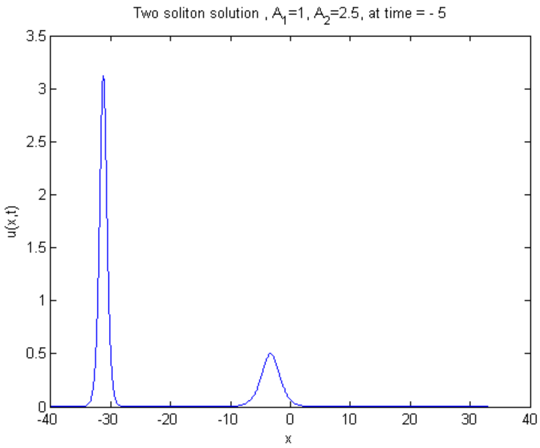
Figure 9b shows the two solitary waves as they separate after they pass through each other at time $t=0$. Referring to Figure 9b if the superposition principle did exist, the larger wave's peak should lie at $x=20$ as at time $t=-5$ the larger peak lay at $x=-20$. This would imply a constant speed, thus a symmetry in the position of each wave would occur. However, in Figure 9b we can see that the larger peak is actually slightly ahead of $x=20$, hence an absence of symmetry has occurred thus indicating the non-existence of the linear superposition principle. From our knowledge of soliton properties, we know that once two solitary waves have passed through each other, their velocity and shape remains unchanged. A viable explanation for the absence of symmetry in our example is that a phase shift has occurred. This occurrence of a phase shift is not a surprise, but rather a well-known characteristic of solitons. It is this concept which aids us in recognising travelling solitary waves and rules out the existence of the linear superposition principle. Instead it introduces a kind of 'non-linear' superposition principle.

We mentioned earlier that there exists two distinct peaks in our solution. We can suggest that the height of each peak is determined by A_1 and A_2. To see the effect of A_1 and A_2, we will do another example and slightly change these values to $A_1 = 1$ and $A_2 = 2.5$. Solving in the same way as we did previously, we obtain the following solution:

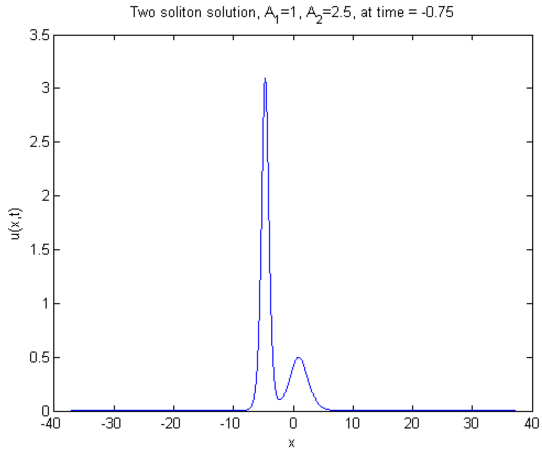
$$u(x,t) = \frac{-2(\eta_{xx}\eta - \eta_x^2)}{\eta^2} \tag{4.93}$$

where $\eta = 49 + 49e^{x-t} + 49e^{2.5x-15.625t} + 9e^{3.5x-16.625t}$ and η_x, η_{xx} are the 1^{st} and 2^{nd} derivatives with respect to x.

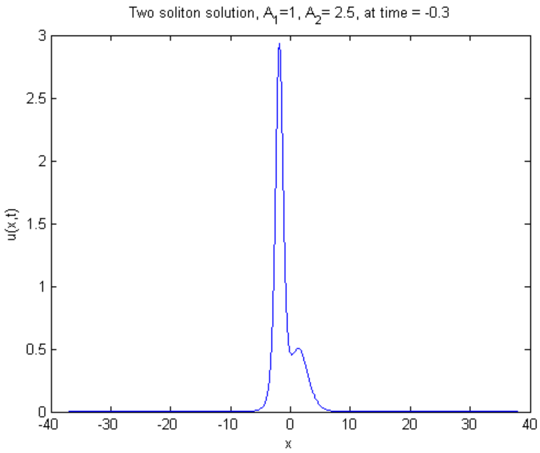
We can recall that the shape and size of solitary waves does not change as it travels with time, therefore at all times the waves peaks will remain constant. Comparing solution (4.92) with (4.93), we observe the significant changes in the soliton peaks. These changes occur as a result of increasing A_2 by 0.5 and keeping A_1 the same. This is demonstrated in Figure 10 below, which illustrates the plot of solution (4.93) at progressive times.



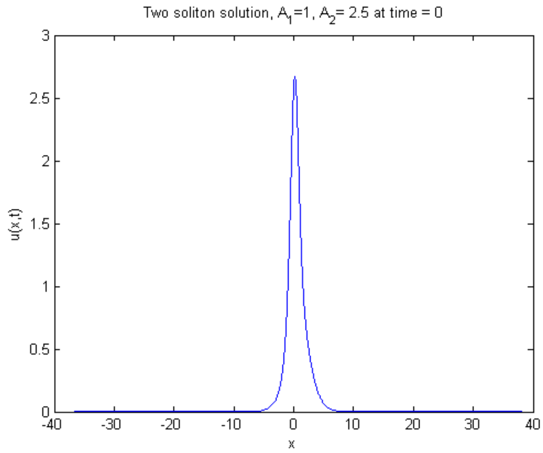
(a) Time= -5



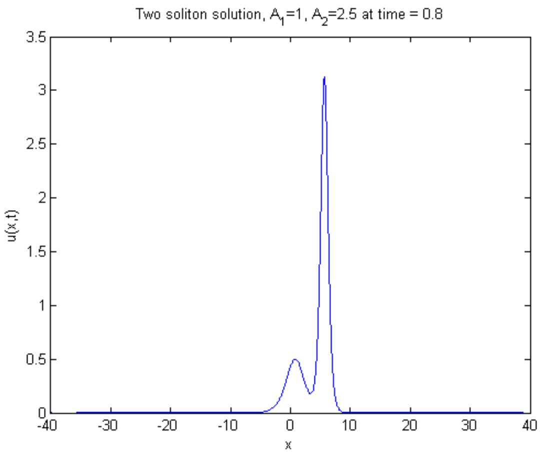
(b) Time= -0.75



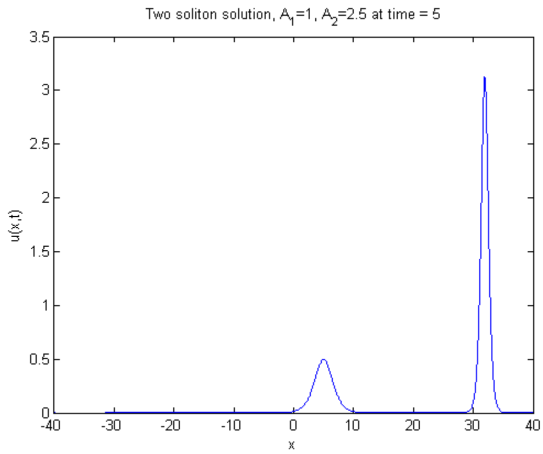
(c) Time= -0.3



(d) Time= 0



(e) Time= 0.8



(f) Time= 5

Figure 10: This figure illustrates how two solitons travel in our second two-soliton solution (4.93) at times $t = -5$, $t = -0.75$, $t = -0.3$, $t = 0$, $t = 0.8$ and $t = 5$. The constants in our solution are $A_1 = 1$, $A_2 = 2.5$ and $C_1 = C_2 = 0$.

Looking at Figure 10, we can see that in comparison with Figure 8 the peak of the larger wave is greater and is also at a different position at the same time. From this we can deduce that the values of A_1 and A_2 determine the height of the smaller and larger solitary waves and also their positions in time. Making this observation and looking at one time in particular, say time, $t = -5$ in Figure 10a, we can see that keeping A_1 unchanged has resulted in the smaller wave remaining the same size, and in the same position as in Figure 8a. Comparatively, looking at the larger wave, increasing A_2 by 0.5 has resulted in an increase in the amplitude and also a change in its position. In fact, A^2 actually refers to the speed of the respective soliton [17].

Comparing Figures 8 and 10 we can see that as time progresses, the larger of the two solitons travels faster, as it covers a greater distance in the same amount of time. Whereas the smaller soliton has covered the same distance in both figures, as is expected since A_1 remains unchanged. This is a rather significant observation of our solutions.

By understanding the effect of the constants on the speed and shape of our solitons, we can manipulate these values and produce several variations of the two solitons in question. A couple of observations we can afford to make are how long it takes for two solitons of a particular size to interact with each other, whilst travelling at particular speeds. As well as how long it takes n-solitons to interact with one other whilst travelling at same or different speeds. This gives a considerable amount of freedom to observing such solutions.

5 Conclusion

To conclude our observations, we will summarise the findings we have made throughout this thesis.

We have studied three partially differentiated wave equations; the linear wave equation, the Schrödinger equation and the Korteweg-de Vries equation.

From our study of the wave equation, we were able to deduce that the linear wave equation exhibited a linear dispersion relation. Using Fourier analysis we calculated a general solution to this equation. And from the illustration of this we observed that such wave solutions represented waves which had constant amplitude as they travelled, indicating the presence of these in tsunamis.

Conversely, we deduced that the Schrödinger equation satisfied a non-linear dispersion relation. Upon calculating a general solution to this equation, we observed that waves represented by these solutions produced illustrations in which the waves possessed a decreasing amplitude, as they travelled with time.

Taking our knowledge of waves further, we studied a very particular, non-linear, partial differential equation which models shallow water waves; the Korteweg-de Vries equation. Linearising this equation and calculating its non-linear dispersion relation, we very quickly recognised that this linearised equation, exhibited solutions of dispersive waves. However, upon calculating one and two-soliton solutions of the non-linear KdV equation, we were able to make some interesting findings on the propagation of such solitary waves.

We observed a phase-shift upon the interaction of two solitons. Due to this phase-shift and the non-linearity of the KdV equation which governed our soliton solutions, we disregarded the existence of the linear superposition principle. This meant that we were not able to use Fourier analysis as we were required to calculate a particular solution rather than a general one. To do this, we calculated a one-soliton solution using direct integration. We then used Hirota's method to obtain a two-soliton solution. Using this method we observed the significant contributions of the arbitrary constants on the soliton's speed and shape. We then further discussed observations we could make upon varying these constants.

When studying the Hirota bilinear D-operator, an attempt was made to use this to obtain a two-soliton solution, however this proved to be rather challenging. If we were afforded more time, we could use the D-operator as an alternative method to finding two-soliton solutions. Hence we could use this to confirm the correctness of the two-soliton solution we obtained. We can also adopt this method to derive n-soliton solutions. In illustrating these solutions, we will be able to observe that the solitons should eventually line up in order of decreasing amplitude [22]. In addition to this, we can vary the constants to see how solitons of different speed and size interact with one another. This could enable us to apply our examples to real-world situations, and be able to make predictions and assumptions based on these results.

6 Appendices

6.1 Appendix 1

Full calculation for lines between (3.10) – (3.13)

$$
\begin{aligned}
-\frac{k^2}{4}+i(kx-\frac{1}{2}k^2t) &= -\frac{k^2}{4}+ikx-\frac{1}{2}k^2it \\
&= \left(-\frac{1}{4}-\frac{1}{2}it\right)k^2+ikx \\
&= \left(\frac{-1-2it}{4}\right)k^2+ikx \\
&= \frac{-1-2it}{4}\left[k^2+\left(\frac{4ix}{-1-2it}\right)k\right] \\
&= \frac{-1-2it}{4}\left[k^2+\left(\frac{4ix}{-1-2it}\right)k+\left(\frac{2ix}{-1-2it}\right)^2-\left(\frac{2ix}{-1-2it}\right)^2\right] \\
&= \frac{-1-2it}{4}\left[\left(k+\left(\frac{2ix}{-1-2it}\right)\right)^2-\left(\frac{2ix}{-1-2it}\right)^2\right] \\
&= \frac{-1-2it}{4}\left(k+\left(\frac{2ix}{-1-2it}\right)\right)^2+\left(\frac{1+2it}{4}\right)\left(\frac{2ix}{-1-2it}\right)^2
\end{aligned} \tag{6.1}
$$

In the above calculations we have taken the exponent from (3.9) and rearranged it in such a way were we can apply a the sum of two squares. This method has been used to enable us to make a suitable substitution.

Now we can make a substitution as shown below

$$
\begin{aligned}
\text{let} y &= \sqrt{\frac{1+2it}{4}}\left(k+\frac{2ix}{-1-2it}\right) \\
\frac{dy}{dk} &= \frac{\sqrt{1+2it}}{2} \\
\frac{2dy}{\sqrt{1+2it}} &= dk.
\end{aligned}
$$

6.2 Appendix 2

Below is the full length of calculations from (3.17) – (3.18).

In order to be able to plot the solution in question, we are required to take the absolute value of $\psi(x,t)$ in order to get rid of the complex valued number, i.

$$|\psi(x,t)| = |\frac{1}{\sqrt{(1+2it)}}|e^{\frac{2itx^2}{4t^2+1}}||e^{-\frac{x^2}{4t^2+1}}|$$

We can take the absolute value of each term individually and then multiply them together. We will make use of the following rule to find the absolute value; $|a+bi| = \sqrt{a^2+b^2}$.

Applying this, we obtain the following terms

$$\begin{aligned}
\text{1st term} &= \left|\frac{1}{\sqrt{(1+2it)}}\right| = \frac{1}{\sqrt{\sqrt{(1^2+(2t)^2}}} = \frac{1}{\left((1+4t^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}} = \left(1+4t^2\right)^{-\frac{1}{4}} \\
\text{2nd term} &= |e^{-\frac{x^2}{4t^2+1}}| \\
&= e^{-|\frac{x^2}{4t^2+1}|} \\
&= e^{-\frac{x^2}{4t^2+1}}.
\end{aligned}$$

For the following term, we will make use of Euler's identity, $e^{ix} = \cos x + i \sin x$

$$\begin{aligned}
\text{3rd term} &= |e^{\frac{2itx^2}{4t^2+1}}| \\
&= \left|\cos\left(\frac{2tx^2}{4t^2+1}\right) + i\sin\left(\frac{2tx^2}{4t^2+1}\right)\right| \\
&= \sqrt{\cos^2\left(\frac{2tx^2}{4t^2+1}\right) + \sin^2\left(\frac{2tx^2}{4t^2+1}\right)} \\
&= 1
\end{aligned}$$

Above, we have applied the trigonometric identity $\sin^2 x + \cos^2 x = 1$ to simplify this term.

Finally, multiplying the results together gives us

$$|\psi(x,t)|^2 = \left(\left(1+4t^2\right)^{-\frac{1}{4}}\right)^2 \left(e^{-\frac{x^2}{1+4t^2}}\right)^2$$

Re-writing this, a solution of the Schrödinger equation with a non-linear dispersion relation is

$$|\psi(x,t)|^2 = \tfrac{1}{\sqrt{1+4t^2}} e^{-\frac{2x^2}{1+4t^2}}$$

6.3 Appendix 3

Now we can calculate all the relevant derivatives according to the ones present in Hirota's form of KdV as follows

$$\begin{aligned}
\eta &= 1 + e^{\theta_1} + e^{\theta_2} + ae^{\theta_1+\theta_2} \\
\eta_x &= A_1e^{\theta_1} + A_2e^{\theta_2} + a(A_1+A_2)e^{\theta_1+\theta_2} \\
\eta_t &= -A_1^3e^{\theta_1} - A_2^3e^{\theta_2} - (A_1^3+A_2^3)ae^{\theta_1+\theta_2} \\
\eta_{xx} &= A_1^2e^{\theta_1} + A_2^2e^{\theta_2} + (A_1+A_2)^2ae^{\theta_1+\theta_2} \\
\eta_{xxx} &= A_1^3e^{\theta_1} + A_2^3e^{\theta_2} + (A_1+A_2)^3ae^{\theta_1+\theta_2} \\
\eta_{xxxx} &= A_1^4e^{\theta_1} + A_2^4e^{\theta_2} + (A_1+A_2)^4ae^{\theta_1+\theta_2} \\
\eta_{xt} &= -A_1^4e^{\theta_1} - A_2^4e^{\theta_2} - (A_1^3+A_2^3)(A_1+A_2)ae^{\theta_1+\theta_2}
\end{aligned}$$

We can obtain each term in Hirota's KdV equation by multiplying the above derivatives were necessary and collecting like terms where required.

$$\begin{aligned}
\eta_{xt}\eta = {} & -A_1^4e^{\theta_1} - A_2^4e^{\theta_2} - A_1^4e^{2\theta_1} - A_2^4e^{2\theta_2} + (-A_1^4 - A_2^4 - a(A_1+A_2)(A_1^3+A_2^3))e^{\theta_1+\theta_2} + \\
& (-A_1^4 - (A_1+A_2)(A_1^3+A_2^3))ae^{2\theta_1+\theta_2} + (-A_2^4 - (A_1+A_2)(A_1^3+A_2^3))ae^{\theta_1+2\theta_2} - \\
& (A_1+A_2)(A_1^3+A_2^3)a^2e^{2\theta_1+2\theta_2}
\end{aligned}$$

$$\begin{aligned}\eta_x\eta_t &= -A_1^4e^{2\theta_1} - A_2^4e^{2\theta_2} + (-A_1A_2^3 - A_1^3A_2)e^{\theta_1+\theta_2} + (-A_1(A_1^3+A_2^3) - A_1^3(A_1+A_2))ae^{2\theta_1+\theta_2}\\ &\quad +(-A_2(A_1^3+A_2^3) - A_2^3(A_1+A_2))ae^{\theta_1+2\theta_2} - a^2(A_1+A_2)(A_1^3+A_2^3)e^{2\theta_1+2\theta_2}\end{aligned}$$

$$\begin{aligned}\eta\eta_{xxxx} &= A_1^4e^{\theta_1} + A_2^4e^{\theta_2} + A_1^4e^{2\theta_1} + A_2^4e^{2\theta_2} + (A_1^4 + A_2^4 + a(A_1+A_2)^4)e^{\theta_1+\theta_2} +\\ &\quad a(A_1^4 + (A_1+A_2)^4)e^{2\theta_1+\theta_2} + a(A_2^4 + (A_1+A_2)^4)e^{\theta_1+2\theta_2}a^2(A_1+A_2)^4e^{2\theta_1+2\theta_2}\end{aligned}$$

$$\begin{aligned}\eta\eta_{xxx} &= A_1^4e^{2\theta_1} + A_2^4e^{2\theta_2} + (A_1A_2^3 + A_1^3A_2)e^{\theta_1+\theta_2} + (A_1(A_1+A_2)^3 + A_1^3(A_1+A_2))ae^{2\theta_1+\theta_2}\\ &\quad +(A_2(A_1+A_2)^3 + A_2^3(A_1+A_2))ae^{\theta_1+2\theta_2} + a^2(A_1+A_2)^4e^{2\theta_1+2\theta_2}\end{aligned}$$

$$\begin{aligned}3\eta_{xx}^2 &= 3A_1^4e^{2\theta_1} + 3A_2^4e^{2\theta_2} + 3(A_1^2A_2^2 + A_1^2A_2^2)e^{\theta_1+\theta_2} + 3(A_1^2(A_1+A_2)^2 + A_1^2(A_1+A_2)^2)ae^{2\theta_1+\theta_2}\\ &\quad +3(A_2^2(A_1+A_2)^2 + A_2^2(A_1+A_2)^2)ae^{\theta_1+2\theta_2} + 3a^2(A_1+A_2)^4e^{2\theta_1+2\theta_2}\end{aligned}$$

Now referring back to the Hirota form of the KdV we can collect the coefficients of e^{θ_1}, e^{θ_2}, $e^{\theta_1+\theta_2}$, $e^{2\theta_1}$, $e^{2\theta_2}$, $e^{2\theta_1+2\theta_2}$, $e^{2\theta_1+\theta_2}$, $e^{\theta_1+2\theta_2}$ from all the terms $\eta_{xt}\eta$, $\eta_x\eta_t$, $\eta\eta_{xxxx}$, $\eta\eta_{xxx}$, $3\eta_{xx}^2$ above. What is required of us is that the coefficients of each of the exponential terms must eventually all cancel out, thus equating to 0 in order to satisfy equation (4.77). Below we will show how each term is cancelled out.
$e^{\theta_1} = -A_1^4 + A_1^4 = 0$

$e^{\theta_2} = -A_2^4 + A_2^4 = 0$

$e^{2\theta_1} = -A_1^4 + A_1^4 + A_1^4 - 4A_1^4 + 3A_1^4 = 0$

$e^{2\theta_2} = -A_2^4 + A_2^4 + A_2^4 - 4A_2^4 + 3A_2^4 = 0$

$$\begin{aligned}e^{\theta_1+\theta_2} &= -A_1^4 - A_2^4 - a(A_1+A_2)(A_1^3+A_2^3) + A_1A_2^3 + A_1^3A_2 + A_1^4 + A_2^4 + a(A_1+A_2)^4\\ &\quad -4A_1A_2^3 - 4A_1^3A_2 + 6A_1^2A_2^2\end{aligned} \tag{6.2}$$

making the substitution $a = \left(\frac{A_1-A_2}{A_1+A_1}\right)^2$ gives the following

$$e^{\theta_1+\theta_2} = -A_1^4 - A_2^4 - \frac{(A_1-A_2)^2}{(A_1+A_2)}(A_1^3+A_2^3) + A_1A_2^3 + A_1^3A_2 + A_1^4 + A_2^4 + (A_1-A_2)^2(A_1+A_2)^2$$

expanding all the brackets gives us the following

$$\begin{aligned}e^{\theta_1+\theta_2} &= \frac{-A_1^3A_2^2 - A_2^5 - A_1^5 - A_1^2A_2^3 + 2A_1^4A_2 + 2A_1A_2^4}{A_1+A_2} + A_1A_2^3 + A_1^3A_2 +\\ &\quad A_1^4 + A_2^4 - 2A_1^2A_2^2 - 4A_1A_2^3\\ &\quad -4A_1^3A_2 + 6A_1^2A_2^2\end{aligned}$$

This can be simplified slightly by collecting the like terms $6A_1^2A_2^2 - 2A_1^2A_2^2$, and then we can multiply throughout by $(A_1 + A_2)$ to remove the fraction

$$\begin{aligned}e^{\theta_1+\theta_2} &= -A_1^3A_2^2 - A_2^5 - A_1^5 - A_1^2A_2^3 + 2A_1^4A_2 + 2A_1A_2^4 + 4A_1^3A_2^2 + 4A_1^2A_2^3 - 3A_1^4A_2\\ &\quad -3A_1^3A_2^2 - 3A_1^2A_2^3 - 3A_1A_2^4 + A_1^5 + A_1^4A_2 + A_1A_2^4 + A_2^5\end{aligned}$$

Now we can collect like terms and everything cancels itself out and so $e^{\theta_1+\theta_2} = 0$.

$$\begin{aligned} e^{2\theta_1+\theta_2} &= a(-A_1^4 - (A_1+A_2)(A_1^3+A_2^3)) + a(A_1(A_1^3+A_2^3) + A_1^3(A_1+A_2)) + \\ &\quad a(A_1^4 + (A_1+A_2)^4) - 4a(A_1(A_1+A_2)^3 + A_1^3(A_1+A_2)) + \\ &\quad 3a(A_1^2(A_1+A_2)^2 + A_1^2(A_1+A_2)^2. \end{aligned}$$

Making the substitution for a gives the following

$$\begin{aligned} e^{2\theta_1+\theta_2} &= -A_1^4(\frac{A_1-A_2}{A_1+A_2})^2 - \frac{(A_1-A_2)^2(A_1^3+A_2^3)}{(A_1+A_2)} + \frac{A_1(A_1-A_2)^2(A_1^3+A_2^3)}{(A_1+A_2)^2} + \\ &\quad \frac{A_1^3(A_1-A_2)^2}{(A_1+A_2)} + A_1^4(\frac{A_1-A_2}{A_1+A_2})^2 + (A_1+A_2)^2(A_1-A_2)^2 - \\ &\quad 4A_1(A_1-A_2)^2(A_1+A_2) - \frac{4A_1^3(A_1-A_2)^2}{(A_1+A_2)} + 6A_1^2(A_1-A_2)^2. \end{aligned}$$

To get rid of the denominator we can multiply throughout by $(A_1+A_2)^2$ which gives us

$$\begin{aligned} e^{2\theta_1+\theta_2} &= -(A_1-A_2)^2(A_1^3+A_2^3)(A_1+A_2) + A_1(A_1-A_2)^2(A_1^3+A_2^3) - \\ &\quad 3A_1^3(A_1-A_2)^2(A_1+A_2) + (A_1+A_2)^4(A_1-A_2)^2 - \\ &\quad 4A_1(A_1-A_2)^2(A_1+A_2)^3 + 6A_1^2(A_1-A_2)^2(A_1+A_2)^2 \end{aligned} \tag{6.4}$$

multiplying out all the brackets and collecting the like terms within them gives the following

$$\begin{aligned} e^{2\theta_1+\theta_2} &= -A_1^6 - A_2^6 - 2A_1^3A_2^3 + A_1^5A_2 + A_1^2A_2^4 + A_1^4A_2^2 + A_1A_2^5 \\ &\quad +A_1^6 + A_1^3A_2^3 - 2A_1^5A_2 - 2A_1^2A_2^4 + A_1^4A_2^2 + A_1A_2^5 \\ &\quad -3A_1^6 + 3A_1^5A_2 + 3A_1^4A_2^2 - 3A_1^3A_2^3 \\ &\quad +A_1^6 + 2A_1^5A_2 - A_1^4A_2^2 - A_1^2A_2^4 + 2A_1A_2^5 + A_2^6 - 4A_1^3A_2^3 \\ &\quad -4A_1^6 - 4A_1^5A_2 + 8A_1^4A_2^2 + 8A_1^3A_2^3 - 4A_1^2A_2^4 - 4A_1A_2^5 \\ &\quad +6A_1^6 + 6A_1^2A_2^4 - 12A_1^4A_2^2 \\ &= 0. \end{aligned} \tag{6.5}$$

Above we have collected the remaining like terms and we eventually obtain 0 as everything cancels out.

$$\begin{aligned} e^{\theta_1+2\theta_2} &= a(-A_2^4 - (A_1+A_2)(A_1^3+A_2^3)) + a(A_2(A_1^3+A_2^3) + A_2^3(A_1+A_2)) + \\ &\quad a(A_2^4 + (A_1+A_2)^4) - 4a(A_2(A_1+A_2)^3 + A_2^3(A_1+A_2)) + \\ &\quad 3a(A_2^2(A_1+A_2)^2 + A_2^2(A_1+A_2)^2 \end{aligned}$$

making the substitution for a gives the following

$$\begin{aligned} e^{\theta_1+2\theta_2} &= -A_2^4(\frac{A_1-A_2}{A_1+A_2})^2 - \frac{(A_1-A_2)^2(A_1^3+A_2^3)}{(A_1+A_2)} + \frac{A_2(A_1-A_2)^2(A_1^3+A_2^3)}{(A_1+A_2)^2} + \\ &\quad \frac{A_2^3(A_1-A_2)^2}{(A_1+A_2)} + A_2^4(\frac{A_1-A_2}{A_1+A_2})^2 + (A_1+A_2)^2(A_1-A_2)^2 - \\ &\quad 4A_2(A_1-A_2)^2(A_1+A_2) - \frac{4A_2^3(A_1-A_2)^2}{(A_1+A_2)} + 6A_2^2(A_1-A_2)^2. \end{aligned}$$

To get rid of the denominator we can multiply throughout by $(A_1 + A_2)^2$ which gives us

$$
\begin{aligned}
e^{\theta_1+2\theta_2} \quad = \quad & -(A_1 - A_2)^2(A_1^3 + A_2^3)(A_1 + A_2) + A_2(A_1 - A_2)^2(A_1^3 + A_2^3) - \\
& 3A_2^3(A_1 - A_2)^2(A_1 + A_2) + (A_1 + A_2)^4(A_1 - A_2)^2 - \\
& 4A_2(A_1 - A_2)^2(A_1 + A_2)^3 + 6A_2^2(A_1 - A_2)^2(A_1 + A_2)^2
\end{aligned} \tag{6.6}
$$

multiplying out all the brackets and collecting the like terms within them gives us the following

$$
\begin{aligned}
e^{\theta_1+2\theta_2} \quad = \quad & -A_1^6 - A_2^6 - 2A_1^3A_2^3 + A_1^5A_2 + A_1^2A_2^4 + A_1^4A_2^2 + A_1A_2^5 \\
& +A_2^6 + A_1^3A_2^3 - 2A_1A_2^5 - 2A_1^4A_2^2 + A_1^5A_2 + A_1^2A_2^4 \\
& -3A_2^6 + 3A_1A_2^5 + 3A_1^2A_2^4 - 3A_1^3A_2^3 \\
& +A_1^6 + 2A_1^5A_2 - A_1^4A_2^2 - A_1^2A_2^4 + 2A_1A_2^5 + A_2^6 - 4A_1^3A_2^3 \\
& -4A_2^6 - 4A_1^5A_2 + 8A_1^2A_2^4 + 8A_1^3A_2^3 - 4A_1^4A_2^2 - 4A_1A_2^5 \\
& +6A_2^6 + 6A_1^4A_2^2 - 12A_1^2A_2^4 \\
= \quad & 0.
\end{aligned} \tag{6.7}
$$

Above we have collected the remaining like terms and we eventually obtain 0 as everything cancels out.

Finally looking at the last term we can collect the coefficients

$$
\begin{aligned}
e^{2\theta_1+2\theta_2} \quad = \quad & -a^2(A_1 + A_2)(A_1^3 + A_2^3) + a^2(A_1 + A_2)(A_1^3 + A_2^3) \\
& +a^2(A_1 + A_2)^4 - 4a^2(A_1 + A_2)^4 + 3a^2(A_1 + A_2)^4 \\
= \quad & 0
\end{aligned} \tag{6.8}
$$

Thus we have shown that (4.77) is satisfied by η.

6.4 Appendix 4

Applying the quotient rule for differentiation gives the following fraction

$$\frac{(-18e^{x-t}-72e^{2x-8t}-18e^{3x-9t})(9+9e^{x-t}+9e^{2x-8t}+e^{3x-9t})-(-18e^{x-t}-36e^{2x-8t}-6e^{3x-9t})(9e^{x-t}+18e^{2x-8t}+3e^{3x-9t})}{(9+9e^{x-t}+9e^{2x-8t}+e^{3x-9t})^2}$$

The brackets can be expanded and simplified and the coefficients of e^{2x-2t}, e^{4x-16t} and e^{6x-18t} are cancelled out. This leaves us with the following solution of u

$$u(x,t) = \frac{-162e^{x-t} - 324e^{3x-9t} - 72e^{4x-10t} - 648e^{2x-8t} - 18e^{5x-17t}}{(9 + 9e^{x-t} + 9e^{2x-9t} + e^{3x-9t})^2}$$

6.5 Appendix 5

The following are the intermediate calculations for the second two-soliton solution. The calculation we need to carry out is written as follows:

$$u(x,t) = -2\left(\frac{49e^{x-t}+122.5e^{2.5x-15.625t}+31.5e^{3.5x-16.625t}}{49+49e^{x-t}+49e^{2.5x-15.625t}+9e^{3.5x-16.625t}}\right)_x$$

To this we can apply the quotient rule for differentiation which gives us

$$\frac{(-98e^{x-t}-612.5e^{2.5x-15.625t}-220.5e^{3.5x-16.625t})(49+49e^{x-t}+49e^{2.5x-15.625t}+9e^{3.5x-16.625t})}{(49+49e^{x-t}+49e^{2.5x-15.625t}+9e^{3.5x-16.625t})^2} -$$

$$\frac{(-98e^{x-t}-245e^{2.5x-15.625t}-63e^{3.5x-16.625t})(49e^{x-t}+122.5e^{2.5x-15.625t}+31.5e^{3.5x-16.625t})}{(49+49e^{x-t}+49e^{2.5x-15.625t}+9e^{3.5x-16.625t})^2}.$$

The brackets above can be expanded, and the terms can be simplified. This give us the following required solution

$$u(x,t) = \frac{2401e^{x-t}+10804.5e^{3.5x-16.625t}+15006.25e^{2.5x-15.625t}+2756.25e^{4.5x-17.625t}+441e^{6x-32.25t}}{(49+49e^{x-t}+49e^{2.5x-15.625t}+9e^{3.5x-16.625t})^2}$$

References

[1] *TutorVista.Com Answers* http://answers.tutorvista.com/768047/how-does-the-crest-and-trough-relate-to-wavelength-and-wave-height-on-a-wave.html Last Accessed 20/08/2014

[2] *Traveling Waves vs. Standing Waves*, http://www.physicsclassroom.com/class/waves/Lesson-4/Traveling-Waves-vs-Standing-Waves Accessed on 15/08/2014

[3] Palais R. S. *An Introduction to Wave Equations and Solitons*, The Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing, Summer 2000, http://www.ma.utexas.edu/users/uhlen/solitons/notes.pdf Accessed on 15/08/2014

[4] *GCSE, Bitesize, Geography, Tsunamis*, http://www.bbc.co.uk/schools/gcsebitesize/geography/natural_hazards/tsunamis_rev1.shtml Last accessed at 16/08/2014

[5] *The 10 most destructive tsunamics in history* http://www.australiangeographic.com.au/topics/science-environment/2011/03/the-10-most-destructive-tsunamis-in-history/ Last accessed at 16/08/2014

[6] *The Fourier Transform* http://www.thefouriertransform.com/transform/fourier.php Last accessed at 16/08/2014

[7] *How Quantum Suicide Works* http://science.howstuffworks.com/innovation/science-questions/quantum-suicide4.htm Last accessed 18/08/2014

[8] *Schrödinger's Equation- what is it?* http://plus.maths.org/content/schrodinger-1 Last accessed 18/08/2014

[9] *Korteweg-de Vries Equation* http://mathworld.wolfram.com/Korteweg-deVriesEquation.html Last accessed 18/08/2014

[10] Curry J.M. *Soliton Solutions of Integrable Systems and Hirota's Method* Massachusetts Institute of Technology '08

[11] Brauer K. *The Korteweg-de Vries Equation: History, exact Solutions, and graphical Representation*, University of Osnabrück/Germany, May 2000

[12] http://www.answers.com/topic/group-velocity Last Accessed on 23/08/2014

[13] Druitt F. *Hirota's Direct Method and Sato's Formalism in Soliton Theory*, The University of Melbourne http://www.ms.unimelb.edu.au/publications/druitt.pdf Last Accessed on 24/08/2014

[14] Soliton http://www.scholarpedia.org/article/Soliton

[15] *Lecture 2: The Korteweg-de Vries Equation* http://math.arizona.edu/~mcl/Miller/MillerLecture02.pdf Last Accessed 25/08/2014

[16] *Conservation Laws for the KdV* http://www.wikiwaves.org/Conservation_Laws_for_the_KdV Last Accessed 25/08/2014

[17] *Lecture 2: The KdV and CH Equations and their soliton solutions; the method of traveling waves* http://u.cs.biu.ac.il/~schiff/Teaching/914/c2.pdf Last Accessed 26/08/2014

[18] *WAVE PACKETS* `http://www.ems.psu.edu/~bannon/Packet.html` Last Accessed 02/09/2014

[19] Professor J Lidsey MTH6129 *6.3 d'Alembert's solution to the one-dimensional wave equation*, Queen Mary, University of London `http://www.maths.qmul.ac.uk/~jel/MTH6129/OWP/notes/MTH6129part3a.pdf` Last Accessed 08/09/2014

[20] *Quantum Vortices* `http://www.physics.umd.edu/courses/Phys726/The_Quantum_Vortex.htm` Last Accessed 08/09/2014

[21] Hietarinta J. *HirotaâĂŹs bilinear method and soliton solutions* Department of Physics, University of Finland `http://cis01.central.ucv.ro/pauc/vol/selected/12.Hietarinta_PAUC15(partI)31-37.pdf`

[22] *Infield E and Rowlands G* Nonlinear waves, solitons and chaos

[23] *Drazin P. G. and Johnson R. S.* Solitons: an introduction, Cambridge Texts in Applied Mathematics