Computing The Matrix Exponential Through Spectral Decomposition

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Abstract

There seems to be a lack of easy to follow material which introduces the matrix exponential without overloading the reader with an unnerving amount of information. In fact there may be 'Nineteen Dubious Ways to Compute the Exponential of a Matrix' [6] however this project will focus on a single way, namely, through the use of spectral decomposition. Working over a closed, complex field, we look at two classes of matrices - diagonalizable and defective - and find an effective method for each. This requires the introduction of new concepts such as generalized eigenvectors and the Jordan Canonical form.

1 What is e?

First we start with a very brief history of the number e itself. It is thought to have naturally arisen through mathematical experiment with compound interest, pre-calculus. In fact it was already referred to in Edward Wright's English translation of John Napier's work on logarithms, published in 1618. The number itself was not denoted as e until Leonhard Euler's work in the first half of the eighteenth century which gave us it's more familiar role in calculus. It is now known as Euler's number $e = 2.71828182845904...$[5]. (If one wishes to learn more about this number they could read "e: The Story of a Number", by Eli Maor).

We move swiftly on to the natural exponential function itself.

Definition 1.1. *The **natural exponential function** exp(x), is defined for all $x \in \mathbb{R}$ as the following Taylor series:*

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{1}$$

where the base e, is Euler's transcendental constant.[8]

The graph of this function is shown below:

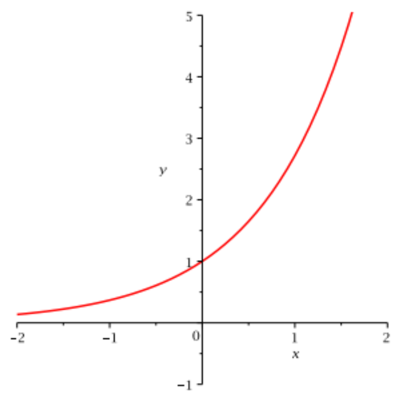

Figure 1: Graph of $y = e^x$

Note: We shall refer to the natural exponential function simply as the exponential function from now onwards.

For completeness we shall now provide properties and applications of the exponential function.

Proposition 1.2. *(Properties of the exponential function).[8]*

1. *The exponential function (1) is continuous with domain $\mathbb{R}$ and range (0,∞).This means that $e^x > 0$ for all x. Consequently we get the limits*

$$\lim_{x\to-\infty} e^x = 0 \ \textit{and} \ \lim_{x\to\infty} e^x = \infty, \tag{2}$$

   hence the x-axis is a horizontal asymptote of the exponential function - one can see this from figure 1.

2. *exp(x) is an inverse to the natural logarithm ln(x).*

3. *The n-th derivative of the exponential function with respect to x, given any $k, x \in \mathbb{R}$ is:*

$$\frac{d^n}{dx^n} e^{kx} = k^n e^{kx}$$

4. *For all $a, b \in (-\infty, \infty)$*

$$e^{a+b} = e^a e^b \, and \; e^{a-b} = \frac{e^a}{e^b}$$

5.

$$(e^a)^b = e^{ab}$$

Proof. Proofs of these properties can be found in Thomas' Calculus textbook.[8] □

The exponential function is an imperative notion with applications ranging from mathematics, statistics, natural sciences, and economics. In general e is the base rate of growth shared by all continually growing processes. It lets you take a simple growth rate (where all change happens at the end of the year) and find the impact of continuously compounded growth. The exponential function is found in all continuously growing systems: population, radioactive decay, interest calculations, and more. It is the exponent x that determines the scale of e by which a process increases.

To illustrate the broad reach of the exponential function, we can use an example from political economist Thomas Malthus. In Layman's terms, Malthus stated that, if left unchecked, the human population would grow exponentially at a rate λ for time t until a natural disaster occurred.[4] This has come to be known as the Malthusian Catastrophe. The initial exponential growth in population can be modelled as:

$$P = P_0 e^{\lambda t}$$

Where, P_0 is the initial population size and P is the population size after time t.

2 Meet the Matrices

Before delving straight into the matrix exponential it is essential that we discuss the two forms of matrices we will be dealing with and how they differ - since each will require different tools for computation of their exponential.

2.1 Diagonalizable Matrix

Traditionally one may have been introduced to the notion of a diagonalizable matrix with the following definition,

Definition 2.1. *A matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *is said to be* ***diagonalizable*** *if and only if there exists an invertible matrix* $\boldsymbol{P}$ *such that*

$$\mathbf{D} = \mathbf{P^{-1}AP} \tag{3}$$

where $\boldsymbol{D}$ *is a diagonal matrix.[3]*

However we shall now give a more applicable definition in terms of the multiplicity of eigenvalues belonging to a matrix.

First we will define the algebraic and geometric multiplicities of a matrix.

Definition 2.2. *Let* $\mathbf{A}$ *be a complex* $n \times n$ *matrix with eigenvalue* λ.

1. *The* ***algebraic multiplicity*** *of* λ *is the number of times it is repeated as root of the characteristic polynomial* $p_\lambda(x)$*.[3] Let us denote the algebraic multiplicity of* λ *as*

$$h_\lambda(\mathbf{A}) = max[h : p_\lambda(x) = (x-\lambda)^h k(x)].$$

2. *the* ***geometric multiplicity*** *of* λ *is the dimensions of the eigenspace of* λ *i.e. the dimensions of the nullspace of* $(A - \lambda I)$*.[3] Let us denote the geometric multiplicity of lambda as*

$$g_\lambda(\mathbf{A}) = dimN(\mathbf{A} - \lambda \boldsymbol{I}).$$

Definition 2.3. *Let* $\mathbf{A}$ *be a complex* $n \times n$ *matrix.* $\mathbf{A}$ *is said to be* ***diagonalizable*** *if and only if each eigenvalue of* $\mathbf{A}$ *has an algebraic multiplicity equal to it's geometric multiplicity.[1]*

[Note that for the following examples we have omitted the explicit calculation of the characteristic polynomials, eigenspaces and eigenvectors. If one would like to familiarize themselves with these notions, one can look at chapter 9, Schaum's Outline of Linear Algebra. [3]]

Example 2.4. *Let*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

Since

$$p_\lambda(x) = (\lambda + 1)(\lambda + 2)$$

Thus, the eigenvalues of $\mathbf{A}$ *are* $\lambda_1 = -1; \lambda_2 = -2$. *Corresponding to eigenspaces*

$$E_{-1} = span \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } E_{-2} = span \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Each eigenvalue has $h_\lambda = g_\lambda = 1$ *which means* $\mathbf{A}$ *is diagonalizable (by definition 2.3).*

Example 2.5. *Let*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix}.$$

we get,

$$p_\lambda(x) = (1 - \lambda)(2 - \lambda)(4 - \lambda)$$

Thus, the eigenvalues of $\mathbf{A}$ *are* $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$.

corresponding to eigenspaces

$$E_1 = span \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} E_2 = span \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \text{ and } E_4 = span \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively. Hence $\mathbf{A}$ *is diagonalizable since each eigenvalue has* $h = g = 1$.

Remark: For simplicity we shall use notation h and g for the respective algebraic and geometric multiplicity of an eigenvalue.

2.2 Defective Matrix

In order to define a non-diagonalizable (defective) matrix in a similar fashion, we must give the following proposition.

Proposition 2.6. *The following is true;*

$$h_\lambda(\mathbf{A}) \geq g_\lambda(\mathbf{A})$$

i.e. the algebraic multiplicity of λ is at least as large as it's geometric multiplicity. [1]

Proof. A proof of this result can be found in chapter 8 of the Advanced Linear Algebra textbook. [7] □

Definition 2.7. *Let $\mathbf{A}$ be a complex $n \times n$ matrix. $\mathbf{A}$ is said to be **non-diagonalizable** if there exists at least one eigenvalue λ for which*

$$h_\lambda(\mathbf{A}) > g_\lambda(\mathbf{A})$$

i.e. there is at least one eigenvalue with an algebraic multiplicity greater than its geometric multiplicity.[1]

Example 2.8. *Let*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We have,

$$p_\lambda(x) = (1-\lambda)(1-\lambda).$$

Thus, there is only one distinct eigenvalue of $\mathbf{A}$, namely, $\lambda_1 = 1$ with algebraic multiplicity $h = 2$ and geometric multiplicity $g = dimE_1 = 1$. Now since $h > g$, $\mathbf{A}$ is non-diagonalizable (by definition 2.7).

Example 2.9.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

where

$$p_\lambda(x) = (1-\lambda)(2-\lambda)^2.$$

The characteristic polynomial gives us eigenvalue $\lambda_1 = 1$ and defective eigenvalue $\lambda_2 = 2$ with $h = 2$ and $g = dimE_2 = 1$ therefore $h > g$ and $\mathbf{A}$ is non-diagonalizable.

Example 2.10.

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix},$$

where

$$p_\lambda(x) = (\lambda - 3)^4.$$

Hence our only eigenvalue is $\lambda = 3$ *with* $h = 3 > g = 2$ *hence* $\mathbf{A}$ *is non-diagonalizable.*

3 The Matrix Exponential

Now we arrive at the matrix exponential itself. The exponential function of a matrix is analogous to the ordinary exponential function, the difference simply being that the exponent is instead a square matrix instead of a real number.

Definition 3.1. *Given a square matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, the exponential of* $\boldsymbol{A}$*, denoted* $e^{\mathbf{A}t}$*, is the* $n \times n$ *matrix given by the power series,*

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!},$$

where t *is a constant.[1]*

Proposition 3.2. *(Properties of the matrix exponential function)[2]*

Given that $\mathbf{A}$ *and* $\mathbf{B}$ *are square matrices,* $\mathbf{P}$ *is a non-singular matrix and* t *is a real number, we have:*

1. *If* $\mathbf{AB} = \mathbf{BA}$ *then* $e^{\mathbf{A}t}e^{\mathbf{B}t} = e^{(\mathbf{A}+\mathbf{B})t}$*;*
2. *If* $\mathbf{AB} = \mathbf{BA}$ *then* $e^{\mathbf{A}t}\mathbf{B} = \mathbf{B}e^{\mathbf{A}t}$*;*
3. $(e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$*;*
4. $e^{\mathbf{PAP^{-1}}t} = \mathbf{P}e^{\mathbf{A}t}\mathbf{P^{-1}}$*;*

Proof. The proofs of these properties can be found in *Hall (2003)*. [2] □

Definition 3.3. *If* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *has n distinct eigenvalues, each with the property that* $h_\lambda(\mathbf{A}) = g_\lambda(\mathbf{A})$ *then we let matrix* $\boldsymbol{S}$ *form the basis of the linearly independent eigenvectors of* $\boldsymbol{A}$*.[3]*

$$\mathbf{S} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix},$$

i.e. $\boldsymbol{S}$ *has columns constructed from the eigenvectors of* $\boldsymbol{A}$*.*

Proposition 3.4. *Given a diagonalizable matrix* $\mathbf{A} \in \mathbb{C}^{n\times n}$*, with its basis of linearly independent eigenvectors* $\mathbf{S}$*, then*

$$(\mathbf{S^{-1}AS})^n = \mathbf{S^{-1}A^nS},$$

holds true for all $n \in \mathbb{R}$*.[2]*

Proof.

$$\begin{aligned}(\mathbf{S^{-1}AS})^n =&(\mathbf{S^{-1}AS})(\mathbf{S^{-1}AS})\cdots(\mathbf{S^{-1}AS})\\ =&\mathbf{S^{-1}A(SS^{-1})A(SS^{-1})}\cdots(\mathbf{SS^{-1}})\mathbf{AS}\\ =&\mathbf{S^{-1}A(I)A(I)}\cdots(\mathbf{I})\mathbf{AS}\\ =&\mathbf{S^{-1}A^nS}.\end{aligned}$$

□

The matrix exponential is most applicable as the solution to initial value problems in differential equations. Namely, given

$$\vec{x}'(t) = \mathbf{A}\vec{x}(t),$$

where $\mathbf{A}$ is a known, fixed, matrix in $\mathbb{C}^{n\times n}$. One seeks the solution vector which satisfies the initial condition

$$\vec{x}(0) = \vec{x}_0.$$

This solution can be obtained as

$$\vec{x}(t) = e^{\mathbf{A}t}\vec{x}_0,$$

where $e^{\mathbf{A}t}$ can be defined as the power series in definition 3.1.[6]

Remark: This project will not focus on the application of the matrix exponential but the computation only. However we compute the matrix exponential with constant t, so that this project can be applied, by the reader, to differential equations if needed.

4 Solving the Matrix Exponential

We shall split the following section into the diagonalizable and defective cases, providing a concise method to solving the matrix exponential for each case, and finally showing how they are intertwined.

4.1 Case 1: Diagonalizable Matrix

The simplest case is when matrix $\mathbf{A}$ is already diagonal. One can compute the exponential of an arbitrary diagonal matrix as shown below:

Proposition 4.1. *Given a square $n \times n$ matrix in the complex field,*

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

the exponential of $\mathbf{A}$ is computed as follows:

$$\begin{aligned} e^{\mathbf{A}t} &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k t^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\lambda_2^k t^k}{k!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{\lambda_n^k t^k}{k!} \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}. \end{aligned}$$

Proof. Directly from definition 3.1. $\square$

However $\mathbf{A}$ may not be diagonal but diagonalizable. To compute the exponential of such a matrix one can use the following theorem.

Theorem 4.2. *Given a diagonalizable matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$. The matrix exponential of $\mathbf{A}$ is given by*

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{D}t}\mathbf{S}^{-1}. \tag{4}$$

where $\mathbf{D}$ is a diagonal matrix similar to $\mathbf{A}$ and $\mathbf{S}$ is a basis of $\mathbf{A}$.[6]

Proof. If $\mathbf{A}$ is diagonalizable then by definition 2.3 all eigenvalues λ have algebraic multiplicities equal to their geometric multiplicities which means there corresponds n linearly independent eigenvectors $\vec{v}_i$, for $i = 1 \cdots n$. Hence we can form a basis $\mathbf{S}$ and find diagonal matrix $\mathbf{D}$ through spectral decomposition to be

$$\mathbf{D} = \mathbf{S^{-1}AS} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{5}$$

where $\lambda_1, \cdots \lambda_n$ are constants.

Now as with any arbitrary diagonal matrix we can calculate the exponential of matrix $\mathbf{D}$ as given in proposition 4.1:

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}.$$

For the next step one needs to remember the following fact:

Using Definition 3.1 with Proposition 3.4 we can show

$$e^{\mathbf{D}t} = \sum_{n=0}^{\infty} \frac{t^n(\mathbf{S^{-1}AS})^n}{n!} = \mathbf{S}^{-1} \sum_{n=0}^{\infty} \frac{t^n \mathbf{A}^n}{n!} \mathbf{S} = \mathbf{S}^{-1} e^{\mathbf{A}t} \mathbf{S}$$

rearranging this we finally get

$$e^{\mathbf{A}t} = \mathbf{S} e^{\mathbf{D}t} \mathbf{S}^{-1}.$$

□

We will illustrate this result using the earlier examples stated in section 2.1.

Example 4.3. *Given matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

with eigenvectors

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Hence our basis and its inverse are

$$\mathbf{S} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \mathbf{S}^{-1} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$$

Now diagonalising $\mathbf{A}$*, as shown in equation 3, we arrive at*

$$D = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Next we use the matrix exponential function on diagonal matrix $\mathbf{D}$ *to get*

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

Finally using equation 4 we find the matrix exponential of $\mathbf{A}$ *to be*

$$\begin{aligned} e^{\mathbf{A}t} &= \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}. \end{aligned}$$

Example 4.4. *Given matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix}$$

with eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence our basis and its inverse are:

$$\mathbf{S} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{S}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}.$$

Now we find diagonal matrix $\mathbf{D}$ *such that*

$$\mathbf{D} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Using the matrix exponential function on $\mathbf{D}$ *we get*

$$e^{\mathbf{D}t} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix}$$

Finally we find the matrix exponential of $\mathbf{A}$ *to be*

$$e^{\mathbf{A}t} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} - e^t & 0 \\ 0 & e^{2t} & 0 \\ 0 & \frac{e^{2t}-e^{4t}}{2} & e^{4t} \end{pmatrix}$$

4.2 Case 2: Non-diagonalizable Matrices

As we can see in the previous section, when a matrix is diagonalizable there exists a basis of linearly independent eigenvectors. However when a given matrix is non-diagonalizable, by definition, there is an insufficient quantity of linearly independent eigenvectors to form a basis. In this case we must find ‘generalized eigenvectors’ to complete the basis.

Definition 4.5. *Given a complex* $n \times n$ *matrix* $\mathbf{A}$*, the vector* $\vec{u}$ *is a* ***generalized eigenvector of rank m*** *corresponding to eigenvalue* λ *if it satisfies*

$$(\mathbf{A} - \lambda \boldsymbol{I})^m \vec{u} = 0, \tag{6}$$

but

$$(\mathbf{A} - \lambda \boldsymbol{I})^{m-1} \vec{u} \neq 0,$$

where $m \in \mathbb{Z}^+$*, and the* ***generalized eigenspace of rank m*** *is the null space of* $(\mathbf{A} - \lambda \boldsymbol{I})^m$*, denoted by* K_λ^m*.[1]*

Key Lemma 4.6. *One can denote the basis of eigenvectors and basis completed by generalized eigenvectors, both by* $\mathbf{S}$*.*

Proof. By definition 4.5 if we set m equal to 1 then the eigenvectors of $\mathbf{A}$ are also generalized eigenvectors. By this fact $\mathbf{S}$ is the basis of generalized eigenvectors for both diagonalizable and non-diagonalizable matrices. $\square$

For simplicity - and in order to produce a replicable method for calculating the exponential of a defective matrix - we do not want our choice of generalized eigenvectors to be an arbitrary matter. We want to choose generalized eigenvectors that will enable us to write $\mathbf{A}$ in a specific form. As we already know, defective matrices do not have an equivalent diagonal representation however we can find the next best canonical form, namely the Jordan canonical form.

Before defining the Jordan canonical form, we must introduce the concept of a nilpotent matrix.

Definition 4.7. *A matrix $\mathbf{A}$ for which $\mathbf{A}^\rho = \mathbf{0}$, where ρ is a positive integer, is called* ***nilpotent****. If ρ is the least positive integer for which $\mathbf{A}^\rho = \mathbf{0}$, then $\mathbf{A}$ is said to be nilpotent of* ***index*** *ρ.[3]*

Proposition 4.8. *If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is a strictly upper triangular matrix, then it is nilpotent of index n (corresponding to its dimension).[3]*

Proof. The characteristic polynomial of any $n \times n$ strictly upper triangular matrix $\mathbf{A}$ is $p_\lambda(x) = \lambda^n$ therefore by the Cayley-Hamilton Theorem we get $\mathbf{A}^n = \mathbf{0}$ □

Definition 4.7 and proposition 4.8 will not only help us define the Jordan canonical form but also help in computing its exponential.

Definition 4.9. *An $n \times n$ matrix $\mathbf{J}$ is in* ***Jordan Canonical Form*** *if it is a block diagonal matrix such that*

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}_k \end{pmatrix}$$

where each $\mathbf{J}_i$ is a ***Jordan Block*** *of form*

$$\mathbf{J}_i = \lambda_i \boldsymbol{I} + \mathbf{N} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \ddots & 0 & 0 \\ 0 & 0 & \lambda_i & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}$$

where each λ_i is the only eigenvalue of each $\mathbf{J}_i$, $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{N}$ is a nilpotent matrix with ones on the super-diagonal and zeros everywhere else.[1]

Theorem 4.10. *Given any $n \times n$ complex matrix $\mathbf{A}$. There exists an $n \times n$ basis $\mathbf{S}$ of n linearly independent generalized eigenvectors, such that*

$$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

where $\mathbf{J}$ is in Jordan Canonical form of $\mathbf{A}$.[1]

Proof. Detailed proof can be found in chapter 7.1 of *Friedberg et al. (2003).*[1] □

The following is a method to compute such a basis $\mathbf{S}$ from 4.10 that has been simplified and adapted, for this project, from the works of *Friedberg et al. (2003).*[1]

Algorithm 4.11. *Given a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, using the following steps one can create an ordered basis $\mathbf{S}$ of specific generalized eigenvectors, that will transform $\mathbf{A}$ to it's associated Jordan canonical form $\mathbf{J}$.*

Step 1

Calculate the distinct eigenvalues λ_i of $\mathbf{A}$ and each of their corresponding algebraic and geometric multiplicities, h_{λ_i} and g_{λ_i} respectively.

Step 2

For each eigenvalue λ_i with algebraic and geometric multiplicities h_{λ_i} and g_{λ_i} respectively, calculate the generalized eigenspaces and their corresponding dimensions until we get to $K^j_{\lambda_i}$ which has dimension equal to the algebraic multiplicity of λ_i:

$$dim K^j_{\lambda_i} = h_{\lambda_i}.$$

Step 3

At this step we introduce new notation in the form of a dot plot to help visualise how we calculate our generalized eigenvectors. To start we calculate:

$$
\begin{aligned}
r_1 =& dim K_{\lambda_i}^1 \\
r_2 =& dim K_{\lambda_i}^2 - dim K_{\lambda_i}^1 \\
&\vdots \\
r_j =& dim K_{\lambda_i}^j - dim K_{\lambda_i}^{j-1}
\end{aligned}
$$

Now we make a plot of r_1 dots in row 1, r_2 dots in row 2, and so on until r_j dots in the j-th row.

We now want find a generalized eigenvector corresponding to each dot.

Step 4

Working from the bottom of the plot, for each dot in row j, find linearly independent generalized eigenvectors belonging to $K_{\lambda_i}^j$ but not $K_{\lambda_i}^{j-1}$.

Everytime a dot has a corresponding generalized eigenvector $\vec{u}$, the dot directly above corresponds to $(\mathbf{A} - \lambda_i \boldsymbol{I})\vec{u}$.

All dots must have a corresponding generalized eigenvector which is not in the generalized eigenspace of the row above, and is linearly independent to all other generalized eigenvectors in its row.

Step 5

We now form a matrix $\mathbf{P}_i$ corresponding to the the dot plot of λ_i in the following way.

Starting at the top of the first column on the left and working downwards, each generalized eigenvector becomes a column of $\mathbf{P}_i$. After we reach the bottom of a column we move on to the next until all generalized eigenvectors in each dot plot have been exhausted.

Step 6

We can now form our basis $\mathbf{S}$ of generalized eigenvectors using our matrices $\mathbf{P}_i$ from above. Namely the k columns of $\mathbf{P}_1$ become the first k columns of $\mathbf{S}$ and so on until all $\mathbf{P}_i$ are exhausted.

Step 7

Now that we have our completed basis $\mathbf{S}$ we can find the Jordan Canonical

Form of matrix $\mathbf{A}$ *as follows*

$$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}.$$

Now we know how to find the Jordan Canonical form of a matrix, we can introduce key concepts which will help us find the exponential of $\mathbf{J}$ and therefore the exponential of our initial non-diagonalizable matrix.

Definition 4.12. *Let $\mathbf{A_1}, \mathbf{A_2}, \cdots, \mathbf{A_m}$ be square matrices of respective order $n_1, n_2, \cdots, n_m$. The block diagonal matrix $\mathbf{A}$ is called the **direct sum** of all $\mathbf{A_i}$, for $i = 1, 2, \cdots, m$.[3]*

We denote this as

$$\mathbf{A} = \bigoplus_{i=1}^{m} \mathbf{A_i} = diag(\mathbf{A_1}, \mathbf{A_1}, \cdots, \mathbf{A_m}) = \begin{pmatrix} \mathbf{A_1} & 0 & \cdots & 0 \\ 0 & \mathbf{A_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A_m} \end{pmatrix}.$$

Key Lemma 4.13. *The Jordan Canonical matrix $\mathbf{J}$, is the direct sum of each Jordan Block $\mathbf{J}_{\lambda_i}$. Therefore $e^{\mathbf{J}}$ is the direct sum of each $e^{\mathbf{J}_{\lambda_i}}$.*

Proof. By definition 4.12. □

Proposition 4.14. *The exponential of a Jordan block can be written as*

$$e^{J_{\lambda_i}t} = e^{\lambda_i \mathbf{I}} \sum_{k=1}^{n-1} \frac{\mathbf{N}^k t^k}{k!}.$$

Proof. By definition 4.9 we have that,

$$e^{J_{\lambda_i}t} = e^{(\lambda_i \mathbf{I} + \mathbf{N})t}.$$

We know that all square matrices commute with the identity matrix i.e.

$$(\lambda_i \mathbf{I})\mathbf{N} = \lambda_i \mathbf{I}\mathbf{N} = \lambda_i(\mathbf{I}\mathbf{N}) = \lambda_i(\mathbf{N}\mathbf{I}) = \lambda_i \mathbf{N}\mathbf{I} = (\lambda_i \mathbf{N})\mathbf{I}.$$

Hence, by proposition 3.2,

$$e^{(\lambda_i \mathbf{I} + \mathbf{N})t} = e^{\lambda_i \mathbf{I} t} e^{\mathbf{N}t}.$$

Now, since $\lambda_i \mathbf{I}$ is a diagonal matrix, $e^{\lambda_i \mathbf{I}}$ will simply follow from proposition 4.1. The $n \times n$ matrix $\mathbf{N}$ is neither diagonal nor diagonalizable, however it is nilpotent and we know, by proposition 4.8, that $\mathbf{N}^\rho = \mathbf{0}$ when $\rho = n$. This means that $e^{\mathbf{N}t}$ can be written as the power series in definition 3.1, but with finite upper limit $n-1$,

$$e^{\mathbf{N}t} = \sum_{k=0}^{n-1} \frac{\mathbf{N}^k t^k}{k!}$$

this is because every entry in the infinite power series, with $k \geq n$, will equal 0. Finally, we have our desired result,

$$e^{J_{\lambda_i}t} = e^{\lambda_i \mathbf{I}} \sum_{k=0}^{n-1} \frac{\mathbf{N}^k t^k}{k!}.$$

□

We now hold necessary information to define the exponential of a non-diagonalizable matrix.

Theorem 4.15. *Given a non-diagonalizable matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, the exponential of $\mathbf{A}$ is calculated as,*

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}.$$

where $\mathbf{J}$ is the Jordan Canonical matrix similar to $\mathbf{A}$ and $\mathbf{S}$ is the basis formed from linearly independent generalized eigenvectors of $\mathbf{A}$.[6]

Proof. The proof is analogous with the diagonalizable case, except now we have a Jordan Canonical matrix $\mathbf{J}$ instead of a diagonal matrix $\mathbf{D}$.

$$e^{\mathbf{A}t} = e^{\mathbf{SJS}^{-1}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{SJS^{-1}})^k t^k}{k!} = \mathbf{S}\sum_{k=0}^{\infty} \frac{\mathbf{J}^k t^k}{k!}\mathbf{S^{-1}} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}$$

□

We shall culminate all of these new concepts and illustrate them in the following examples from section 2.2.

Example 4.16. *Given*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The first thing we want to do is find the matrix $\mathbf{J}$ similar to $\mathbf{A}$ using algorithm 4.11.

Step 1: We have our only eigenvalue $\lambda_1 = 1$ with $h = 2$ and $g = 1$.

Step 2: Now we find our generalized eigenspaces and compute their dimensions.

Solving $(\mathbf{A} - \mathbf{I})\vec{v} = 0$ gives

$$K_1^1 = span \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

therefore $dim K_1^1 = 1$.

Solving $(\mathbf{A} - \mathbf{I})^2\vec{u} = 0$ gives

$$K_1^2 = span\{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\},$$

So $dim K_1^2 = 2 = h_\lambda = 2$ therefore we stop.

Step 3: We now create our dotplot by finding

$$r_1 = dim k_1^1 = 1 and\ r_2 = dim K_1^2 - dim k_1^1 = 2 - 1 = 1.$$

Hence our dot plot has a single column, with two rows containing one dot each,

$$\begin{matrix} \bullet \\ \bullet \end{matrix}$$

Step 4: Now we assign a generalized eigenvector to the bottom dot. It must be in K_1^2 but not K_1^1. It is clear to see that,

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

complies with this rule.

Now we find the eigenvector $(\mathbf{A} - \mathbf{I})\vec{u}_1$ corresponding to our top dot.

$$\vec{u}_2 = \vec{v}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 5: Hence our dot plot now looks like

$$\begin{matrix} \bullet\, \vec{v}_1 \\ \bullet\, \vec{u}_1 \end{matrix}$$

and so we form our matrix $\mathbf{P}_1$ *as*

$$\mathbf{P}_1 = [\vec{v}_1 \ \vec{u}_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Step 6: Since we only had 1 eignevalue our basis $\mathbf{S} = \mathbf{P}_1$.

Step 7: We now find our Jordan Canonical matrix $\mathbf{J}$ *similar to* $\mathbf{A}$ *through spectral decomposition as follows,*

$$\mathbf{J} = \mathbf{S^{-1}AS} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \mathbf{I} + \mathbf{N},$$

where $\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Now that we have $\mathbf{J}$ *we find its matrix exponential to be*

$$e^{\mathbf{J}t} = e^{\mathbf{I}t} \sum_{k=0}^{1} \frac{\mathbf{N}^k t^k}{k!} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} (\mathbf{I} + \mathbf{N}t) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$$

and finally we have

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1} = \begin{pmatrix} e^t & 0 \\ te^t & e^t \end{pmatrix}.$$

Example 4.17. *Given*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

where

$$p(\lambda) = (1 - \lambda)(2 - \lambda)^2.$$

We have simple eigenvalue $\lambda_1 = 1$ *with h=g=1 and defective eigenvalue* $\lambda_2 = 2$ *with h=2, g=1.*

Dealing with $\lambda_1 = 1$ *first, we can skip straight to finding its corresponding eigenvalue and then set it equal to* $\mathbf{P}_1$. *Solving* $(\mathbf{A} - \mathbf{I})\vec{v} = 0$ *we get,*

$$\vec{v}_1 = \mathbf{P}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Next we deal with our degenerate eigenvalue $\lambda_2 = 2$.

Solving $(\mathbf{A} - 2\mathbf{I})\vec{v} = 0$ *gives us,*

$$K_2^1 = span \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

therefore $dim K_2^1 = 1$.

Solving $(\mathbf{A} - 2\mathbf{I})^2\vec{u} = 0$ *gives us,*

$$K_2^2 = span\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \},$$

therefore $dim K_2^2 = 2$, *so we stop.*

As in the previous example, we find $r_1 = 1 and\ r_2 = 1$. *Hence our dot plot is identically,*

$$\begin{matrix} \bullet \\ \bullet \end{matrix}$$

Now we seek a generalized eigenvector in K_2^2 *that is linearly independent to* K_2^1, *namely* $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. *Consequently we find* $\vec{u}_2 = (\mathbf{A} - 2\mathbf{I})\vec{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. *Hence we have*

$$\mathbf{P}_2 = [\vec{u}_2\ \vec{u}_1] = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which completes our basis,

$$\mathbf{S} = [\mathbf{P}_1\ \mathbf{P}_2] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Computing our Jordan Canonical Matrix gives us,

$$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{J}_1 \bigoplus \mathbf{J}_2,$$

where our first Jordan block is the 1×1 matrix $\mathbf{J}_1 = \begin{pmatrix} 1 \end{pmatrix}$ and our second Jordan block is the 2×2 matrix $\mathbf{J}_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

By lemma 4.13 we know that the exponential of our Jordan canonical matrix $\mathbf{J}$ is the direct sum of the exponential of each Jordan block $\mathbf{J}_1$ and $\mathbf{J}_2$.

Since $\mathbf{J}_1$ is a diagonal matrix we can directly compute its exponential as

$$e^{\mathbf{J}_1 t} = \begin{pmatrix} e^t \end{pmatrix}.$$

Next we rewrite our second Jordan block as

$$\mathbf{J}_2 = \lambda_2 \mathbf{I} + \mathbf{N} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

therefore

$$e^{\mathbf{J}_2} = e^{(2\mathbf{I}+\mathbf{N})t} = e^{2\mathbf{I}t} e^{\mathbf{N}t} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

By direct sum, we have

$$e^{\mathbf{J}} = e^{\mathbf{J}_1} \bigoplus e^{\mathbf{J}_2} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}.$$

Finally, we can conclude that the exponential of $\mathbf{A}$ is,

$$e^{\mathbf{A}t} = \mathbf{S} e^{\mathbf{J}t} \mathbf{S}^{-1} = \begin{pmatrix} e^t & e^{2t} - e^t & 0 \\ 0 & e^{2t} & 0 \\ 0 & te^{2t} & e^{2t} \end{pmatrix}.$$

N.B. *If we had constructed our basis as $\mathbf{S} = [\mathbf{P}_1\ \mathbf{P}_2]$, we would have had an alternative Jordan Canonical Matrix where $\mathbf{J}_1$ and $\mathbf{J}_2$ had swapped positions on the diagonal, namely,*

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

However our $e^{\mathbf{A}t}$ would still have remained the same.

For simple matrices we may not need our dot plot to visualize the generalized eigenvectors but it will be of more use when dealing with matrices of higher dimensions such as our next example.

Example 4.18. *Given*

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix},$$

where

$$p(\lambda) = (\lambda - 3)^4.$$

We have defective eigenvalue $\lambda = 3$ *with* $h = 4$ *and* $g = 2$.

Solving $(\mathbf{A} - 3\mathbf{I})\vec{v} = 0$ *we get,*

$$K_3^1 = span\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\},$$

hence $dim K_3^1 = 2$.

Next solving $(\mathbf{A} - 3\mathbf{I})^2\vec{u} = 0$ *yields,*

$$K_3^2 = span\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\},$$

therefore $dim K_3^2 = h = 4$, *so we can stop.*

We find $r_1 = r_2 = 2$ *therefore our dot plot has two columns of two dots,*

$$\begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$$

Now starting with the bottom left dot we find a corresponding generalized eigenvector in K_3^2 *but not in* K_3^1. *It is easy to see that* $\vec{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ *is suitable.*

Consequently we find $\vec{u}_2 = (\mathbf{A} - 3\mathbf{I})\vec{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Now moving on to our second column, we need to find, for the bottom right dot, a generalized eigenvector in K_3^2, but not in K_3^1 and linearly independent to $\vec{u}_1$. Once again this is easily found to be $\vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. So our last eigenvalue is therefore $\vec{u}_4 = (\mathbf{A} - 3\mathbf{I})\vec{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

We can now redraw our dot plot as,

$$\begin{array}{cc} \bullet\, \vec{u}_2 & \bullet\, \vec{u}_4 \\ \bullet\, \vec{u}_1 & \bullet\, \vec{u}_3 \end{array}$$

which means our basis is,

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Our Jordan canonical matrix can now be computed as:

$$\mathbf{J} = \mathbf{S^{-1}AS} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \mathbf{J}_1 \bigoplus \mathbf{J}_2,$$

where our Jordan blocks are $\mathbf{J}_1 = \mathbf{J}_2 = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

The next step is to compute the exponential of each Jordan block,

$$\begin{aligned} e^{\mathbf{J}_1 t} = e^{\mathbf{J}_2 t} =& e^{3\mathbf{I}t}\left(\sum_{k=0}^{1} \frac{\mathbf{N}^k t^k}{k!}\right) \\ =& \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \\ =& \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix} \end{aligned}$$

therefore

$$e^{\mathbf{J}t} = \begin{pmatrix} e^{3t} & te^{3t} & 0 & 0 \\ 0 & e^{3t} & 0 & 0 \\ 0 & 0 & e^{3t} & te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{pmatrix}.$$

Finally, we compute the exponential of our defective matrix as,

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1} = \begin{pmatrix} e^{3t} & 0 & 0 & 0 \\ 0 & e^{3t} & te^{3t} & 0 \\ 0 & 0 & e^{3t} & 0 \\ te^{3t} & 0 & 0 & e^{3t} \end{pmatrix}.$$

Remarks:

1. Algorithm 4.11, and the Jordan canonical form as a whole, are not unique to defective matrices. In fact, diagonalizable matrices too have a Jordan Conical form, where each diagonal entry represents a 1×1 Jordan block. To illustrate this, look back to example 4.4. The matrix $\mathbf{A}$ is similar to the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

   which is actually a Jordan canonical matrix and the direct sum of 3 Jordan blocks of dimension 1: $\mathbf{J}_1 = (1)$, $\mathbf{J}_2 = (2)$ and $\mathbf{J}_3 = (4)$.

2. We don't have to explicitly compute $\mathbf{J} = \mathbf{S^{-1}AS}$ to find $\mathbf{J}$. Instead we use algorithm 4.11 intuitively as follows. The number of blocks in $\mathbf{J}$ will equal the total number of columns in all our dot plots. The diagonal entries of each block are given by the corresponding eigenvalue, and the dimension of the block is equal to the number of dots in the equivalent column. The blocks are then ordered along the diagonal of $\mathbf{J}$ in the same way that we ordered our basis $\mathbf{S}$. To illustrate, take our last example. Here we see that the dot plot has two columns both containing two dots, hence our Jordan canonical matrix has two blocks both of dimension two. We can also see in this example that the number of blocks corresponding to a single eigenvalue is equal to the geometric multiplicity of said eigenvalue.

5 conclusion

In this project we have determined a method by which to solve the exponential of diagonalizable matrices and by introducing the concept of generalized eigenvectors and Jordan canonical form, we have been able to adapt this method for defective matrices. Specifically we have shown that there is an intuitive way to find non-arbitrary generalized eigenvectors and order them correctly into a basis, so that any square matrix, on a complex field, can be transformed into its Jordan canonical form.

Future work could lead down two paths, namely exploring the applications of the matrix exponential in more detail. Or expanding upon the matrix exponential itself. An aim of this project was to focus on the matrix exponential itself instead of introducing it as a concept in differential equations, as much literature does. For that reason I would most likely explore the second route and introduce concepts such as functions of the matrix exponential e.g. the matrix-matrix exponential. I would also look into finding methods that wouldn't be restricted to matrices on a closed, complex field e.g. using the rational canonical forms of matrices.

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