The Kuramoto Model: A Paradigm Of Synchronisation

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Abstract

The Kuramoto Model is a mathematical model which describes synchronisation. It is a model for the behaviour of a large set of oscillators. Order emerging spontaneously is becoming increasingly popular amongst scientists. The phenomenon itself incurred a co-alignment amongst the varied fields of study only recently when it started to emerge that scientists were looking at the same thing. This project takes the Kuramoto model for a large set of oscillators but seeks specifically to analyse the stability of synchronised states in the two and three oscillator cases.

Contents

1. Introduction

Synchrony is simultaneous occurrence. The phenomenon of synchronisation surrounds us everywhere. Such spectacles can be seen most beautifully in nature. Observing how collections of animals or other living organisms behave and react in their natural environments demonstrates why the phenomenon has an increasing charm on research today.

With animals, there are typical behaviours when they function as a group. For instance some schools of fish stick together and swim in line with each other, and move so similarly it looks as though each fish moves exactly like their neighbour. They're all in phase with one another. Swarms of bees also move collectively in a similar pattern.

A very classic example of phase synchronisation or spontaneous order is of a phenomenon at the river banks in Southeast Asia (also in some other parts of the world). Here tropical fireflies are seen to flash perfectly in phase with one another. There have been a few theories as to why this occurs, such as male fireflies trying to attract female fireflies by being the first to flash; consequently what happens is that they all flash at the same time in phase with one another. In the early 20thcentury Phillip Laurent thought he'd solved the enigma "*the apparent phenomenon was caused by the twitching or sudden lowering and raising of my eyelids. The insects had nothing whatsoever to do with it*" [1]. Today we know better.

The popular science has widespread relevance to many if not all subject areas including biology, physics, astronomy, mathematics, engineering, social sciences, ecology etc. Synchrony can be seen in many ways; it has applications in neuroscience when looking at the way in which neurons behave, in the cells of pacemakers, it is even present in the light of a laser beam which has many atoms emitting light waves in synchrony, without which we wouldn't have laser eye surgery.

Taking things closer to home with us humans, synchrony can also be observed with the way in which we commute, there are 'rush hours' where everyone seems to commute at the same time during the same times of day. Our sleeping patterns seem to have synchrony to them as do the menstrual cycles of women. When the Millennium Bridge first opened in London it would resonate as pedestrians walked across it, consequently through the unconscious efforts of these people to steady themselves on the bridge, they all stepped in phase with one another in their efforts to get to the other side.

There are many examples to exhibit the phenomenon of synchronisation [2], but one may ask how does this relate to mathematics? How can one convert what they see with their eyes and describe it in terms of mathematical symbols?

The Kuramoto Model describes each oscillator, the oscillator being the body in question, be it a firefly, a person, a light wave etc. The oscillator $\boldsymbol{\theta}$ depends linearly on time and has its own natural frequency denoted by $\boldsymbol{\omega}$ which is coupled equally by a coupling constant $\boldsymbol{k}$ to all other oscillators.

$$\dot{\boldsymbol{\theta}}_i = \boldsymbol{\omega}_i + \boldsymbol{k} \sum_j \boldsymbol{sin}(\boldsymbol{\theta}_i - \boldsymbol{\theta}_j) \qquad \textbf{(1.1)}$$

The above equation is the general equation for the models I study in my project. The coupling constant is attached to a trigonometric function giving the phase relationship between the oscillators. What I am trying to achieve is finding the conditions for which each oscillator can synchronise with the oscillators and analyse the stability of solutions to determine when $\boldsymbol{\theta}_i = \boldsymbol{\theta}_j$ i.e. when the oscillators are at the same position at the same time moving at the same speed; in phase with one another.

Applying the equation to an instance, we can look at a group of joggers, represented by some theta, they each have their own natural jogging pace, and this would be the omega. The phase difference between each jogger in relation to another would be the difference in velocities between them. The coupling strength would denote the ability for them to synchronise. So some may be faster or slower than others, and they look around to see whether they need to slow down or speed up to jog with the others. If the joggers' speeds are too different then they won't synchronise as their own natural speeds would be too diverse for this to happen, if however they do synchronise then the coupling strength is strong enough for this to occur.

What I researched in my project models the above general equation for two and three oscillators, and analysed under what conditions they can synchronise and the stability of the solutions for this. The Kuramoto Model itself looks at a large number of N oscillators so equation (1.1) would be different in that the coupling constant $\boldsymbol{k}$ would instead be $\frac{\boldsymbol{k}}{\boldsymbol{N}}$ [3]. A variation of the model can also be used to study the cases for an N with an infinite limit.

2. Two Oscillator Model

2.1. Description of the model

Suppose there is an oscillator with a linearly time dependent angle $\boldsymbol{\theta_1(t)}$, with omega, $\boldsymbol{\omega}$ its *angular frequency* and $\boldsymbol{\theta_1(t) = \omega_1 t}$ where $\boldsymbol{\omega} \in \mathbb{R}$. The rate of change of the angle at which it turns would be $\dot{\boldsymbol{\theta}}_1 = \boldsymbol{\omega_1}$. Now suppose there is another oscillator with angle $\boldsymbol{\theta_2(t) = \omega_2 t}$. The two systems do not necessarily move with the same angular velocity nor do they have the same position at any given time, they are *uncoupled*.

The *coupling* of two oscillators is a bit like if they were attached by a spring, where the spring acts as the interaction between the oscillators that led to the synchrony (Fig 2.1). If we suppose that $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$ move with the same angular velocity and have the same positions at the same time, i.e. are coupled, we need to determine under which conditions this coupling can occur so that $\boldsymbol{\theta_1 = \theta_2}$.

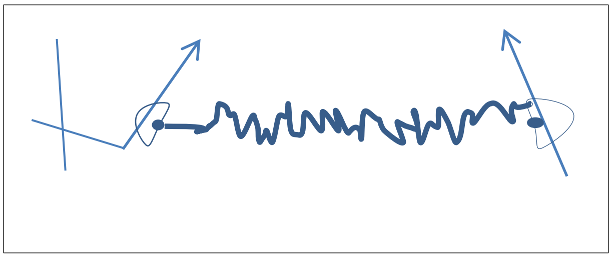

Figure 2.1

We can write the *equations of motion* describing how the two oscillators behave as two time dependent differential equations:

$$\dot{\boldsymbol{\theta}}_1 = \boldsymbol{\omega_1 + k \sin\left(\theta_1 - \theta_2\right)} \tag{2.1}$$

$$\dot{\boldsymbol{\theta}}_2 = \boldsymbol{\omega_2 + k \sin\left(\theta_2 - \theta_1\right)} \tag{2.2}$$

We have $\dot{\boldsymbol{\theta}}_1$and $\dot{\boldsymbol{\theta}}_2$, the rates of change of the time dependent angles in terms of the *natural frequency* of each oscillator $\boldsymbol{\omega_1}$and $\boldsymbol{\omega_2}$ as well as the *coupling constant* $\boldsymbol{k}$ which determines the strength of the interaction between the oscillators, and there is a phase relationship between these angles.

For a *synchronised state* to exist for the oscillators we can write $\boldsymbol{\theta_1}$and $\boldsymbol{\theta_2}$ in the general form:

$$\boldsymbol{\theta_1 = \Omega\, t + \vartheta_a} \tag{2.3}$$

$$\boldsymbol{\theta_2 = \Omega\, t + \vartheta_b} \tag{2.4}$$

The equations are both a function of the same time dependent angular frequency $\boldsymbol{\Omega}$, so they move with the same speed, and they both have some phase difference $\boldsymbol{\vartheta_a}, \boldsymbol{\vartheta_b}$. They are in synchrony as they move with the same angular frequency. The next section describes how we can get from equations (2.1) and (2.2) to something like equations (2.3) and (2.4) in order to study the synchrony.

2.2. Reduction of variables

Equations (2.1) and (2.2) contain three unknown parameters, namely $\boldsymbol{\omega_1}$, $\boldsymbol{\omega_1}$ and $\boldsymbol{k}$. By reducing the number of parameters they become simpler to solve. We can apply a *coordinate transformation* and introduce new theta variables $\boldsymbol{\vartheta_1}$and $\boldsymbol{\vartheta_2}$to describe $\boldsymbol{\theta_1}$and $\boldsymbol{\theta_2}$in terms of another frame of reference. This coordinate transformation maintains the same speed for the oscillators; however the coordinates differ by a time independent phase shift.

$$\boldsymbol{\theta_1(t) = \vartheta_1(t) + A\,t} \tag{2.5}$$

$$\boldsymbol{\theta_2(t) = \vartheta_2(t) + A\,t} \tag{2.6}$$

There is an introduction of an arbitrary constant $\boldsymbol{A}$; this will be useful in reducing the parameters as we shall see shortly.

$$\boldsymbol{\dot{\theta}_1 = \dot{\vartheta}_1(t) + A = \omega_1 + k\sin(\vartheta_1 - \vartheta_2)} \tag{2.7}$$

$$\boldsymbol{\dot{\theta}_2 = \dot{\vartheta}_2(t) + A = \omega_2 + k\sin(\vartheta_2 - \vartheta_1)} \tag{2.8}$$

We equate the original equations of motion and the coordinate transformations of these equations. This helps to begin to group the equations (as below) so that the first and second parts have common terms which can be eliminated in order to *reduce* the equations.

$$\boldsymbol{\dot{\vartheta}_1 = (\omega_1 - A) + k\sin(\vartheta_1 - \vartheta_2)} \tag{2.9}$$

$$\boldsymbol{\dot{\vartheta}_2 = (\omega_2 - A) + k\sin(\vartheta_2 - \vartheta_1)} \tag{2.10}$$

As mentioned before $\boldsymbol{A}$ is an arbitrary constant so we are able to choose it to be the mean of the angular frequencies, therefore we can let $\boldsymbol{A = \frac{\omega_1+\omega_2}{2}}$. We can now simplify our equations further so that we obtain a new constant $\boldsymbol{\delta}$ having reduced the 3 parameters ($\boldsymbol{\omega_1}$, $\boldsymbol{\omega_1}$ and $\boldsymbol{k}$) to just one ($\boldsymbol{\delta}$).

$$\boldsymbol{\omega_1 - A = \frac{\omega_1-\omega_2}{2} = \delta} \tag{2.11}$$

$$\boldsymbol{\omega_2 - A = \frac{\omega_2-\omega_1}{2} = -\delta} \tag{2.12}$$

The new constant $\boldsymbol{\delta}$ is the frequency difference between the two rotators, it measures how different the frequencies were and how they determine the dynamical behaviour of the oscillators. This helps to obtain simplified versions of (2.9) and (2.10). We now have only two unknown parameters in our equations of motion as desired. These are some of the key equations that will be utilised in this section to analyse the conditions of stability for two coupled systems. So we can put the new variable into our two equations of motion.

$$\dot{\boldsymbol{\vartheta}}_1 = \boldsymbol{\delta} + \boldsymbol{k}\sin(\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2) \tag{2.13}$$

$$\dot{\boldsymbol{\vartheta}}_2 = -\boldsymbol{\delta} + \boldsymbol{k}\sin(\boldsymbol{\vartheta}_2 - \boldsymbol{\vartheta}_1) \tag{2.14}$$

By taking the difference of the above equations we reduce the differential equations to one degree of freedom, and obtain one differential equation.

$$\dot{\boldsymbol{\vartheta}}_1 - \dot{\boldsymbol{\vartheta}}_2 = 2\boldsymbol{\delta} + 2\boldsymbol{k}\sin(\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2) \tag{2.15}$$

The difference between the rates of change of the angles as they turn $\dot{\boldsymbol{\vartheta}}_1 - \dot{\boldsymbol{\vartheta}}_2$ can then in turn be denoted by $(\dot{\boldsymbol{\Delta\vartheta}})$ further reducing the number of parameters for the angles of the rotators to just one.

$$(\dot{\boldsymbol{\Delta\vartheta}}) = 2\boldsymbol{\delta} + 2\boldsymbol{k}\sin(\Delta\,\boldsymbol{\vartheta}) \tag{2.16}$$

Now, we have reduced the number of equations of motion, have eliminated $\boldsymbol{\omega}_1$and $\boldsymbol{\omega}_2$to obtain $\boldsymbol{\delta}$ and represented the difference between the angles $\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2$ to a much simpler $\boldsymbol{\Delta\vartheta}$.

2.3. Synchronised state

We are trying to obtain conditions for when synchronisation occurs, when the oscillators are moving at the same pace and position. For a *synchronised state* we may look at the equations in terms of $\boldsymbol{\Omega}$, the same angular frequency which was described in equations (2.3) and (2.4) in section 2.1 where the model is described. Equating the derivative of these equations to (2.13) and (2.14) we obtain reduced equations for the angular frequency, having eliminated the variables $\dot{\boldsymbol{\vartheta}}_1$, $\dot{\boldsymbol{\vartheta}}_2$, $\boldsymbol{\omega}_1$and $\boldsymbol{\omega}_2$ and replaced them with only $\boldsymbol{\delta}$ and $\boldsymbol{\Omega}$ and making them simpler to analyse in terms of the same angular frequency.

$$\boldsymbol{\Omega} = \boldsymbol{\delta} + \boldsymbol{k}\sin(\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2) \tag{2.17}$$

$$\boldsymbol{\Omega} = -\boldsymbol{\delta} + \boldsymbol{k}\sin(\boldsymbol{\vartheta}_2 - \boldsymbol{\vartheta}_1) \tag{2.18}$$

The addition of both these angular frequencies gives us:

$$\begin{aligned} \mathbf{2\Omega} &= \boldsymbol{k}\,\mathbf{sin}(\boldsymbol{\vartheta_1} - \boldsymbol{\vartheta_2}) + \boldsymbol{k}\,\mathbf{sin}(\boldsymbol{\vartheta_2} - \boldsymbol{\vartheta_1}) \\ &= \boldsymbol{k}(\mathbf{sin}(\boldsymbol{\vartheta_1} - \boldsymbol{\vartheta_2}) + \mathbf{sin}(\boldsymbol{\vartheta_2} - \boldsymbol{\vartheta_1})) \\ &= \mathbf{0} \\ \therefore \mathbf{\Omega} &= \mathbf{0} \end{aligned} \tag{2.19}$$

The rotators are stationary.

The difference between the angular frequencies (2.17) and (2.18) is our simple equation of motion equation (2.16) but equated to zero. From this equation the region where the values for $\boldsymbol{\delta}$ and $\boldsymbol{k}$ may exist can be found. The *equilibrium* points are when $\mathbf{2}\boldsymbol{\delta} + \mathbf{2}\boldsymbol{k}\,\mathbf{sin}(\boldsymbol{\Delta\vartheta}) = \mathbf{0}$, this would be when there is no difference between the angular frequencies i.e. they have the same angular frequency. The equation helps work out the region where synchronisation can occur by a simple rearrangement.

$$\mathbf{sin}(\boldsymbol{\Delta\vartheta}) = -\frac{\boldsymbol{\delta}}{\boldsymbol{k}},\ \ \boldsymbol{k} \neq \mathbf{0} \tag{2.20}$$

This tells us that equilibrium points are determined by the sine of the difference between the rates of the change of the angles, and the coupling constant cannot have zero strength. The equilibrium points are a ratio of the frequency difference between the two oscillators $\boldsymbol{\delta}$ and the coupling strength $\boldsymbol{k}$**.**

$$-\mathbf{1} \leq \mathbf{sin}(\boldsymbol{\Delta\vartheta}) \leq \mathbf{1} \tag{2.21}$$

$$-\mathbf{1} \leq -\frac{\boldsymbol{\delta}}{\boldsymbol{k}} \leq \mathbf{1} \tag{2.22}$$

From further rearrangements we see that a synchronised state can exist when $|\boldsymbol{\delta}| < |\boldsymbol{k}|$; when the (absolute value of the) oscillators' frequency difference is less than (the absolute value of the) strength of the interaction between the oscillators. This mathematical inequality is demonstrated in the positive shaded regions of Figure 2.2.

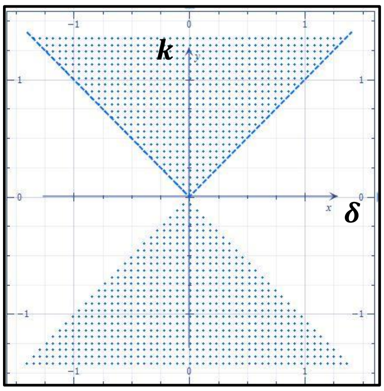

Figure 2.2

Using the above figure, we can see that if the oscillators are too diverse i.e. if the difference between their frequencies is much too great, and the coupling strength is weaker in comparison, then the oscillators will not synchronise. Of course from our inequality we know that if however the coupling strength is greater than the difference between the frequencies, then a synchronised state exists. We now know that the region for synchronised values exists when $|\boldsymbol{\delta}| < |\boldsymbol{k}|$**,** however the equilibrium points are yet to be determined. In order to find the points where oscillators are at the same position at the same time now entails some trivial mathematical calculations. We use equation (2.20) and can again represent this graphically to better see where these points may lie:

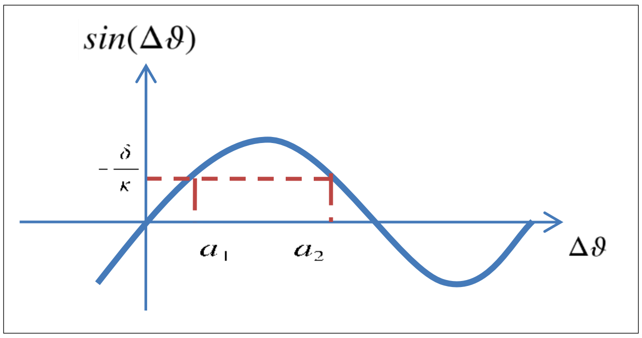

Figure 2.3

From Figure2.3 it is evident that there are 2 *equilibrium points* for $(-\frac{\delta}{k})$, namely $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$, where $\boldsymbol{a_1} = \boldsymbol{\pi} - \boldsymbol{a_2}$.

2.4. Stability analysis

Having determined the region of values where the oscillators will synchronise as well as the equilibrium points, the points within this region need to be analysed qualitatively. Looking at these points, their stability must be determined to give a clearer picture of what's going on, whether the points are unstable or stable. We use *linear stability analysis* [4] to determine when we have these *stable* or *unstable* solutions for our unknown parameters $\boldsymbol{\delta}$ and $\boldsymbol{k}$, and see how the oscillators behave**.** We do this by working out the Jacobian of $(\dot{\boldsymbol{\Delta\vartheta}})$. The Jacobian here is the derivative of $\boldsymbol{\Delta\vartheta}$ (our reduced equation). Whether or not this value is positive determines the stability of the points $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$.

$$\boldsymbol{J}(\dot{\boldsymbol{\Delta\vartheta}}) = \mathbf{2}\boldsymbol{kcos}(\,\boldsymbol{\Delta\vartheta}) \qquad \textbf{(2.23)}$$

Using this result we can work out the stability of the equilibrium points. Figure 2.4 helps to illustrate the value of the equilibrium points in reference to the Jacobian.

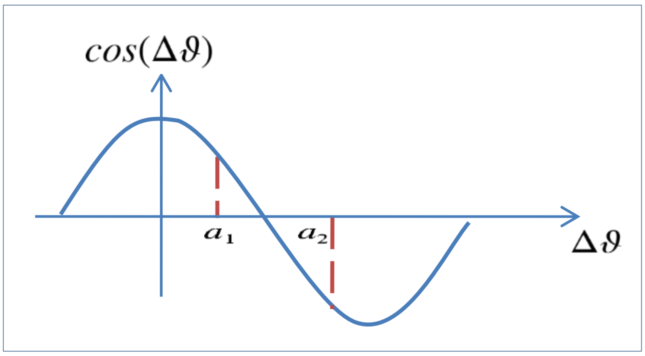

Figure 2.4

When the Jacobian is positive ($\boldsymbol{J(\Delta\dot{\vartheta})} > 0$) there is an *unstable point*, from figure 2.4 this would be $\boldsymbol{J(a_1)}$. Intuitively when the Jacobian is negative, it is a *stable point*, hence $\boldsymbol{J(a_2)}$ is the stable point. Near the *equilibrium points* we can judge the qualitative behaviour of $\boldsymbol{ce^{J(a)t}}$ where $\boldsymbol{a}$ represents the points $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$. This analysis can be done by looking at the limits as $\boldsymbol{t} \to \infty$ whereby the points can then be determined to be stable or unstable. Looking at the point $\boldsymbol{a_2}$**,** the solutions tend to 0 as $\boldsymbol{t} \to \infty$ so this has a stable solution, then looking at the point $\boldsymbol{a_1}$, the solutions tend to ∞ as $\boldsymbol{t} \to \infty$ and so comes at no surprise that the solution for this point is unstable. See Figure 2.5.

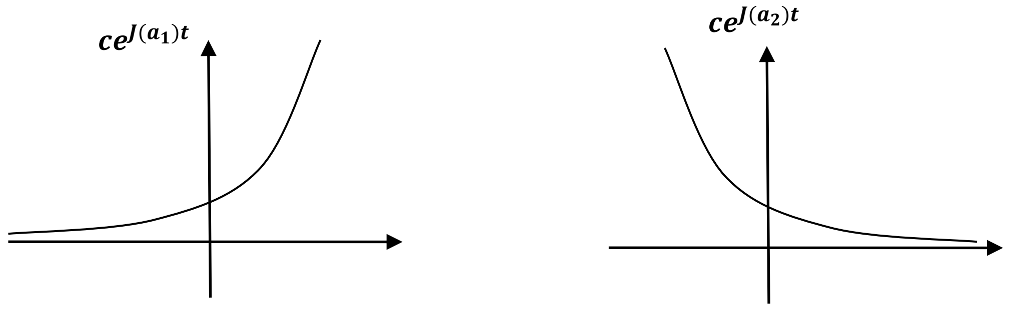

Figure 2.5

Going back to Figure 2.2 we can model the behaviour of two such points $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ and look at the values of the equilibirum points and see how they behave in the region where they synchronise. As the difference between these two points decreases, they get closer to each other as they approach the boundary where $\boldsymbol{\delta = k}$**,** when these points reach the boundary they collide and as they cross it they annihilate each other and disappear. This is known as a Saddle-node bifurcation; a local bifurcation where two fixed points (the equilibirum points) of a dynamical system collide and annihilate each other.

Within all points of the region where synchronised solutions can exist there are always two synchronised states, one of these is stable, and the other is unstable, outside of this region there is no stability.

3. Three Oscillator Model

The two oscillator model paints a fairly neat and tidy picture of the coupling of two rotators. What would happen if another oscillator were to be added? This chapter builds on the two oscillator model, and there are now more things to consider in the conditions for coupling with the additional rotator, as will hopefully be made apparent through the proceeding sections of this chapter.

3.1. Description of the model

Having seen the two oscillator model and determined the region for synchronisation and the stability, it is interesting to look at the stability for the three oscillators model and more so to see how the two oscillator model evolves.

For three coupled systems we start at the same point as that for two coupled systems, with the equations of motion for this model.

$$\dot{\theta}_1 = \omega_1 + k\sin(\theta_1 - \theta_2) + k\sin(\theta_1 - \theta_3) \tag{3.1}$$

$$\dot{\theta}_2 = \omega_2 + k\sin(\theta_2 - \theta_1) + k\sin(\theta_2 - \theta_3) \tag{3.2}$$

$$\dot{\theta}_3 = \omega_3 + k\sin(\theta_3 - \theta_2) + k\sin(\theta_3 - \theta_1) \tag{3.3}$$

There are now three oscillators θ_1, θ_2 *and* θ_3 so there are three equations. Each rotator has its own angular frequency ω_1, ω_2 and ω_3 and there is the coupling constant k which determines the strength of the interactions between the oscillators. The phase relationship between an oscillator and the other two oscillators must be taken into consideration, hence there are now 3 terms for each equation describing its oscillator compared to just two in the previous section.

Similarly to the model for two oscillators, for a *synchronised state* we can write θ_1, θ_2 and θ_3 in the following general form, with the same time dependent angular frequency Ω and the phase differences of the oscillators ϑ_a, ϑ_b *and* ϑ_c.

$$\theta_1 = \Omega t + \vartheta_a \tag{3.4}$$

$$\theta_2 = \Omega t + \vartheta_b \tag{3.5}$$

$$\theta_3 = \Omega t + \vartheta_c \tag{3.6}$$

3.2. Reduction of variables

In order to find when the oscillators may synchronise, as before the equations are reduced so that they become simpler to breakdown for analysis. Following the same process as with the two oscillator model, we try to reduce the equations using a coordinate transformation and introduce new variables $\boldsymbol{\vartheta_1}, \boldsymbol{\vartheta_2}$ and $\boldsymbol{\vartheta_3}$ describing $\boldsymbol{\theta_1}, \boldsymbol{\theta_2}$ and $\boldsymbol{\theta_3}$ in another frame of reference**.** The speeds here are the same, but the coordinates differ from before by a time independent phase shift $\boldsymbol{At}$**.**

$$\boldsymbol{\theta_1(t) = \vartheta_1(t) + At} \tag{3.7}$$

$$\boldsymbol{\theta_2(t) = \vartheta_2(t) + At} \tag{3.8}$$

$$\boldsymbol{\theta_3(t) = \vartheta_3(t) + At} \tag{3.9}$$

This equated into (3.1)-(3.3) gives similar equations to (2.7) and (2.8), except the additional coupled phase differences.

$$\boldsymbol{\dot\theta_1 = \dot\vartheta_1(t) + A = \omega_1 + k\sin(\vartheta_1 - \vartheta_2)\ + k\sin(\vartheta_1 - \vartheta_3)} \tag{3.10}$$

$$\boldsymbol{\dot\theta_2 = \dot\vartheta_2(t) + A = \omega_2 + k\sin(\vartheta_2 - \vartheta_1)\ + k\sin(\vartheta_2 - \vartheta_3)} \tag{3.11}$$

$$\boldsymbol{\dot\theta_3 = \dot\vartheta_2(t) + A = \omega_3 + k\sin(\vartheta_2 - \vartheta_1) + k\sin(\vartheta_3 - \vartheta_1)} \tag{3.12}$$

The above are then grouped in terms of the rate of change of the coordinate transformation, so that all the angles in the equations are in terms of the new theta and there aren't two different types of theta. The first term is a constant; the difference between the angular frequency and our new arbitrary constant $\boldsymbol{A}$. The second and third terms are then the positions of the oscillators with reference to the oscillator that the equation identifies it to.

$$\boldsymbol{\dot\vartheta_1 = (\omega_1 - A) + k\sin(\vartheta_1 - \vartheta_2)\ + k\sin(\vartheta_1 - \vartheta_3)} \tag{3.13}$$

$$\boldsymbol{\dot\vartheta_2 = (\omega_2 - A) + k\sin(\vartheta_2 - \vartheta_1)\ + k\sin(\vartheta_2 - \vartheta_3)} \tag{3.14}$$

$$\boldsymbol{\dot\vartheta_2 = (\omega_3 - A) + k\sin(\vartheta_2 - \vartheta_1) + k\sin(\vartheta_3 - \vartheta_1)} \tag{3.15}$$

By choosing $A = \frac{\omega_1+\omega_2+\omega_3}{3}$, the mean angular frequency since A is an arbitrary constant; the three equations above can now have the 4 parameters $\boldsymbol{A}, \boldsymbol{\omega_1}, \boldsymbol{\omega_2}, \boldsymbol{\omega_3}$ reduced to be just two $\boldsymbol{\delta_1}$ and $\boldsymbol{\delta_2}$**.** In Chapter 2, $\boldsymbol{\delta}$ was the frequency difference between the two rotators, whereas here it is the difference between a rotator and the mean frequency. The third oscillator has its constant term as $\boldsymbol{\delta_3}$, although this can be put in terms of $\boldsymbol{\delta_1}$ and $\boldsymbol{\delta_2}$ minimising the number of unknown parameters (as shown in (3.18)). The new parameters $\boldsymbol{\delta_1}$ and $\boldsymbol{\delta_2}$ help to determine the dynamical behaviour of the rotators.

$$\boldsymbol{\omega_1 - A = \frac{2\omega_1 - \omega_2 - \omega_3}{3} = \delta_1} \quad \textbf{(3.16)}$$

$$\boldsymbol{\omega_2 - A = \frac{-\omega_1 + 2\omega_2 - \omega_3}{3} = \delta_2} \quad \textbf{(3.17)}$$

$$\boldsymbol{\omega_3 - A = \frac{-\omega_1 - \omega_2 + 2\omega_3}{3} = \delta_3 = -(\delta_1 + \delta_2)} \quad \textbf{(3.18)}$$

An interesting thing to note is that the addition of the new parameters equate to zero. This tells us that in the new frame of reference the mean frequency is zero.

$$\boldsymbol{\delta_1 + \delta_2 + \delta_3 = 0} \quad \textbf{(3.19)}$$

Substituting in the new parameters $\boldsymbol{\delta_1}$and $\boldsymbol{\delta_2}$ reduces the 3 equations of motion to give the following:

$$\boldsymbol{\dot{\vartheta}_1 = \delta_1 + k\sin(\vartheta_1 - \vartheta_2) + k\sin(\vartheta_1 - \vartheta_3)} \quad \textbf{(3.20)}$$

$$\boldsymbol{\dot{\vartheta}_2 = \delta_2 + k\sin(\vartheta_2 - \vartheta_1) + k\sin(\vartheta_2 - \vartheta_3)} \quad \textbf{(3.21)}$$

$$\boldsymbol{\dot{\vartheta}_3 = -(\delta_1 + \delta_2) + k\sin(\vartheta_3 - \vartheta_2) + k\sin(\vartheta_3 - \vartheta_1)} \quad \textbf{(3.22)}$$

The above equations can be simplified even further into just 2 equations. This can be done because the similarities of the phase differences enable the use of a trigonometric identity for the second and third parts of the equations of motion. We can take the difference between the first two equations with the third; the third equation contains the $\boldsymbol{\delta}$ parameters which can relate to either the first or second equation making it easier to eliminate.

$$\boldsymbol{\dot{\vartheta}_1 - \dot{\vartheta}_3 = 2\delta_1 + \delta_2 + k\sin(\vartheta_1 - \vartheta_2) + 2k\sin(\theta_1 - \theta_3) - k\sin(\vartheta_3 - \vartheta_2)} \quad \textbf{(3.23)}$$

$$\boldsymbol{\dot{\vartheta}_3 - \dot{\vartheta}_2 = -(\delta_1 + 2\delta_2) + 2k\sin(\vartheta_3 - \vartheta_2) + \sin(\theta_3 - \theta_1) - k\sin(\vartheta_2 - \vartheta_1)} \quad \textbf{(3.24)}$$

We also took the difference of our equations of motion in chapter 2 and introduced a simplification of $\boldsymbol{\vartheta_1 - \vartheta_2}$ to a much simpler $\boldsymbol{\Delta\vartheta}$**.** The same can be applied here making it easier to follow how the equations will be used. Thus to simplify the notation of the two above equations, we let:

$\boldsymbol{\varrho_1 = \vartheta_1 - \vartheta_3}$
$\boldsymbol{\varrho_2 = \vartheta_3 - \vartheta_2}$

We now have obtained the equations of motion reduced from three equations to two, having simplified the parameters for the fixed terms and the phase differences and introduced simplified notation.

$$\boldsymbol{\dot{\varrho}_1 = 2\delta_1 + \delta_2 + 2k\sin(\varrho_1) - k\sin(\varrho_2) + k\sin(\varrho_1 + \varrho_2)} \quad \textbf{(3.25)}$$

$$\boldsymbol{\dot{\varrho}_2 = -(\delta_1 + 2\delta_2) - k\sin(\varrho_1) + 2k\sin(\varrho_2) + k\sin(\varrho_1 + \varrho_2)} \quad \textbf{(3.26)}$$

Up till now most of the steps taken in obtaining equations that have been simplified to make them suitable for analysis have pretty much been the same as for the two oscillator model. After this the analysis as will be shown in the next section will start to become more interesting to work with.

3.3. Synchronised state

In order to see when there can be a synchronised state we look at when the oscillators are positioned at the same place; there is no difference between the phases of all three oscillators. In the previous section this was when we equated the angular frequencies $\boldsymbol{\Omega}$, equation (2.17) to equation (2.19), giving the oscillators at the same position. Previously this was the same as the equation $\boldsymbol{2\delta + 2k\sin(\Delta\vartheta) = 0}$ reduced from the two equations of motion for the two rotators.

Equations (3.4)-(3.6) describe the general form for a synchronised state. We already have reduced the number of equations of motion down to two ((3.25) and (3.26)). These two equations of motion have been reduced to have two variables $\boldsymbol{\varrho_1}$and $\boldsymbol{\varrho_2}$**,** so figuring out the region for a synchronised state starts to get trickier. We add and subtract our equations from one another and they are both different ways of describing that there is no difference between the oscillators, ergo are synchronised. These now are the equations which we will use in order to determine our region for the existence of synchrony.

$$\boldsymbol{\dot{\varrho}_1 + \dot{\varrho}_2 = 0 = \delta_1 - \delta_2 + k\sin(\varrho_1) + k\sin(\varrho_2) + 2k\sin(\varrho_1 + \varrho_2)} \quad \textbf{(3.27)}$$

$$\boldsymbol{\dot{\varrho}_1 - \dot{\varrho}_2 = 0 = 3\delta_1 + 3\delta_2 + 3k\sin(\varrho_1) + 3k\sin(\varrho_2)} \quad \textbf{(3.28)}$$

As in the section 2.3 we take all the constants $\boldsymbol{k}$, $\boldsymbol{\delta_1}$and $\boldsymbol{\delta_2}$ to one side and work with the trigonometric functions, which consist of up to two variable parameters $\boldsymbol{\varrho_1}$and $\boldsymbol{\varrho_2}$.

$$\Delta_+ = \boldsymbol{\frac{\delta_1 - \delta_2}{k} = -(\sin(\varrho_1) + \sin(\varrho_2) + 2\sin(\varrho_1 + \varrho_2))} \quad \textbf{(3.29)}$$

$$\Delta_- = \boldsymbol{\frac{\delta_1 + \delta_2}{k} = -\sin(\varrho_1) + \sin(\varrho_2)} \quad \textbf{(3.30)}$$

By labelling the equations Δ_+ and Δ_-**,** the equations are described as a ratio of the change in diversity of the oscillators ($\boldsymbol{\delta_1}$and $\boldsymbol{\delta_2}$) and their coupling strength $\boldsymbol{k}$.

This can be plotted as a region where synchronisation can occur. For a general picture one could plot points using values for $\boldsymbol{\varrho_1}$and $\boldsymbol{\varrho_2}$ by choosing random values which may paint a vague idea of what the region for synchronisation looks like. These points can be chosen logically so that roughly where the further most points lie can tell us the boundary. Although of course this would be the least analytical method to implement.

A more analytical approach would give more results. For example, by using (3.29) and (3.30) as parametric equations we can find the ellipses which would be contained in the region where the solutions for a synchronised state may exist. This could be done by hand to find individual ellipses by computing the eigenvalues and eigenvectors etc. This process is rather tiresome and prone to many calculation errors if done solely by hand. The most efficient method would be to do this on mathematics programming software. I used Maple to produce the parametric plots (see Figure 3.1). I used equations (3.29) and (3.30) as the functions for my parametric plot (labelling them slightly differently i.e. as X and Y). I plotted them as a function where one of the two parameters varies as the others stays constant, and then used this idea to plot a range of fixed points with the other variable providing a range of values of each fixed point. This was computed by creating a sequence.

Figure 3.1 then gives the region for synchronicity. You can clearly see that the ellipses together appear to be bounded by a bigger ellipse which crosses the y-axes at approximately $y = \pm 2$, and the x axes somewhere between ± 3 and ± 4. This ellipse which acts as a bound for all the other ellipses inside of it is the *envelope*.

```
> X := (rho1, rho2) → -sin(rho1) - sin(rho2) - 2 * (sin(rho1) * cos(rho2) + sin(rho2) * cos(rho1)) :
> Y := (rho1, rho2) → -sin(rho1) + sin(rho2);
> plot([seq([X(t, 0.05 * Pi * k), Y(t, 0.05 * Pi * k), t = 0 ..2 * Pi], k = 0 ..40)]);
```

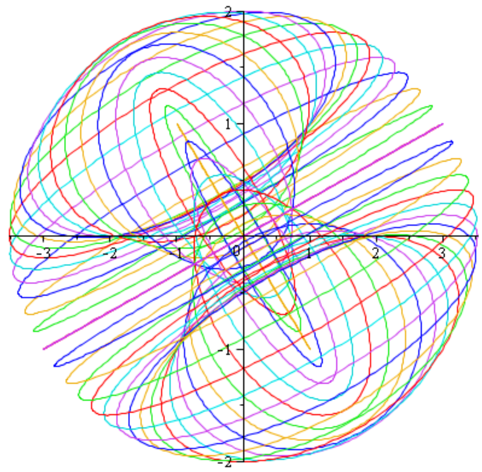

Figure 3.1

3.4 Envelope

The *Envelope* [5], more formally, is a curve which at some point is the tangent to each member of a family of curves. In 3.3 we saw a computerised approach to see the region where synchronisation can occur. From Figure 3.1 we deduced that this region which looks like it's contained in an ellipse can be bound by a single curve, which is the Envelope. The envelope can be computed by a formula which will have equations (3.29) and (3.30) substituted into it, and we let these equations be the functions $\boldsymbol{X(\varrho_1, \varrho_2)}$ and $\boldsymbol{Y(\varrho_1, \varrho_2)}$ respectively, so that the envelope is found by solving the equation from the Envelope Theorem [6] which is largely used for optimisation problems in microeconomics. The equation comes about from maximizing curves represented by $\boldsymbol{f(X(\varrho_1, \varrho_2), Y(\varrho_1, \varrho_2))}$ and taking their derivative with respect to $\boldsymbol{\varrho_1}$ and $\boldsymbol{\varrho_2}$ to give:

$$\frac{\partial X(\varrho_1, \varrho_2)}{\partial \varrho_1}\frac{\partial Y(\varrho_1, \varrho_2)}{\partial \varrho_2} - \frac{\partial Y(\varrho_1, \varrho_2)}{\partial \varrho_1}\frac{\partial X(\varrho_1, \varrho_2)}{\partial \varrho_2} = 0 \tag{3.31}$$

Thus substituting the equations Δ_+ and Δ_- (or $\boldsymbol{X(\varrho_1, \varrho_2)}$ and $\boldsymbol{Y(\varrho_1, \varrho_2)}$) into the above we obtain the equation which defines the boundary of the synchronised region.

$$\cos(\varrho_1)\cos(\varrho_2) + (\cos(\varrho_1) + \cos(\varrho_2))\cos(\varrho_1 + \varrho_2) = 0 \tag{3.32}$$

Now that the region for synchronicity has been established the qualitative behaviour of solutions within it can be analysed to explain their stability.

3.5 Stability

The stability region gives another relation between the two parameters $\boldsymbol{\varrho_1}$, and $\boldsymbol{\varrho_2}$. To determine the regions of stability and instability inside of the envelope we can use linear stability analysis as in section 2.4. We can work out the Jacobian, as we have two equations instead of the one in terms of two variables; the Jacobian is a 2 by 2 matrix. For an equation $\boldsymbol{y = f(x)}$, the Jacobian is $\boldsymbol{J(x) = y'}$, this is what we did for Equation (2.23), now for equations $\boldsymbol{\dot{\varrho}_1}$ and $\boldsymbol{\dot{\varrho}_2}$ the Jacobian is

$$J(\varrho_1, \varrho_2) = \begin{pmatrix} \frac{\partial \dot{\varrho}_1}{\partial \varrho_1} & \frac{\partial \dot{\varrho}_1}{\partial \varrho_2} \\ \frac{\partial \dot{\varrho}_2}{\partial \varrho_1} & \frac{\partial \dot{\varrho}_2}{\partial \varrho_2} \end{pmatrix}$$

Where $\boldsymbol{\varrho_1}$, and $\boldsymbol{\varrho_2}$ in the Jacobian refer to the parameters which we differentiate with respect to. So for equations for $\boldsymbol{\dot{\varrho}_1}$ and $\boldsymbol{\dot{\varrho}_2}$ the Jacobian is

$$\boldsymbol{J(\varrho_1,\varrho_2) =}$$
$$\begin{pmatrix} 2kcos\varrho_1 + kcos\varrho_1 cos\varrho_2 - ksin\varrho_1 sin\varrho_2 & -kcos\varrho_2 + kcos\varrho_1 cos\varrho_2 - ksin\varrho_1 sin\varrho_2 \\ -cos\varrho_1 + kcos\varrho_1 cos\varrho_2 - ksin\varrho_1 sin\varrho_2 & 2kcos\varrho_2 + kcos\varrho_1 cos\varrho_2 - ksin\varrho_1 sin\varrho_2 \end{pmatrix} \quad \textbf{(3.33)}$$

The stability can be determined using the Jacobian by using Bifurcation Theory [7] which is useful in studying dynamical systems. As we are dealing with models of two and three oscillators the analysis will be that of local bifurcations where there is a change in stability of a fixed point (instead of global bifurcations which refer to a nonlocal change of the phase portrait). We look at what happens when the parameter values from our equations of motion cross certain thresholds which causes stability changes. In the previous section we saw a Saddle-node bifurcation; this will be described for three oscillator model also later on in this section. As the Jacobian is a 2 by 2 matrix we can employ use of the Hopf bifurcation as for this we look at eigenvalues.

The Hopf bifurcation is a local bifurcation which occurs when there is a loss of stability when a boundary is crossed. The Routh Hurwitz criterion is the necessary criterion for this bifurcation.
The *Hurwitz Criterion* looks at how to determine the stability of linearized equations of motion of a system. An equation has a stable solution if real parts of the eigenvalues are negative. For complex numbers $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \in \mathbb{C}$ the eigenvalues are conjugates of each other $\boldsymbol{\lambda_1 = \overline{\lambda_2}}$ with negative real parts. The formulas for the eigenvalues get quite messy, so to analyse the stability conditions we can use knowledge of the properties of the determinant and trace in relation to the eigenvalues to significantly simplify the work load.

The Determinant is the product of the eigenvalues so $\boldsymbol{Det(A) = \lambda_1 \lambda_2}$ (where A is a matrix), then for stability having $\boldsymbol{\lambda_1} < 0$ and $\boldsymbol{\lambda_2} < 0$ (negative eigenvalues), the determinant being the product of these is thus positive, so $\boldsymbol{Det(A)} > 0$, this also holds true for complex conjugates. Below is the determinant.

$$\boldsymbol{Det(J(\varrho_1,\varrho_2)) = 3k^2 \cos\varrho_1 cos\varrho_2 + 3k^2 \cos\varrho_1 cos(\varrho_1+\varrho_2) + 3k^2 \cos\varrho_2 cos(\varrho_1+\varrho_2)} \quad \textbf{(3.34)}$$

As we want this to be positive we can reduce the equation.

$$\boldsymbol{Det(J(\varrho_1,\varrho_2)) = \cos\varrho_1 cos\varrho_2 + \cos\varrho_1 cos(\varrho_1+\varrho_2) + \cos\varrho_2 cos(\varrho_1+\varrho_2) > 0} \quad \textbf{(3.35)}$$

This turns out to be the equation of the envelope, and we now have a condition which tells us that for the synchronised region this equation is positive.

The trace of a matrix is the addition of the eigenvectors $\boldsymbol{Tr(A) = \lambda_1 + \lambda_2}$, and as the eigenvalues are negative, the trace is also negative $\boldsymbol{Tr(A)} < 0$, this is also true for complex numbers as the conjugates would cancel out.

$$\boldsymbol{Tr(J(\varrho_1,\varrho_2)) = 2k(\cos\varrho_1 + \cos\varrho_2) + cos(\varrho_1+\varrho_2) < 0} \quad \textbf{(3.36)}$$

From the Hopf bifurcation the trace can give us the region where the stability region changes by equating this to zero; this would also be where the eigenvalues are zero. The envelope, when crossed (to outside of it) has no stability, there are no values. The stability region is inside of the envelope, and where the trace is equated to zero gives us the boundary inside of the envelope. When this boundary is crossed the stability can change from stable to unstable, so either side of this boundary there are stable and unstable points.

We can derive the stability boundary by determining the curve in a parameter plane for when stability change occurs. We can use equations (3.29) and (3.30) and call them X and Y respectively and also use $\boldsymbol{Tr(J(\varrho_1, \varrho_2)) = 0}$ as this would give us the line for the boundary itself.

$$\boldsymbol{X = -(\sin(\varrho_1) + \sin(\varrho_2) + 2\sin(\varrho_1 + \varrho_2))} \qquad \textbf{(3.37)}$$

$$\boldsymbol{Y = -\sin(\varrho_1) + \sin(\varrho_2)} \qquad \textbf{(3.38)}$$

$$\boldsymbol{0 = (\cos\varrho_1 + \cos\varrho_2) + cos(\varrho_1 + \varrho_2)} \qquad \textbf{(3.39)}$$

We can after some rearrangement and substitutions of the above equations find Y in terms of $\boldsymbol{\varrho_1}$

$$\boldsymbol{Y = \frac{-(X + \sin(\varrho_1)}{3} - \frac{\sin(\varrho_1)(\cos(\varrho_1) + 3)}{3(\cos(\varrho_1) + 1)}}$$

(3.40)

We can use the fact that $\boldsymbol{cos^2(\varrho_2) + sin^2(\varrho_2) = 1}$ and rearrange (3.38) for $\boldsymbol{sin^2(\varrho_2)}$and find an expression for$\boldsymbol{cos^2(\varrho_2)}$ derived from a similar process used to find (3.40) by finding everything in terms of $\boldsymbol{\cos\varrho_2}$ in (3.39) and squaring it to give the following ellipse.

$$\boldsymbol{(Y + \sin(\varrho_2))^2 + (\frac{(X + \sin(\varrho_1) + (Y + \sin(\varrho_2))(1 + 2cos(\varrho_1))}{-2\sin(\varrho_1)})^2 - 1 = 0}$$

(3.41)

Using the software Maple, I worked out the two solutions for $\boldsymbol{X}$ from (3.41), this gives two of the parametric equations. Since they are for different values of $\boldsymbol{X}$ we also need the parametric equations for $\boldsymbol{Y}$ by substituting these values for $\boldsymbol{X}$ back into (3.37).

$$\boldsymbol{X = \frac{cos^2(\varrho_1) - 2 + cos(\varrho_1) \pm 3sin(\varrho_1)\sqrt{3 - (1 - cos(\varrho_1))^2}}{2sin(\varrho_1)}}$$

(3.42)

$$Y = \frac{3cos^2(\varrho_1) - 2 - cos(\varrho_1) \pm sin(\varrho_1)\sqrt{3 - (1 - cos(\varrho_1))^2}}{2sin(\varrho_1)}$$

(3.43)

We Denote the two solutions for X by $X_1 and\ X_2$, and these substituted back into Y to give $Y_1$and Y_2. The following are the graphs of the parametric equations.

```
> plot( [X1p(t), Y1p(t), t=-arccos(1 - sqrt(3)) ..arccos(1 - sqrt(3))]);
```

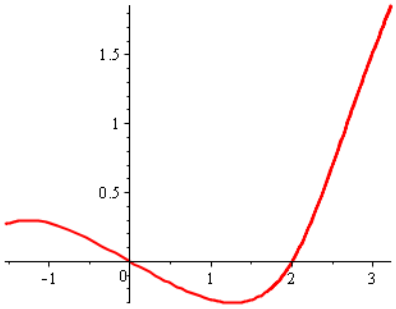

Figure 3.2

Figure 3.2 is the plot of $X_1$and Y_1. It shows a minimum point in the lower right quadrant of the x-y Cartesian plane, as t increases the curve also increases from the minimum. On the other side of the minimum the curve also rises as t decreases till it reaches a point which in Figure 3.4 will show a maximum.

```
> plot( [X2p(t), Y2p(t), t=-arccos(1 - sqrt(3)) ..arccos(1 - sqrt(3))]);
```

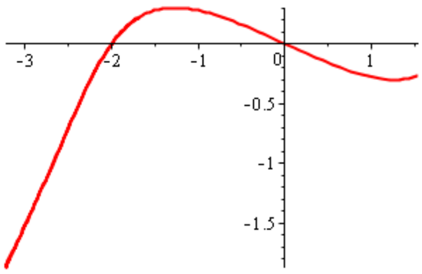

Figure 3.3

Figure 3.3 is similar to Figure 3.2 except reflected in the x axis and then the y axis. So there is a maximum in the upper left quadrant of x-y Cartesian plane, and as t decreases the curve decreases from the minimum. However on the other side of the minimum the curve also decreases but approaches a minimum which is more visible in Figure 3.4 below.

Plotting the two curves together gives one curve for the boundary of the trace (Figure 3.4). It looks a bit like a cubic graph centred at the origin. From the Hopf bifurcation as this is a boundary curve for when a change in stability occurs, points above the curve would have a different stability to that of points below it.

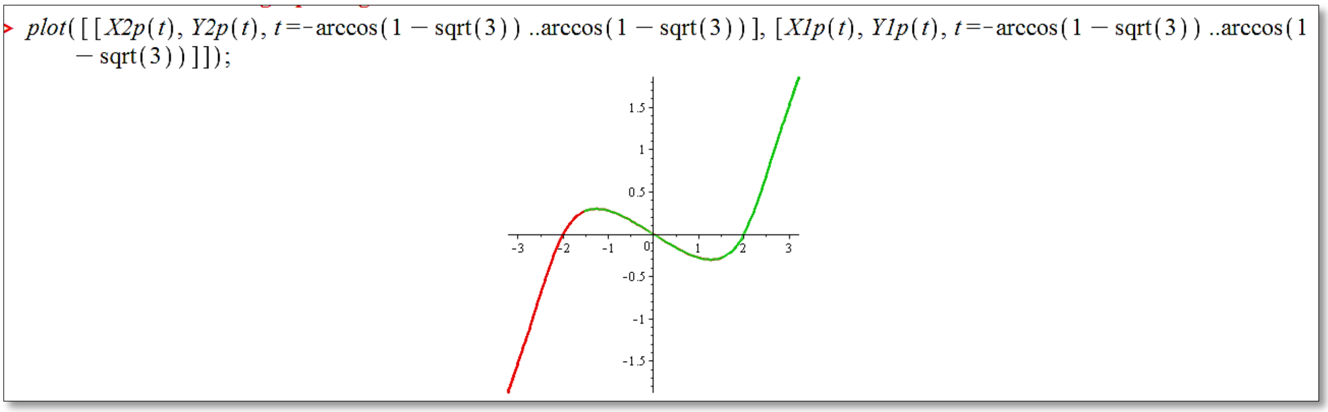

Figure 3.4

We can input a point in the graph either side of the curve above to determine whether there is stability above or below the curve. The point $(\frac{\pi}{2}, 0)$ which is above the curve for example when put into the trace gives $\boldsymbol{Tr}\left(\boldsymbol{J}((\frac{\pi}{2}, 0))\right) = \mathbf{1} > \mathbf{0}.$ As the trace must be negative for stability, we have determined that the point $(\frac{\pi}{2}, 0)$ is in the unstable region, thus, the region below the curve is where stability occurs. We can graph the diagram for the envelope and the trace together to produce a graph that contains all the information on the stable regions where synchronisation occurs.

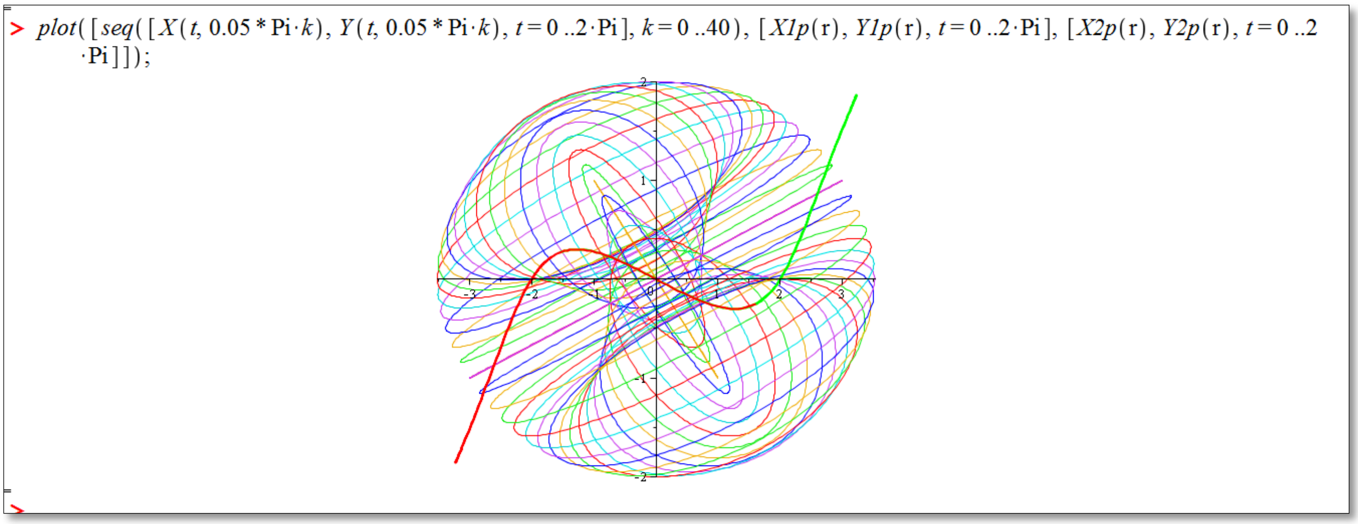

Figure 3.6

In the final diagram output of the region of stability (Figure 3.5) there is a boundary where $\boldsymbol{Det}(\boldsymbol{J}(\varrho_1, \varrho_2)) = \mathbf{0}$ which is also the envelope and inside of this curve there is stability. The curve for when $\boldsymbol{Tr}(\boldsymbol{J}(\varrho_1, \varrho_2)) = \mathbf{0}$ is the curve for the change of stability, when points cross this boundary they go from stable to unstable or vice versa, this is the phenomenon from the Hopf bifurcation. The region for stable solutions is below this curve and a region for unstable solutions is above it. When points within the curve approach the boundary of the determinant, the solutions disappear as they cross it, there is no stability outside of the curve; this is the Saddle-node bifurcation.

A finale note on the curve with the family of ellipses which has an interesting characteristic. Closer to the origin it can be seen that the ellipses may cross each other many times, this could be due to multiple solutions for stability.

4. Summary

This paradigm of the Kuramoto Model looks at the synchronisation of models of two and three oscillators. For each oscillator model there is a similar process undergone to analyse its stability. They each go through a general process of having the equations of motion reduced so that the region for the synchronised state can be found, and then have the stability within this region evaluated. Below I describe this process in a little more detail to sum up the results from the analysis.

For the Two Oscillator Model there were two equations of motion for linearly time dependent rotators. In order to analyse this model to see when they may synchronise I represented these equations in terms of a coordinate transformation to see the dynamical behaviours of the systems in terms of a different frame of reference. This introduced an arbitrary parameter $\boldsymbol{A}$ which could then be used to manipulate the equations to reduce the number of constant parameters $\boldsymbol{\omega_1}$and $\boldsymbol{\omega_2}$ which represented the natural frequencies of the oscillators. These parameters were replaced with a new one $\boldsymbol{\delta}$, which decribed the frequency difference between these parameters.

I wanted to achieve a synchronised state where the two oscillators move with the same speed at the same position, so I took the difference between these equations so that there was only one equation of motion $(\dot{\boldsymbol{\Delta\vartheta}}) = \boldsymbol{2\delta + 2k\sin(\Delta\vartheta)}$. Equating this to zero found the equilibrium points by showing that there is no difference between the motions of the oscillators. I found the points of equilibirum where the oscillators are at the same position at the same time. The value of these points are called $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ **(for**$-\frac{\delta}{\boldsymbol{k}}$**).**

Using linear stability analysis to determine the stability of the oscillators I found the Jacobian and analysed the jacobian at $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$. The sign of these points (positive/negative) told me the qualitative behaviour in time, and also showed which of these points gave stable and unstable values. Having found the stable solutions, I could use the Saddle-node bifurcation to describe what happens to the stable and unstable points in the region where synchronisation occurs, and found that solutions disappear outside this region and there is no stability.

For the Three Oscillator Model the equations are similar to that of the Two Oscillator Model except that now there is an additional equation and there are two terms for the phase differences between each oscillator instead of just one term for the phase difference. A coordinate transformation was applied to reduce the equations in order to represent the equations in such a way that they could be simplified so that they can be analysed in relation to one another. This meant that the terms $\boldsymbol{A}, \boldsymbol{\omega_1}, \boldsymbol{\omega_2}$and $\boldsymbol{\omega_3}$ could be reduced to just $\boldsymbol{\delta_1}$and $\boldsymbol{\delta_2}$**.** I then obtained two new equations by taking the difference between two sets of equations and then reduced the notation so that the equations were represented in terms of $\boldsymbol{\varrho_1}$and $\boldsymbol{\varrho_2}$ instead of three different theta variables.

The new equations were then simplified to only be in terms of the new parameters so that the equations of motion were represented in terms of the ratio between the frequencies and the coupling strength Δ_+ and Δ_-. These could be utilised to find the region for synchronisation by finding ellipses in the parameter plane from these equations, this region was bounded by an envelope.

For the stability analysis I computed the Jacobian of Δ_+ and Δ_- and used the Hopf bifurcation and Routh Hurwitz Criterion to determine the stability. From the conditions for the eigenvalues from the Hurwitz Criterion, the determinant needed to be positive and the trace had to be negative for stability. The determinant is the same equation as that of the envelope reinforcing the stability region being inside of this curve. The trace gave the boundary within the curve where if points crossed this boundary their stability would change. I found through the condition of the trace being negative for stability that the region within the envelope below the trace boundary was where the solutions were stable, and so the solutions above the curve were thus unstable. Using the Saddle-node bifurcation again, as solutions cross the envelope from within the region where solutions exists to outside of it, there is no stability.

The main difference between the two models is the stability analysis. As for the three oscillator model this gets more involved since there are two parametric equations, the Jacobian is a 2 by 2 matrix so it is trickier to work with and analyse. The Hopf Bifurcation and Routh Hurwitz Criterion are useful in describing what happens to the stability, and these weren't there in the two oscillator case, although the Saddle-node bifurcation was used in both of the models.

The results support Kuramoto's findings for the model for a large number of oscillators in that synchronisation can occur within a certain threshold or boundary, however beyond this point there is no stability.

5. Bibliography

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