Fractional Dimension Analysis for Electricity Market Spot Prices

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Abstract

Being opposed to the Efficient Market Hypothesis (EMH) that indicates unpredictable market, due to the independency (uncorrelated) of its historical and future behaviours. The Fractal Market Hypothesis (FMH), by looking at the market from the prospective of fractal geometry, is to reveal the cyclic or periodic patterns of history, which seems especially significant when associated with the electricity market. In this paper, the fitness of traditional Geometric Brownian Motion (GBM) model will be discussed. After that a fractional diffusion model will be built in order to further detect the correlation. Despite taking the stationary assumption, non-stationary circumstances would also be compatible in this model. Finally, results and conclusion will be analysed, as well as the possible improvements.

Content

Introduction

The Persistent periodic behaviour of electricity market can be seen easily from daily spot price, which is essentially related to long term auto-correlations. In other words, it implies short term or long term memory, which is neither independent nor normal distributed. The fractional model is suited for detecting memory, which was first introduced from Fractal Geometry. A fractal shows self-affine patterns that are similar or the same to itself when being observed by different scales. Similarly, as if the electricity prices show similar persistent patterns daily, weekly or monthly, seasonally. There are numerous researches in fractional models for electricity spot prices that have been done for the last thirty years, and the fractional noise model was first introduced for electricity spot prices by JRM. Hosking [1]; While the most widely applied model is the Geometric Brownian Motion (GBM) Process, which has been maturely studied and practised from stock markets to derivative markets, commodity markets, etc. In this paper, we study the fractional dimension of the electricity spot prices to check whether it has memory or it follows GBM.

1. Geometric Brownian Motion Analysis

If a stochastic process flows Geometric Brownian Motion, then it can be formulized as

$$y(t) = y(t_0)e^{m(t-t_0)+\sigma W_t}$$

Where (t), $y(t_0)$ can be denoted as spot prices at time t, t_0 respectively. m is the drift, σ is the square root of volatility. W_t represents the Wiener Process which follows the standard normal distribution with the mean of zero and variance $t - t_0$(generally, $t_0 = 0$) . Alternatively, we can write GBM in another form that

$$\mathrm{Y(t)} \equiv ln\left\{\frac{y(t)}{y(t_0)}\right\} = m(t - t_0) + \sigma W_t$$

$$\mathrm{Y}(\mathrm{t} + \tau) - \mathrm{Y(t)} = \sigma W_\tau$$

Where $\mathrm{Y(t)}$ is the profile we just defined. Since if the differences of the profiles with the same time increments are tested not to follow a standard normal distribution, which is σW_τ, or more specifically, it fails to reject the null hypothesis in some significance value. Then we can conclude that it does not follow the classical Geometric Brownian Motion (Fractional Brownian Motion not included).

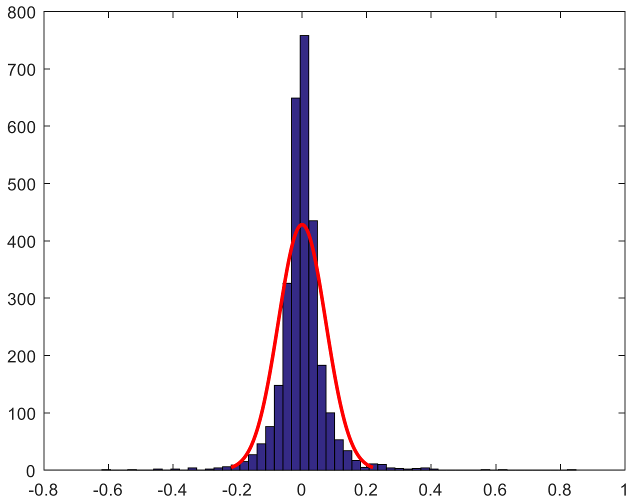


Figure (1): **Red: Gaussian; Blue:** $\Upsilon(t+\tau) - \Upsilon(t)$.

The above graph shows the histogram of profile increments of daily spots. Which does not show any obvious sign of a Gaussian (the curve). Furthermore, In the study of Marathe[2], the normality and independency are being tested, of the monthly U.S electric power consumption data from 1993 to 2002, which is considered relevant to the electricity market spots as the relationship of supply and demand. The result fails to reject the null hypothesis before de-seasonalization. Specifically in which, after computing the log-prices, the de-seasonalization process is to subtract the periodic centre-moving-average data with its mean seasonal residuals. As a result, it rejects the null hypothesis of significant value. However, this is not appropriate for our circumstances as our goal is to study the seasonal effects or particularly the periodic, cyclical effect, in other words, to investigate whether the spots are long term correlated/anti-correlated; While the de-seasonalization process just filters out these features.

2. Fractional Dimension Analysis

2.1. Fractal

Fractals catch the attentions of the public by the famous paradox of the length of the Great Britain's coast line. Which states that when looking at the map, intuitively one the length of the Great Britain's coast line is finite. However, assume that we can infinitely zoom in the map by arbitrary large scale microscope, for say, even to the level of molecules, the length of the "edge" can grows to infinity. Fractal exactly has this property, moreover, it has similar patterns no matter by what distance we look at it.

2.2 Dimension

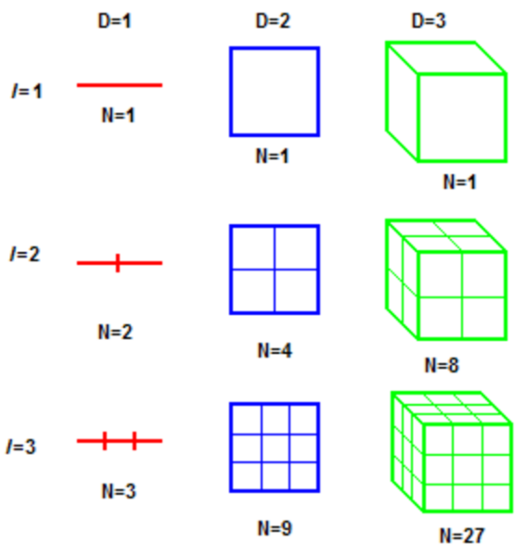
The Wiener process in the last section follows a power law. Which for instance, imagine a two-dimensional square, if its sides halved, the area of the square would be one-fourth of its original. In general, this relationship indicates its traditional dimension, To formalise the previous example of square.

$$l^d = N$$

Where l represents the new length of the square after shrinking/amplifying while the original length is one, respectively, N is the new area. While dimension d, does not change at all whatever the length would be, take the previous shrinking example

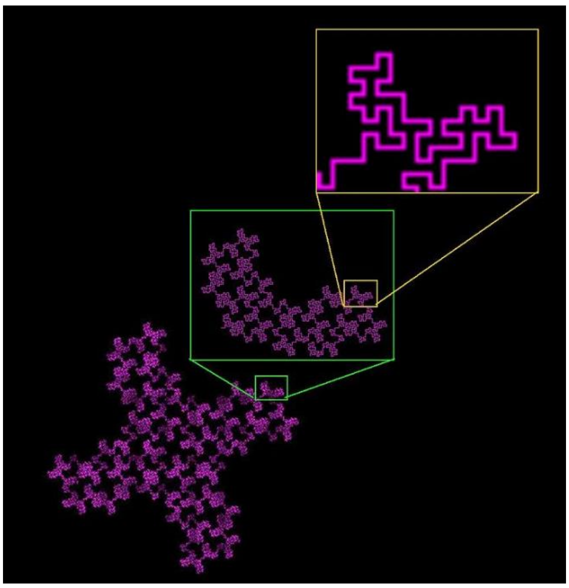
$$(1/2)^2 = \frac{1}{4}, \left(\frac{1}{3}\right)^2 = \frac{1}{9} \dots$$

This relationship holds in one and three dimensions as well, for demonstrations



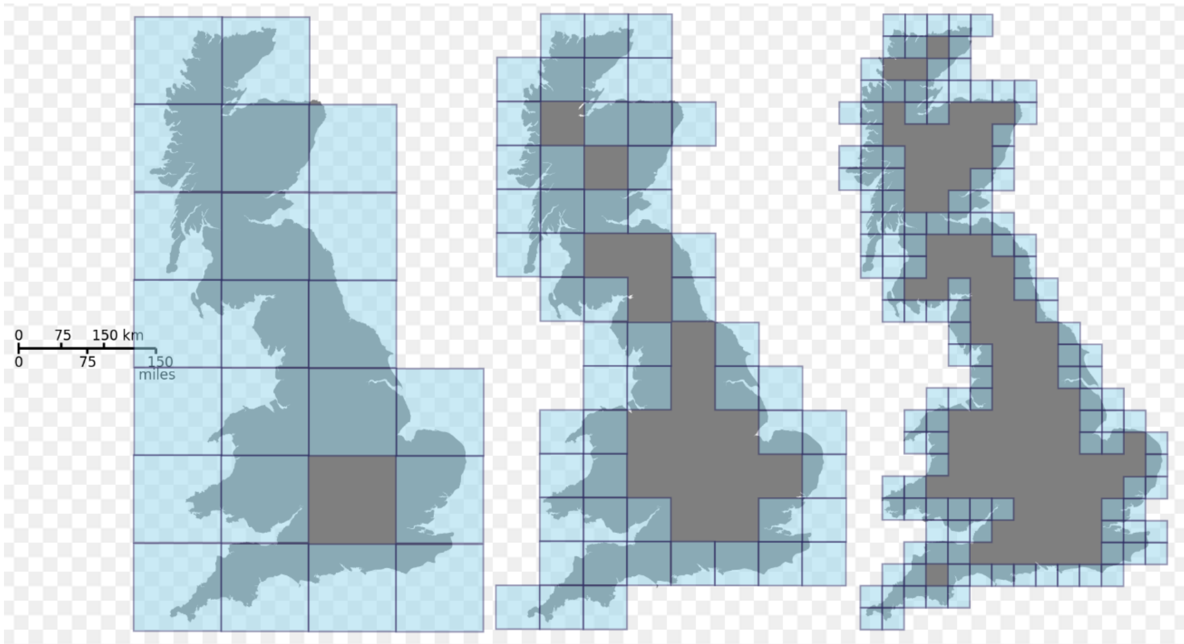
Figure(2): Examples for traditional dimension and scaling.

As an extension, the Hausdorff dimension or fractal dimension indicates this relationship for fractals, which for example, a 32-segment quadric fractal.



Figure(3): 32-segment quadric fractal.

Which looks exactly the same by observing from different scales, it illustrates self-similarity. As the same procedure as the last case, we calculate the dimension of a fractal by measuring the exponent of its new size and area for a two traditional dimensional fractal on a plane. It can be done by several methods, for example, the box-counting method, in which generally approximates its area by putting same sizes of blocks to cover the edge of the fractal. For illustration, to measure the fractal dimension of the coast line of the Great Britain.



Figure(4):Box-counting for coast line of the Great Britain.

As the number of blocks increases to infinity, the box-counting dimension of the coast line can be represented as

$$\mathrm{Dim}_{box}(S) = \lim_{\epsilon \to 0} \frac{lnN(\epsilon)}{ln(1/\epsilon)}$$

Where $N(\epsilon)$ is the number of boxes of length ϵ. And this is just a rough illustration to give a general idea before moving further.

2.3 Hurst Exponent

In previous illustration we approximate the length of coast line with box-counting; to be extended, the trace of a stochastic random process also has a fractal dimension. Like a fractal in geometry, one may wonder what it implies if the electricity spot prices are self-similar. Imagine different investors in the market reacts to information with different time scales. For example, intraday traders react to high-frequency price spikes and financial events with a day; while long term investors only put few trades within a year by considering the prospects of markets in a long term, but do not participant short term fluctuations. Furthermore, company investors trading electricity at particular hot/cold seasons/long periods of a year could be considered as long term investing behaviour, even though they might not hold the position for a relatively long time. Take another example, annually fireworks only take place at top half an hour but it is one-year periodic.

Thus if the spot prices seem to behave in some similar patterns no matter what time scale length is considered. Then this pattern must be somehow discovered. And the Hurst exponent is the first to detect long term memory/persistent correlation of signal, which is denoted as parameter H, which lies within the domain from zero to one. For an $\mathrm{H} > 0.5$, the process is characterized by long memory or positively long-term correlated, that is, if something happened before, then in long term it is very likely to happen again; whereas $\mathrm{H} < 0.5$ represents short memory that the current state of the process is more likely to reversed, for example, the so-called mean-reverting phenomena in financial markets. As for $\mathrm{H} = 0.5$, the next state of the process is independent to its previous counterpart, that is uncorrelated.

Conventionally, the Hurst exponent can be calculated by Rescaled Range Analysis which is also known as R/s method, which is similar to the idea of box-counting being mentioned, and the steps to apply are following.

First, take auto-correlated residuals of log prices' profile previous mentioned. Then apply linear regression to fit a line, by which then compute the residuals. This procedure is so-called linear de-trended [3].

Second, divided the whole series into n sub-intervals of equal length, in which, calculate the mean. Note that n starts from one.

Third, calculate cumulative deviates obtained by subtracting the mean in each sub-interval.

Fourth, find the maximum distance in sub-intervals as the range, which is the deviate of the largest value and the smallest value in that sub-period.

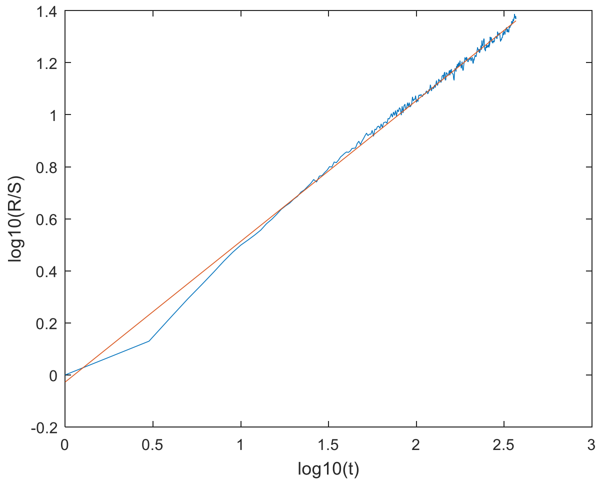
Fifth, for each interval, calculate the standard deviation.

Sixth, for all sub-intervals, compute the mean of all values which, for individual period, is its range divided by its standard deviation.

To be continued, as the length n increases, for each obtained the value in Sixth. Stops when n reaches the half length of the whole time series. Now apply logarithm to all outcomes respects to different n, and the logarithm of all length n.

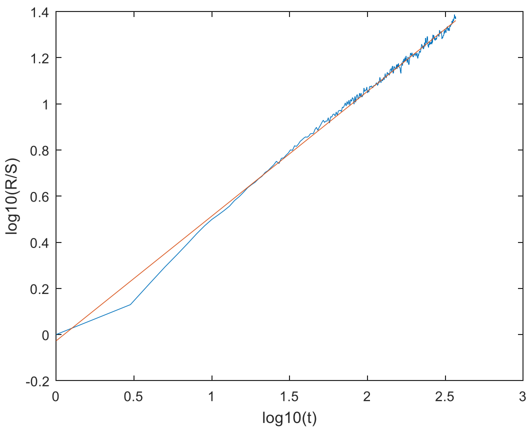
Finally, fit a straight line of this two vectors by linear regression, which maximizes the errors. The slope of the line obtained is an approximation of Hurst exponent.

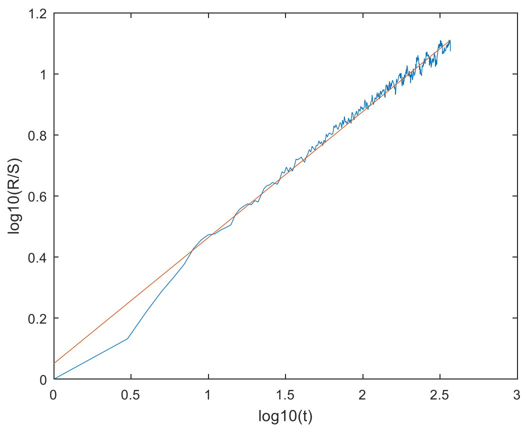
Taking the daily spots of every 0 a.m as input, we have



Figure(5):R/s method for (0 a.m) daily spot; H=0.5408 with 2947 data points.

With certain numbers of data, it is failed to reject the null hypothesis within a significant level of 95 percent (see [4] as the size of lag data increases the upper and lower bounds are more narrow around 0.5) that the data is uncorrelated, which to be exact, it has a long memory. However, as we compute Hurst exponent for other daily data, it is significantly different.





Figure(6) (left:7 a.m) H=0.4128 ; Figure(7) (right:18 p.m) H=0.4773

These two datasets show short term correlations with the Hurst exponents smaller 0.5, while their counterpart (figure 5) has a long memory. Overall, the variation of different states of H is an indication to the potential non-stationarity of the spot prices.

3. Stationarity Analysis

In general, fractional models for the electricity spot prices are based on the assumption that the process is stationary or at least characterized by either long-memory or mean-reverting (short-memory/anti-persistent). While as what the previous computations imply, it could be both. This is also supported by the work of Nielson and Haldrup [5] [6]. Before modelling, in order to verify the stationarity of the electricity market prices. Here we briefly introduce the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) tests, which is regarded as a widely used tool in econometrics.

The KPSS is not only to test the trend-stationarity of the process, that is, in other words, the de-trended data's stationarity, but also to verify whether the process has alternative unit root. By which in our case, for 2947 daily spots (at 0 a.m), it results in being stationary with 0.05 significance level (Matlab code is provided in Appendix, file name: KPSS.m). This result is contradicted to our previous intuition, which is also claimed by other researchers. Thus further investigations are required for the KPSS test.

According to the study of Cappuccio and Nunzio [7], the KPSS test is fragile to reveal potential non-stationarity. Irrespectively of much detailing in the test, briefly say, this paper has addressed the problem that the validation of KPSS test result relies on a much greater size of data than normal economic data size could be. The size distortion would cause over-rejection to null hypothesis that the data is not trend-stationary. Other words, for common financial data that without an adequate amount, the KPSS would mostly result in trend-stationarity even it might actually not be.

Hence, based on the uncertainty of stationarity, we need to primary assume that it has the potential to be non-stationary. Based on which, in next chapter, we first build a stationary model to compute the fractal dimension to further investigate this. Next, to cope with the non-stationarity, we will make a proposition to let the model compatible in such circumstance.

4. Fractional Dimension

4.1 Diffusion Modelling

As Brownian Motion is widely used in financial modelling, which describes the movement of particles. Particle diffusion model can be well applied to our case as well. We presume the price moves like particles. From a prospective of which, we can start with modelling particle diffusion and its future behaviour, by deriving the diffusion equation.

Let $u(x,t)$ be the density (number of particles in a unit length) of particles at location x at time t. Let λ be the distance that the particles have moved in a small time interval t. And we assume $u(x,t)$ and $u(x,t+\tau)$ are independent. Then we have $u(x,t+\tau)$ as an equation as follow; and the coming derivations of diffusion equation are originally provided by Einstein [8]:

$$u(x, t+\tau) = \int_{-\infty}^{+\infty} u(x+\lambda, t)\mathrm{p}(\lambda)d\lambda$$

Where $\mathrm{p}(\lambda)$ is the PDF (probability density function) of λ, we consider it a symmetric distribution. The equation above can be interpreted as that the density of particle at time $t + \tau$ is the integral over all possible number of particles λ being put, and its probability is $\mathrm{p}(\lambda)$.

With a small increment of time τ, by applying Taylor expansion to the term of function u we obtain:

$$u(x, t+\tau) = u(x,t) + \tau\frac{\partial u(x,t)}{\partial t} + \frac{\tau^2}{2!}\frac{\partial^2 u(x,t)}{\partial t^2} + \cdots$$

$$u(x+\lambda, t) = u(x,t) + \lambda\frac{\partial u(x,t)}{\partial x} + \frac{\lambda^2}{2!}\frac{\partial^2 u(x,t)}{\partial \lambda^2} +$$

$$\cdots$$

As τ is considered to be small, then λ is also treated as a small value. Thus we only keep the first three terms and omit the rest so an equation can be formed as:

$$u(x,t) + \tau\frac{\partial u(x,t)}{\partial t} = u(x,t)\int_{-\infty}^{+\infty} \mathrm{p}(\lambda)d\lambda + \frac{\partial u(x,t)}{\partial x}\int_{-\infty}^{+\infty} \lambda\mathrm{p}(\lambda)d\lambda +$$

$$\frac{\partial^2 u(x,t)}{\partial \lambda^2}\int_{-\infty}^{+\infty} \frac{\lambda^2}{2!}\mathrm{p}(\lambda)d\lambda$$

$u(x,t)$ we consider it diffuses the same way in all directions, which means that $\mathrm{p}(\lambda)$ is a PDF as well as an even function, whereas λ is odd function itself (imagine $-\lambda\ and + \lambda$). We then obtain a partial differential equation (PDE) in a form that we will later be majorly focused on to give a solution:

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} \qquad Eq\ 1.1$$

Where D is the diffusion constant denoted in Einstein's famous paper in 1905 [6]:

$$D = \int_{-\infty}^{+\infty} \frac{\lambda^2}{2\tau}\mathrm{p}(\lambda)d\lambda$$

Before we go further in particle diffusion, we need to come back to our price data and see how its statistics behaves as time goes by, and following is the second moments of the spot prices.

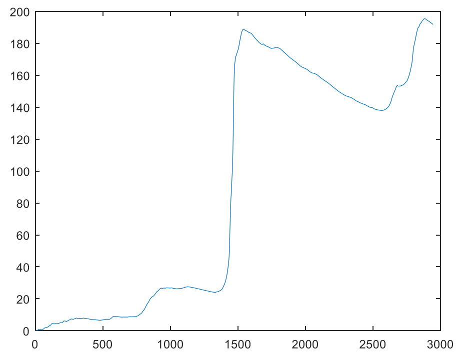


Figure (8): (daily 0 a.m) second moments

As we can see that as time proceeds, the second moments, which are correlated to the volatility, diverge significantly in an upward trend. Also by considering the self-affine behaviour which the series has “memory”, we can propose a model in a Lévy process, which is originally inspired by J. Blackledge [9]. A Lévy distribution has an infinite second moment [10]. And it distributes in a similar way itself after many steps. Particularly it's a self-affine process that meets the requirement of our model. The Gaussian distribution is similar to itself as well, it is considered a special case of Lévy distribution. A symmetric Lévy distribution, of which the characteristic function $P(k)$ is:

$$P(k) = e^{-a|k|^{\gamma}}, \gamma \in (0,2)$$

(Noting that when $\gamma \geq 2$, $P(k)$ is Gaussian)

Where γ is Lévy index and a is a positive constant. Here we have a quick review of Fourier Transform for further derivations. For functions “f” and "g”, according to convolution theorem that :

$$\boldsymbol{\mathcal{F}}\{(f \otimes g)\} = \boldsymbol{\mathcal{F}}\{f\}\boldsymbol{\mathcal{F}}\{g\}$$

Where “$\otimes$”denotes the convolution:

$$(f \otimes g)(t) := \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

And “$\boldsymbol{\mathcal{F}}$” denotes Fourier Transform, which by definition is:

$$\boldsymbol{\mathcal{F}}\{f(\omega)\} := \int_{-\infty}^{+\infty} f(x)e^{-2\pi i x\omega}dx$$

And the inversed Fourier Transform is denoted by:

$$f(x) = \boldsymbol{\mathcal{F}}^{-1}\{\boldsymbol{\mathcal{F}}\{f(\omega)\}\} = \int_{-\infty}^{+\infty} \boldsymbol{\mathcal{F}}\{f(\omega)\}e^{2\pi i x\omega}d\omega$$

By applying Fourier Transform and the convolution theorem to the equation above, then in the Fourier space it becomes:

$$U(k, t+\tau) = U(k,t)P(k) \qquad Eq(1.2)$$

Where"U" and "P" are characteristics function obtained by the Fourier transform of $u(x,t)$ and the PDF $p(x)$. Based on the work of [11], we are about to see that the fractional derivatives with Lévy process are self-affine process, which we consider the diffusion process of $u(x,t)$ to be in terms of modelling. As we recall the characteristics function:

$$P(k) = e^{-a|k|^{\gamma}} = 1 - a|k|^{\gamma} + \cdots, \gamma \in (0,2]$$

In order to derive a generalized form of Eq. (1.1), similarly, we expand Eq. (1.2) in Taylor Series as τ is small.

$$U(k, t+\tau) \approx U(k,t) + \tau \frac{\partial U(x,t)}{\partial t}$$

$$U(k,t)P(k) \approx U(k,t) - a|k|^{\gamma}U(k,t)$$

$$\frac{U(k,t+\tau) - U(k,t)}{\tau} \approx -\frac{a}{\tau}|k|^{\gamma}U(k,t)$$

Noting the Riesz fractional derivative in "classical case" that [12] (this will be then covered):

$$\boldsymbol{\mathcal{F}}\left\{\frac{\partial^{\gamma} u(x,t)}{\partial x^{\gamma}}\right\}(x) = -|k|^{\gamma}U(k,t) \qquad Eq\ (1.3)$$

(Note that k can be interpreted in terms of angular frequency as: $k = \omega$).

$$\Rightarrow \qquad \frac{\partial U(k,t)}{\partial t} = -\frac{1}{\tau}|k|^{\gamma}U(k,t)$$

Next, by performing the inverse Fourier Transform, we obtained a generalized form:

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^{\gamma} u(x,t)}{\partial x^{\gamma}}, \qquad D = \frac{a}{\tau} \qquad Eq\ (1.4)$$

Where D is the generalized diffusion constant. Moreover, the left side of Eq. (1.3) can be derived into:

$$\frac{\partial^{\gamma} u(x,t)}{\partial x^{\gamma}} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} |k|^{\gamma}U(k,t)e^{ikx}dk$$

With a solution of the singular initial condition, as$\to 0$, $u(x,\tau) \to u(x,0) = \delta(x)$, which is given by:

$$u(x,\tau) = \boldsymbol{\mathcal{F}}^{-1}\{U(k,t+\tau-t)\} = \delta(x,t-t) \otimes \boldsymbol{\mathcal{F}}^{-1}\{P(k,t-t)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\left\{\int_{-\infty}^{+\infty} e^{ik(t-t)}P(k,t-t)d(t-t)\right\}dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(x)P(k,t-t)\,dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx-D'|k|^{\gamma}t}dk$$

Where $\delta(x-\alpha)$ is denoted as Dirac delta function which (would be explained later) has an expression:

$$\delta(x-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ip(x-\alpha)}dp$$

Thus the solution can be re-written as:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx-D'|k|^{\gamma}t}dk \qquad Eq\ (1.5)$$

Which is itself Lévy distribution. Thus it is consistent with the fractional derivatives and we can carry on.

For a brief, $\delta(x)$ is the Dirac delta function, introduced by physicist Paul Dirac, is interpreted as:

$$\delta(x) = \begin{cases} +\infty, & x = 1 \\ 0, & otherwise \end{cases}$$

It is not a function in traditional sense as no function defined on real numbers has these properties [13].

4.2. Past Dependency Analysis based on Fractional Integral

According to the model above, we have the Lévy index $\in(0,2]$. However, in order to have a better representation of the dimension of the power market price signal as time passes. We define $q\in(0,1)$ as the order of differentiation respect to time t, which is a part of a fractional differentiator. Before further derivation, the properties of fractional differentiation are required to be analysed.

As a definition, a fractional differentiation respect to time is represented as:

$$D^q f(t) := \frac{d^k}{dt^k}[I^{k-q}f(t)], \quad k > q$$

Where D is a differential operator; k in here is integer whereas q is not an integer but "fractional number", I represents integration. According to Riemann-Liouville integral, I could be expressed as:

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \ \ \alpha > 0 \qquad Eq\ (1.6)$$

Where $\Gamma(\alpha)$ is Gamma function for real number: $\Gamma(\alpha)= (\alpha-1)!$. As we can see that the integral at time t can be interpreted as integrating over all the past values of $f(t)$ in a form of a convolution. Thus this process is consistent with the self-affine property that the price now is somehow correlated or anti-correlated with its historical values.

Nevertheless, as an extension, we introduce Riesz-Caputo fractional derivative [14] (RC represents a Riesz-Caputo):

$$D_{RC}^{q} f(t) = \frac{1}{2}\{D_{+}^{q} f(t) + (-1)^{n} D_{-}^{q} f(t)\}$$

A left Riesz-Caputo fractional derivative:

$$D_{+}^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_{a}^{t} \left\{\frac{d^{n}}{dt^{n}} f(\tau)\right\} (t-\tau)^{n-q-1} d\tau$$

Where

$$n > q > n-1$$

And the right Riesz-Caputo fractional derivative:

$$D_{-}^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_{a}^{t} \left\{-\frac{d^{n}}{dt^{n}} f(\tau)\right\} (t-\tau)^{n-q-1} d\tau$$

Moreover, its Fourier Transform is [15]:

$$\boldsymbol{\mathcal{F}}\left\{D_{x}^{q} f(x,t)\right\}(x) = |\omega|^{2q} F(\omega, t), \qquad Eq\ (1.7)$$

Base on the previous derivation, we see that Eq. (1.3) has self-affine property as well as the fractional derivative/integral has been mentioned, in which, the past time correlation suggests us to do some tiny modification so that the model obtains parts as derivative respect to time to be past time dependent, which is, in addition, also consistent with the self-affinity.

Hence our goal is to find variable q as a dimension indicator that is rather done by investigating its intrinsic property, instead of solving a fractional partial differential equation, which might end up with some complicated terms in time space; Even though, either ways seem inevitable to discuss the fractional PDE, for both real and complex space by Fourier Transform. In which, the integration needs to be discussed.

Specifically, to the concern of our modelling, t, ω and τ are physical values. They are positive and τ is smaller or equal to t, which is a segment of current time t. Thus, in here Fourier Transform is not as it is generally defined as an integral over all real numbers but constrained to corresponding circumstances, for example, the Fourier Transform respect to x:

$$U(\omega) = \frac{1}{2\pi} \int_{0}^{+\infty} u(x) e^{-i\omega x} dx, \qquad 0 < x$$

4.3 Modelling

To propose an extended model based on earlier considerations, we introduce a fractional version of Fokker Planck Equation [16], in which, the PDF of the diffusion process is characterised by: (note that we use our pre-defined variable names for convenience)

$$P(k,t) = e^{-a|k|^{\gamma} t^{q}}, \qquad (0 < \; q \leq 1;\; 0 < \; \gamma \leq 2)$$

While the equation with the PDF (x,t) :

$$\frac{\partial^{q} p(x,t)}{\partial t^{q}} = \frac{\partial^{\gamma}\{p(x,t)s(x,t)\}}{\partial x^{\gamma}} \qquad Eq\ (1.8)$$

Where function s is arbitrary. Since constant q can take any values from zero to one, it is appropriate to define:

$$q = \frac{\gamma}{2}, \quad (0 < \; q \leq 1;\; 0 < \; \gamma \leq 2)$$

By which we can preserve the homogeneity of this equation so that it is consistent with the hypothesis of self-affinity [17]. As an explanation, Assume that the distribution function of our function $u(x,t)$ is $p(t)$, it is homogeneous in a condition if:

$$p(at) = a^{q} p(t)$$

Which can be interpreted that the distribution p at time at is "similar" to the distribution itself at time t. The way how they are similar each other is ruled by the factor a^{q}. This relationship is just so-called self-affinity.

To be continued, as the procedure before, by Taylor expansion to the left side of Eq. (1.2), a differentiated form with $U(k,t)$ can be derived as $\tau \to 0$:

$$\frac{\partial U(k,t)}{\partial t} = -D{*}|k|^{\gamma} U(k,t), \qquad D{*} = \; \frac{t^{q}}{\tau}$$

Which, in frequency space, indicates the asymptotic changes of the magnitude of impulse at frequency zero as the time interval increases its size with a nearly zero magnitude. Where the impulse is described by the function $(x-\alpha)$, known as the Dirac delta function which has been mentioned before. Moreover, it can be interpreted in a way that, it is the change of the number of same price value of every τ, say, for example, every five minutes or three seconds, etc. And τ is so small that it is almost zero. As we imagine a tempered distribution, which means a test distribution approaches its distribution function as the number of trials increases; In particular, we assume there is only one trial being put, such that, the distribution of this single result has a very sharp peak and has the probability of one, which looks like an impulse described by delta function; In a general sense, as the number of trials increases, the peak of the impulse may become less sharp. Thus, how this distributions varies as time proceeds is exactly why we introduced delta function and the derivative of $U(k,t)$ respect to time we mentioned above. Furthermore, the impulse indicates how the tempered distribution varies as the number of trials gets greater; Therefore it is not hard to presume that, the convolution of the initial condition of a distribution and the impulse (delta function), becomes the distribution itself. Denoting as:

$$p = p' \otimes \delta$$

Where p' is the tempered distribution function and p represents the distribution function itself. Then we replace $s(x,t)$ with D'^{q}, as function s is arbitrary. Our model can be written as:

$$\frac{\partial^q u(x,t)}{\partial t^q} = D'^{q} \frac{\partial^\gamma u(x,t)}{\partial x^\gamma}, \qquad (D* = D'^{q}) \quad Eq\ (1.9)$$

Then we immediately find an equivalent relationship linked up each derivative, such that:

$$(i\omega)^{-q} \boldsymbol{\mathcal{F}} \left\{ \frac{\partial^q u(x,t)}{\partial t^q} \right\} (\omega) = -|k|^{-\gamma} \boldsymbol{\mathcal{F}} \left\{ \frac{1}{D*} \frac{\partial^q u(x,t)}{\partial t^q} \right\} (\omega)$$

$$Eq\ (1.91)$$

Which is based on the classical Riesz derivative Eq. (1.3) that:

$$\boldsymbol{\mathcal{F}} \left\{ \frac{\partial^\gamma u(x,t)}{\partial x^\gamma} \right\} (x) = -|k|^\gamma U(k,t)$$

Then we get:

$$\boldsymbol{\mathcal{F}} \left\{ \frac{\partial^q u(x,t)}{\partial t^q} \right\} (\omega) = \boldsymbol{\mathcal{F}} \left\{ \frac{\partial^\gamma u(x,t)}{\partial x^\gamma} \right\} (k)$$

$$\Rightarrow \qquad \frac{(i\omega)^q}{D'^{q}} = -|k|^\gamma$$

Review that homogeneity holds if $\gamma = 2q$, by which, self-affinity also holds. Therefore we have:

$$k^q = \pm \frac{i(i\omega)^{q/2}}{D'^{q/2}}$$

However, there is a problem that this equation holds for homogeneity, but at the same time, it loses its meaning for fractional derivative. In other words, this relationship makes the equation no longer fractional but turns into the classical heat equation, according to the study of [18]. Applying this relationship is inappropriate to model a fractional self-affine distributed signal. The derivation is wrong based on considering the Levy index is twice as the dimension indicator q. In which it would end up with coefficients being cancelled out. Moreover, the "fractional" version of Fokker-Planck Equation is not really "fractional" regarding the previous assumptions. Therefore we have to be careful to re-think of the relationship of q, γ in Eq. (1.9) so that it is both fractional and homogeneous. But it seems un-fixable, as we now demonstrate the example in [17]. If we start from the original equation Eq. (1.1) but with diffusion coefficient $D*$ as below:

$$\frac{\partial u(x,t)}{\partial t} = D_{*}\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad D_{*} = \frac{t^2}{\tau}$$

We change the derivative order respect to time to q, and get:

$$\frac{\partial^q u(x,t)}{\partial t^q} = D_{*}\frac{\partial^2 u(x,t)}{\partial x^2}$$

Then the only way to keep it homogeneous to time is to substitute the order of diffusion coefficient D_{*}, such that:

$$\frac{\partial^q u(x,t)}{\partial t^q} = D'^q\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad D'^q = D_{*} = \frac{t^2}{\tau}$$

However, respect to x, homogeneity does not hold. Thus we have to replace the counterpart regarding x with $2q$. Then we get:

$$\frac{\partial^q u(x,t)}{\partial t^q} = D'^q\frac{\partial^\gamma u(x,t)}{\partial x^\gamma}, \qquad \gamma = 2q$$

Which unfortunately leads us back to the "non-fractional" result that we started with. However, based on the study of [17], we find out a condition, in which the homogeneity and fraction would be both preserved such that:

$$\frac{\gamma}{2} = 1 - q, \qquad (0 < q < 1;\ 0 < \gamma < 2)$$

Thus, to be continued with what we previous derived, we get:

$$\frac{(i\omega)^q}{D'^q} = -|k|^{2-2q}$$

4.4 Solutions

4.4.1 General Method

Conventionally, a partial differential equation in the form of our model can be solved by Green's function. Generally speaking, we need to the find a Green's function $g(x,t)$ such that:

$$\left\{\frac{\partial^q}{\partial t^q} - D'^q\frac{\partial^\gamma}{\partial x^\gamma}\right\} g(x,t) = \delta(x-t)$$

And the solution can be expressed in terms of the form below:

$$u(x,t) = \int_{-\infty}^{+\infty} g(x,t) f(x) dk$$

Nevertheless, we need to consider the homogeneity of our model, which consists two aspects. First, the initial boundary condition is homogeneous; Second, the function itself is homogeneous. Here we consider the second for our case because the homogeneity is consistent with the property of self-affinity.

So again, it features the property of self-affinity. Overall, as the initial boundary condition of $u(x,t)$ is unknown, we consider it inhomogeneous. Thus our goal is to find a Green's function's solution to a homogeneous function with inhomogeneous boundary conditions, under which the equation satisfies

$$u(x,t|x_0) = f(x,t), \qquad \lim_{|x|\to\infty} u(x,t) = 0 \; \forall \; t$$

There are many researches being done for the model in our form. Here we have a look at a very rigorous one, M.M Meerschaert and H.-P. Scheffler.[19], in which, a general solution is provided for a fractional equation in a form that

$$\frac{\partial^q p(x,t)}{\partial t^q} - D'^q \frac{\partial^\gamma p(x,t)}{\partial x^\gamma} = 0$$

Where $p(x,t)$ can be considered the PDF of $u(x,t)$ in our case. And the solution is

$$p(x,t) = \frac{t}{q}\int_0^{+\infty} g(x,\xi) f_q\left(t\xi^{-\frac{1}{q}}\right)\xi^{-\frac{1}{q}-1} d\xi$$

Where $g(x,\xi)$ is the Green's function solution; ξ denotes frequency in Hertz, where $2\pi\xi = \omega$; And $f_q$is the PDF of a stable distribution with the Laplace Transform of $e^{-s^q t}$, which in this case is specified with the operator $\mathcal{L}$ such that

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} f(t)e^{-s^q t} dt$$

And a distribution is stable when the sum of independent random variables of this distribution has the same distribution as itself. (Note that the above interpretation are simplified translation from the original, which is too restrictive for demonstration.) In simple words, this solution is the convolution of the Green's function and a PDF but has additional fractional terms comparing to the general form we introduced earlier, which makes it hard to compute q, even f_q and $g(x,\xi)$ are both known.

More importantly, we can see that if Fourier Transform would be applied, fractional terms must be dealt with, for which, we will then review.

4.4.2 Fourier Space Solution

To recall the extension of fractional differentiation, according to Eq. (1.6) we have the left Riesz-Caputo derivative that:

$$D_+^q f(t) = \frac{1}{\Gamma(n-q)}\int_a^t \left\{\frac{d^n}{dt^n} f(t)\right\}(t-\tau)^{n-q-1} d\tau$$

Where

$$n > q > n - 1$$

Which contains a derivative term $\frac{d^n}{dt^n} f(\tau)$ inside the integration. This makes it difficult to be managed. But thanks to [17], actually we can simplify this term such that:

$$\frac{d^n}{dt^n} f(\tau) = \frac{d}{dt} f(\tau) = f'(\tau)$$

The reason why is introduced by the notation of "Nondimensionalization". Which, in general, indicates that if a system has intrinsic resonance frequency, length, or time constant, then these values can be recovered by nondimensionlization. To recall our model, which illustrates self-affine motions, it can be seen that by different scales it is similar to itself and so does a resonance frequency. Thus it implies that in Fourier space, scaling terms might be cancelled out and end up with a simple form.

Recall that a distribution itself is formed up by the convolution product of its tempered distribution and delta function. Now we can re-interpret Eq. (1.2) as the concern to tempered distribution in Fourier space.

$$U(k, t + \tau) = U(k, t)P(k, t)$$

It illustrates that the function after some time can be expressed by the product of its original and the characteristics function of itself, thus we can re-write it into:

$$U(k, t_0 + t) = U(k, t_0)P(k, t)$$

(Note that in general we can let $t_0 = 0$) hence it is easy to see that if the characteristic function of its distribution is known, q is also acquirable. In this case, as mentioned the PDF $p(x, t)$ we considered, which we now recall that:

$$\frac{\partial^q p(x, t)}{\partial t^q} = \frac{\partial^\gamma \{p(x, t)s(x)\}}{\partial x^\gamma}$$

Where $s(x)$ can be arbitrary and its characteristic function is given by:

$$P(k, t) = e^{-a|k|^\gamma t^q}, \qquad 0 < q \leq 1$$

In order to let the equation be consistent with our self-affine assumption, which is to be homogenous. According to some property, we previously found out about k, γ, q, ω, Eq. (1.2) can be turned into a form with the initial condition that:

$$U(k, t) = U(k, t|t_0) \exp\{-a|k|^\gamma t^q\}$$

$$U(\omega, t) = U(\omega, t|t_0) \exp\left\{a \frac{(i\omega)^q}{D'^q} t^q\right\}$$

$$U(\omega, t) = \frac{U(\omega, t|t_0)}{\exp\{-(i\omega)^q\}} \qquad Eq\ (1.92)$$

Note that in here $D* \neq -\frac{t^q}{\tau}$, as τ is not considered to be very close to zero. Instead, as τ gets greater, $D*$ turns out to be:

$$\frac{U(k,t+\tau)-U(k,t)}{\tau} \approx -\frac{a}{\tau}|k|^{\gamma}t^{q}U(k,t) = D*|k|^{\gamma}U(k,t)$$

$$D* = D'^{q} = -\frac{at^{q}}{\tau}$$

As a result, the homogeneity of $p(x,t)$ has also been recovered in Eq. (1.6), when we let $s(x) = D*$ be a function of t. Furthermore, the PDF $p(x,t)$ of the financial signal in our model is Lévy distributed, and its tempered distribution is similar to its counterpart at some other time. When $\gamma >= 2$ it is Gaussian, which has been mentioned before. In the simplification of Eq. (1.92), the fractional order terms are cancelled out which again reals the significance of homogeneity or what we previous mentioned, nondimensionalization, but moreover, the self-affinity. In other words, the signal has the same structure when being observed by different time scales, and the structure as a property itself that would not be affected by scaling, for example, say, the density of a cube is not affected by its size or shape.

However, in terms of giving an analytical expression of the signal, due to the unknown initial solution, we cannot make further hypothesis but to compute the index q by the real time data to see how it behaves. So in next section, we propose a numerical method to compute the asymptotic value of γ when $u(k,t), t$ are given.

5. Numerical Approach for Non-Stationarity

5.1 Asymptotic Approach

Based on the analysis of Hurst exponent and Kwiatkowski–Phillips–Schmidt–Shin (**KPSS**) test. The stationarity of the price movements cannot be guaranteed, therefore the potential non-stationarity could lead to the variations of the fractal dimension exponent q, which makes it more difficult to scale comparing to a stationary model. Conventionally, non-stationary fractal time series are analysed by De-trended Fluctuation Analysis (DFA), which is first introduced by Peng et al [20], and then improved by [21]. Roughly speaking, it is a “signal stationalizing” process by signal de-trending linearly or non-linearly until it appears to be stationary. As it is previously mentioned, many researches in de-trending methods and obtaining multiple orders of data residuals are quite successful for acquiring stationary and independent data for some specific financial instruments, especially for S&P 500, to which, in recent years, by J Gatheral 2014 [22] linear stationarity of volatility are found which is similar to solution of Lévy flight in harmonics potential [23]. Generally, its key idea to deal with potentially non-stationary time series is to suppose a relatively small time segment in which the signal stays stationary, provided with mathematical proof in corresponding circumstances [22]. Also in [9], the significance of this idea contributes decent work when dealing with non-stationarity. It is quite advantageous not to over-cut-off the auto-correlations of the signal as we consider it somehow shows long term or short term memory, which is consistent with the fractal time series hypothesis.

Overall, irrespectively to a complex de-trending model, we rather asymptotically compute q by regression for a small segment of time as yet it is relatively stationary within. According to J. Blackledge, 2010 [2], in a mathematical sense:

$$\left|\frac{\partial q}{\partial \tau}\right| \ll \left|\frac{\partial u}{\partial t}\right|$$

It is considered as that q in a time window τ changes in a slower rate than the market price u, of which the Fourier Transform respect to time is Eq. (1.92):

$$U(\omega, t) = \frac{U(\omega, t|t_0)}{\exp\{-(i\omega)^q\}}$$

As there are complex terms of i and potential negative values which are not friendly to logarithm we will then apply, we could take the absolute value for both sides. Because q indicates the relationship between time scaling and price magnitude, this relationship is of course not affected by signs. In addition, the complex number i is considered "positive" while interpreted in complex space, and $-i$ vice versa, which means a logarithm operation is also appropriate to be applied. And q can be represented as below:

$$\exp\{-(i\omega)^q\} = \left|\frac{U(\omega, t|t_0)}{U(\omega, t)}\right|$$

$$2q ln(\omega) = ln\left\{ln^2\left\{\left|\frac{U(\omega, t|t_0)}{U(\omega, t)}\right|\right\}\right\}$$

$$q = -\frac{ln\left\{ln^2\left\{\left|\frac{U(\omega, t|t_0)}{U(\omega, t)}\right|\right\}\right\}}{2ln(\omega)}$$

Alternatively, to avoid the complex term, based on another form of Reisz fractional derivative in Eq. (1.6), we see that its "clean" of complex term with i.

$$\boldsymbol{\mathcal{F}}\left\{D_x^q f(x, t)\right\}(x) = |\omega|^{2q} F(\omega, t)$$

By which we re-organize Eq. (1.91) and get

$$(\omega)^{-2q}\boldsymbol{\mathcal{F}}\left\{\frac{\partial^q u(x, t)}{\partial t^q}\right\}(\omega) = -|k|^{-\gamma}\boldsymbol{\mathcal{F}}\left\{\frac{1}{D*}\frac{\partial^q u(x, t)}{\partial t^q}\right\}(\omega)$$

$\Rightarrow$
$$\frac{(\omega)^{2q}}{D'^q} = -|k|^{\gamma}$$

Thus Eq. (1.92) becomes

$$U(\omega, t) = U(\omega, t|t_0)\exp\{(\omega)^{2q}\}$$

We can see that it is not easy to be approximated with this representation, as the unknown initial condition function cannot be separated out. And even if the initial condition can be approached to some extent, still, the error could be gradually dominant due to the large amounts of operations with logarithm and square. It is an approximation due to the unknown somehow. And it is more preferable if the approximation is applied elsewhere so that we can obtain a simple form. Thus we expand the exponential term and apply similar procedures as before.

$$\exp\{(\omega)^{2q}\} = 1 + (\omega)^{2q} + \frac{(\omega)^{4q}}{2!} \ldots \approx 1 + (\omega)^{2q}$$

And the equation can be written as:

$$U(\omega, t) = \frac{U(\omega, t|t_0)}{\{1 + (\omega)^{2q}\}^{-1}}$$

$$l\,n\{1 + (\omega)^{2q}\} = l\,n\{|U(\omega, t)|\} - l\,n\{|U(\omega, t|t_0)|\}$$

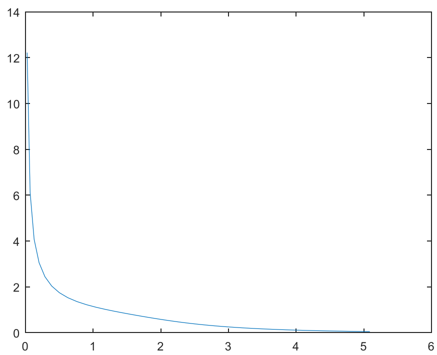
Still, it is not convenient to cancel the complex number neither the exponential. And we propose

$$l\,n\{(\omega)^{2q}\} = l\,n\{|U(\omega, t)|\} - l\,n\{|U(\omega, t|t_0)|\}$$

By considering the convergence of their difference

$$\lim_{|\omega| \to \infty} l\,n\{1 + (\omega)^{2q}\} - l\,n\{(\omega)^{2q}\} = 0 \ \ \forall\, q > 0$$

Which we compute the error for further illustration.



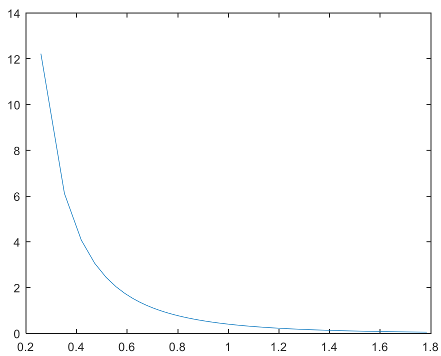


Figure (9-10): it shows that as angular frequency gets larger the error reduces (left: q=1.5; right: q=0.5)

Irrespectively to the value of q, the graph indicates that relatively large angular frequency in the Fourier domain is more preferable to make a well-perform approximation of the equation above. In other words, in the Fourier space we rather take, say, some larger output values or use some method to smoothly deal with, which in here, we consider the Fast Fourier Transform Algorithm.

Therefore, to be continued with our derivation, we can arrive at a simple equation.

$$l\,n\{|U(\omega, t)|\} - l\,n\{|U(\omega, t|t_0)|\} = 2ql\,n(\omega)$$

Thus, as the values of $\omega, U(\omega, t)$ can be obtained by Discrete Fourier Transform with computer for a period of time of data, we can compute q with linear regression methods to fit a line that best optimizes the values.

5.2 Implementation and Results

As we can see that except the initial condition is still left unknown, the whole equation can be treated in a simple linear form such as

$$y = b + ax$$

Where we can consider the constant b corresponding to the initial condition and a to the Levy index γ that we are going to compute; y, x is logarithm of $|U(\omega, t)|^2, \omega$, respectively. The goal of linear regression is to fit a line that minimizing the orthogonal distance apart from each individual data point in the two dimensional plane of market price and local time. As a result the tangent rate of the line is the approximation of γ and b the value respect to local initial condition.

To the beginning of the algorithm, we set a length of moving time interval noted as τ, within which, the distribution of price is assumed to be stationary. For the total data with a length t, we have $t - \tau + 1$ moving intervals, for each we compute a corresponding γ by linear regression. The value of τ should be chosen appropriately, relatively small but adequate for regression.

Next, we Fourier transform the chosen data in the vector of length τ, where the subscript of the scalars in the output vector corresponds to each angular frequency respectively in that time range, so we need another vector to contain those. Be careful that the angular frequencies in frequency space are just scales in the frequency axis, like real numbers in x-axis, thus there is no need for a transformation of time values respected with π as that:

$$\omega = \frac{2\pi}{\tau} = 2\pi f, \qquad 0 < \tau \ll t$$

Where f denotes the general frequency in Hertz. Thus scaling can be ignored, and the angular frequencies are just the elements in subscript vector, which is consistent with the sequence order of inputs that are just integers from one to τ. As previously discussed, values should be taken carefully, as only some of them reals significance and worth to used. The Discrete Fourier Algorithm is considered one of the best numerical methods so far. Before starting the linear regression, the absolute values should be taken, and then comes to logarithm.

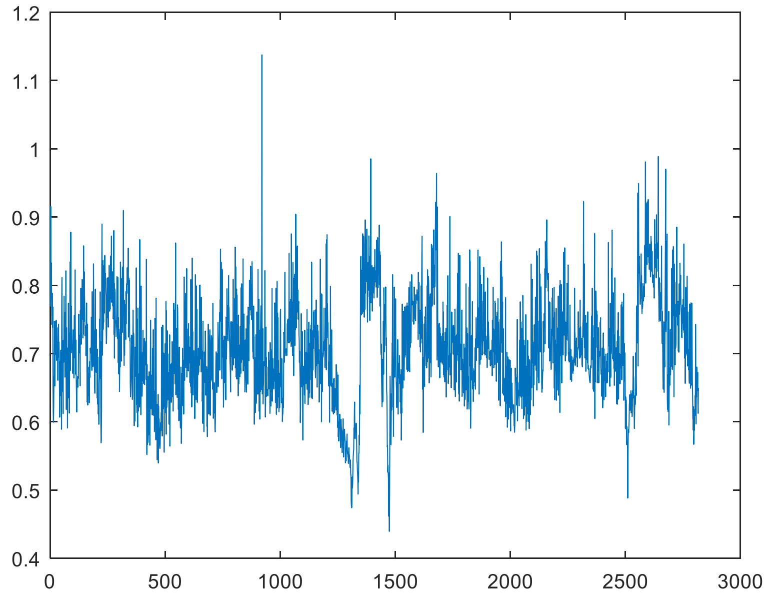


Figure (11): values of q with moving window length 128

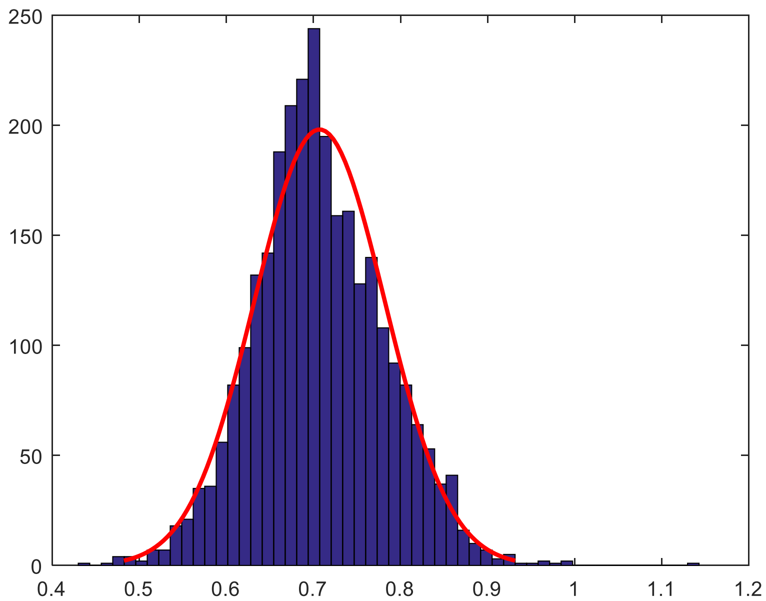


Figure (12): Histogram of q to fit a Gaussian

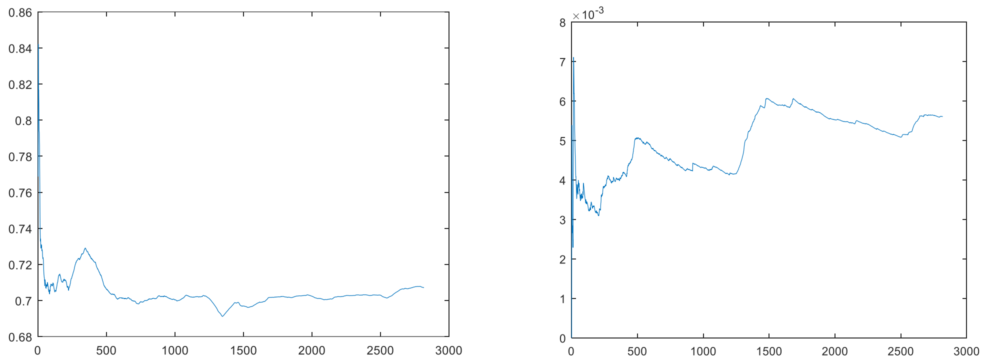


Figure (13-14): left: moving first moment of q; right: moving second moment of q.

As we can see that the q values cannot be concluded to distribute in Gaussian with much sharper peak. In general, it fluctuates between zero and one, while sometimes it gets greater than one. As we know that the Lévy index $\gamma = 2 - 2q$, where $\gamma \geq 2$ can attribute to Gaussian, if $q \leq 0$, which is also possible for real data, because the slope of the regressive fit line could, of course, be negative, in addition the equation does not lose its meaning in that kind of circumstance. A q goes out of the range of $0 < q \leq 1$ could only indicate the non-fitness of this model, which in our case, is rare and it only happens once that can be seen clearly from Figure (12) at the x-axis is around 900.

Overall, it can be concluded as that the raw price (not after cutting residuals or de-seasonalization, etc.) of the electricity market is neither Gaussian as far as we can see; nor is stationary as its dimension indicator q is neither constantly stable nor showing any sign of convergence.

6. Conclusion

From our study we conclude that the spot price of the electricity market is not Gaussian distributed but has memory, in terms of which a fractal model describes it better. Also empirically, the persistent seasonal or cyclic behaviour of the prices can be taken as support. In the model we build, keeping the homogeneity preserves its fractional behaviour, so that we can approximate its dimension indicator q that shows the non-stationarity of the prices, which contradicts the KPSS test; and with a very essential relationship to Lévy index that $\gamma = 2 - 2q$, which means if q is found to be negative in real data, which reveals a Gaussian by $\gamma \geq 2$, and this does not happen in our case, so again the process is not Gaussian.

In fractional modelling, homogeneity is an essential part to hold. Otherwise, the process loses its “memory” to become uncorrelated. Which does not suit our case. However, irrespective to the periodic high spikes of the spots, the small fluctuations could be independent, it is better if our model is not fully fractional so that it can capture the independent occasions, which requires further study. And the accuracy of approximation of q depends on whether the initial condition is known, where further explorations can be done for extension.

Appendix

Matlab codes and data are in file named codes. Where the daily data at 0 a.m is used in this paper, the corresponding hourly data is in the file named BEUR.txt.

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Figures and Plots

Figure(1). Histogram of 0 a.m daily spot, Matlab codes available in Appendix. File name:GBM.m

Figure(2). Brendan Ryan. Available at:

https://en.wikipedia.org/wiki/Fractal_dimension#/media/File:Fractaldimensionexample.PNG

Figure (3). Akarpe. 32-segment quadric fractal. Available at: https://en.wikipedia.org/wiki/Fractal_dimension#/media/File:32_segment_fractal.jpg

Figure (4). Prokofiev.Box-counting for coast line of the Great Britain. Available at:

https://en.wikipedia.org/wiki/Minkowski%E2%80%93Bouligand_dimension#/media/File:Great_Britain_Box.svg

Figure (5-7). R/s method computations for Hurst exponents. Matlab codes available at: file name: Hurst_Exponents.m

Figure (8). (daily 0 a.m) second moments. Matlab codes available at: moving_second_moment.m

Figure (9-10). Error (left: q=1.5; right: q=0.5). Matlab codes available at: log_error.m

Figure (11). values of q with moving widow length 128: Matlab codes available at: compute_q.m

Figure (12). Histogram of q to fit a Gaussian. Matlab codes available at: GBM.m

Figure (13-14). First and second moment of q. Matlab codes available at: q_statistics.m