Synchronisation in Time-Delayed Coupled Oscillators

by Hafsa Nayim

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Author: Hafsa NAYIM
ID: 130812805
Supervisor: Dr Wolfram JUST

Module Code: MTH038
Year 2014/15

MSc MATHEMATICS, School of Mathematical Sciences , Queen Mary
University of London

Abstract

Synchronisation is this deep tendency towards order in the natural world around us and is one of the most persistent drives in the cosmos. The study of synchronisation is an extremely prominent one, its implications being studied in areas of biology, engineering and physics. To model such synchrony, simple time-delayed coupled units are investigated with a view towards synchronisation properties. The behaviour of such oscillators are analysed and conditions upon its coupling are derived in order to bring the system to a synchronised state. For that purpose, we consider two linear complex valued differential equations coupled by a time delayed feedback force. The corresponding linear stability analysis yields a quasipolynomial whose solutions are investigated using a perturbation scheme. Conditions are thus found that enable the system to become stable and the importance of the time delay is evident in bringing synchrony to the coupled units.

Signed Declaration

I hereby declare that this work is entirely my own and all sources have been fully acknowledged.

Name: HAFSA NAYIM

Signed:

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Chapter 1

Introduction

1.1 Synchronisation: A study

Synchronisation is a phenomena that takes place around us every day with both animate and surprisingly, inanimate objects. These objects exhibit beautiful patterns of synchrony; in nature we see the spontaneous synchrony of flocks of birds and fish swimming in organised schools, the fluidity of such movements being truly mesmerising. Taking a trip to South-east Asia, the synchronous behaviour of fireflies that fill the river banks at night can be witnessed, all flashing in rhythm, as male fireflies flash to attract the female fireflies. This precise performance has sparked much interest of biologists and more specifically mathematicians who have studied their synchronisation properties and have thus drawn revolutionary conclusions. These patterns are evident even in biology; the reason you are able to read this thesis is due to the synchrony of thousands of pacemaker cells in your heart that exhibit a rhythm, sending a signal to the heart for it to pump blood and oxygen around the body. In all essence, the human body relies on synchrony to keep alive. (Strogatz, 2003). But why has this sparked so much interest for mathematicians?

Pikovsky et al. (2003) describes synchronisation to be an "adjustment of rhythms of oscillating objects due to their weak interaction." These objects that interact can be seen as individual oscillators, as they change with time and a group of these oscillators can be investigated as coupled units as they exhibit some sort of interaction or communication together, which will be described as its coupling. An example would be the groups of fireflies that communicate with light, or two pendulums that hang from a common support, interacting through the vibrations of this common support. Furthermore, synchrony of coupled units can practically depend on a specific time delay in the coupling. Such behaviour is apparent in many physical systems around us, like the human body and the time delay in neural communication or even delay in radio signal transmission (Klinshov and Nekorkin, 2013). Do such a systems still synchronise even when there is a delay present? Such matters are crucial to the study of synchrony. Non-linear dynamics helps to model such behaviour of coupled oscillators and analyse how and which conditions allow the system to become synchronous.

1.2 Objectives

This dissertation aims to study the behaviour of two self-sustained oscillators, that is systems capable of producing their own rhythm, that adjust their rhythms by the coupling. This will be done by looking at the equations of motion that govern two coupled oscillators and thus obtain conditions on the coupling coefficients that enable the system to become stable. It is important to note that these oscillators are studied in the most mathematical sense. That is, they are analysed independent of their nature, not considering any external physical units, mechanics or electronics. They are given the name oscillators, as the dynamical properties of the units display oscillatory behaviour, relative to the frequency, as will be shown in this thesis.

These systems are further analysed by adding a time delayed feedback force. This is to see how the system will react to time delay and what conditions the coupling will need to have in order for the system to become stable. The systems are reduced to an eigenvalue problem that helps us determine stability. By considering areas of linear algebra and application of perturbation techniques, we delve into stability analysis that allows us to understand the behaviour of a system under change.

1.3 Summary

The thesis is organised as follows:

Chapter 2: This chapter begins by a simple analysis of uncoupled oscillators, resulting in two individual one dimensional systems, which are reduced to an eigenvalue problem. These are then individually analysed by considering real and complex variables and the implications it has. It is interesting to see the qualitative behaviour of an uncoupled system and to then draw necessary comparisons for when synchrony does actually occur like in a coupled case.

Chapter 3: In this chapter, the main focus will be on a coupled system but without time delay. After obtaining an eigenvalue problem, analysis into considering real and complex variables leads us to derive conditions for stability on the coupling. When dealing with complex variables, a perturbation expansion is implored to obtain stability conditions for the oscillators.

Chapter 4: The preceding chapters provide the floor for this chapter as we consider the coupled system but now with a time delay added. A similar perturbation expansion is used, that considers disturbances in the system to get stability conditions on the coupling.

Chapter 5: This chapter will conclude the dissertation whilst summarising its findings. It will also talk about further research that can and possibly has been carried out in this field to further the understanding of the synchronised behaviour in time delayed systems.

Chapter 2

Uncoupled Oscillators

The study begins by looking at a system of uncoupled oscillators and observing its behaviour. The two oscillators, x and y have been modelled as follows:

$$\begin{aligned} \dot{x} &= ax(t) \\ \dot{y} &= by(t) \end{aligned}$$

These two equations are uncoupled, meaning there is no x in the y- equation and vice versa. Physically, this implies that there is no interaction between the two oscillators. In this simple case, the equations may be solved individually. This system is written in matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The solution is:

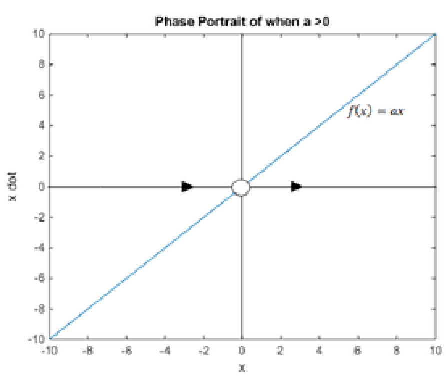
$$\begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{bt} \end{aligned}$$

Where x_0 and y_0 are the initial conditions for x and y respectively. The behaviour of this system depends on the variables a and b. Let's consider the cases when a, b are real and when complex.

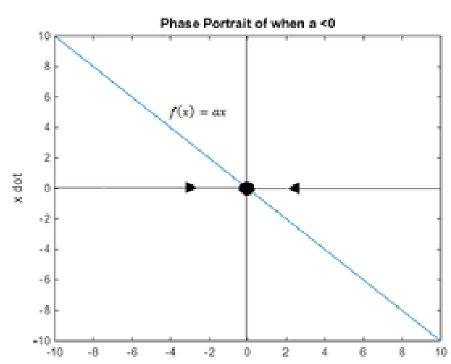
2.1 Real Variables

The first case considered is when $a, b \in \mathbb{R}$. To analyse these systems, consider a fluid that is flowing along the real line of the x axis, whose velocity varies according to the function given by $f(x) = \dot{x}$. At the points where the velocity is 0, there is no flow. These points are called fixed points, satisfying $f(x^*) = 0$ (Strogatz, 2000a). Fixed points help obtain a qualitative image of the system and thus infer the long term behaviour.

Going back to the analysis of the system of the two oscillators, when solving for the fixed points, it is clear to see that a fixed point can always be found at the origin, when $x^* = 0$. To classify this fixed point, the case when the coefficient a is positive or negative is considered. *Figure 2.1* shows the flow of the fluid, which is called the trajectory and this is represented in a phase portrait, shown below. Matlab was used to illustrate this flow and to plot a representation of the phase portrait, showing the different behaviour of the fixed point. By plotting the linear equations in Matlab and then consequently finding fixed points, the resulting qualitative behaviour can be witnessed.



(a) Trajectory for when $a > 0$



(b) Trajectory for when $a < 0$

Figure 2.1: *Phase portraits of system:* $\dot{x} = ax(t)$

The phase portraits above help observe the qualitative behaviour. When $a > 0$, then x is monotonically increasing and the flow is towards the right. Here the fixed point at the origin is unstable as the flow is moving away from this point, depicted by the open dot in *Figure 2.1a*. Furthermore, looking at *Figure 2.1b*, when $a < 0$, then x is monotonically decreasing. The fixed point at the origin here is stable, shown by the solid black dot and here the flow moves towards the fixed point. (Strogatz, 2000a). The results from this analysis can also be applied to the equation governing the second oscillator involving y, since the equations are uncoupled and can be considered independently.

The systems discussed above represent one dimensional flow, where the trajectories either approach a fixed point or diverge. The behaviour of the trajectories in systems of this nature is only monotonic increase or decrease and there is never a periodic solution. Therefore, when the variables $a, b \in \mathbb{R}$ in this uncoupled system, there is never any oscillatory behaviour.

2.2 Complex Variables

Next, the case when both a, b are complex numbers is considered. Let a be a complex number given by:

$$a = a_R + ia_I$$

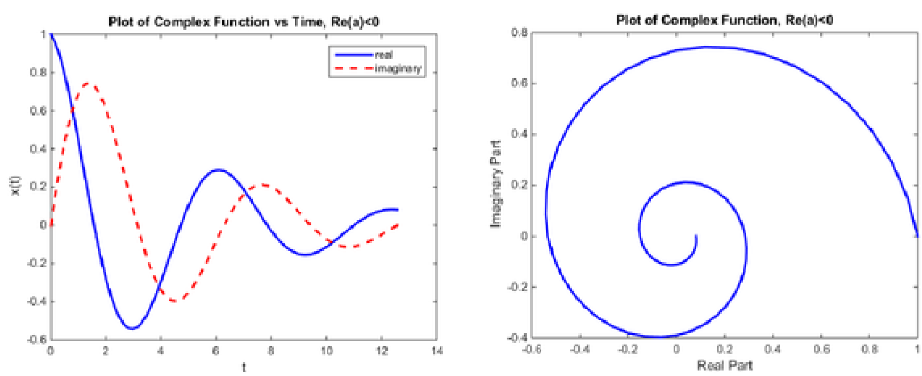
where a_R denotes the real part and a_I denotes the imaginary part of the complex number. Again, since the equations are uncoupled it is sufficient to consider the behaviour of the x equation and the same behaviour will apply to the second oscillator given by the y equation. When a is complex, the solution now is:

$$x(t) = e^{a_R t} e^{a_I t}$$

Using Euler's formula for complex numbers, this can we written as:

$$x(t) = e^{a_R t}[\cos(a_I t) + i\sin(a_I t)]$$

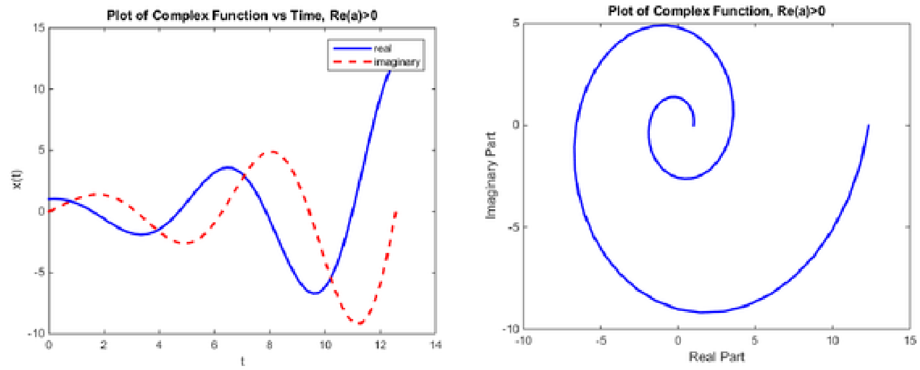
Since the solution now involves trigonometric functions, the system will oscillate with frequency a_I, otherwise it is exponentially increasing or decreasing depending on the sign of the real part. First, we consider the case when $a_R < 0$. Matlab was used to plot the real and imaginary part of the function, using the commands *real(x)* and *imag(x)* and to plot the spiral. Upon taking arbitrary values for a_R, a_I, the behaviour of such a system can be seen in *Figure 2.2,* where a plot of the time series of the solution and the behaviour of fixed point at the origin is shown. The time series in *Figure 2.2a* shows decaying oscillations due to the negative real part and the origin is stable in *Figure 2.2b*, where the flow is moving in towards the fixed point (Strogatz, 2000a).



(a) Time Series (b) Stable Spiral

Figure 2.2: *Behaviour of complex variables for* $a_R = -0.2$, $a_I = 1$

Next, the behaviour of the system is observed when the real part is now positive. From *Figure 2.3a* below, it is clear to see that the time series shows growing oscillations. Again $a_I = 1$ was chosen and with $a_R > 0$, the fixed points will be an unstable spiral, shown in *Figure 2.3b*, with the flow moving away from the origin. In general, if $a_R < 0$, the oscillations will be damped while if $a_R > 0$, the oscillations will be amplified.



(a) Time Series (b) Unstable Spiral

Figure 2.3: *Behaviour of complex variables for* $a_R = 0.2, a_I = 1$

A brief introduction on a system of two uncoupled oscillators was discussed in this chapter, with a view towards looking at the stability and qualitative behaviour. It is interesting to see how the behaviour of the system changes when the variables change from real to complex. In one instance there is monotonic behaviour, whereas in the other, oscillatory patterns can be observed. This will be something that will be perceived in the cases discussed for coupled systems, where changing the variables from real to complex give different conditions on synchronisation. This chapter provides a framework for the subsequent chapters, in which similar equations will be utilised to model the oscillators and synchronisation will be looked at.

Chapter 3

Coupled Oscillators

This chapter seeks to study the same system as in Chapter 2, but now when the two oscillators are coupled by coupling coefficients K_1 and K_2. The following model is considered:

$$\begin{aligned}\dot{x} &= ax(t) - K_1(y(t) - x(t)) \\ \dot{y} &= by(t) - K_2(x(t) - y(t))\end{aligned}$$

The aim here is to find conditions on the coupling coefficients that enable the system to become stable. Practically, stability can be seen as some sort of stimuli, a small disturbance that shakes the system and so stability is the system coming back to its fixed state. It is important to note that the system always begins as unstable, because if it began as stable, the oscillators would already be in synchrony and thus there would be no need to stabilise.

Therefore, the idea is to bring this system to stability by placing conditions on K_1 and K_2. The two-dimensional linear system above can be written in matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a + K_1 & -K_1 \\ -K_2 & b + K_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Stability of linear systems can be found by reducing the above system to an eigenvalue problem. This is important because, analysis into the eigenvalues shows that the system is stable if the real part of the eigenvalue is negative (Strogatz, 2000a).

As shown in Chapter 2, linear systems like this give exponential solutions, so for the above system, general solutions of the form below are sought:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Substituting this into the two-dimensional system and cancelling out the non-zero scalar factor $e^{\lambda t}$ yields:

$$\lambda \mathbf{x} = A\mathbf{x}$$

Where the boldface x represents the matrix $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, which is an eigenvector of matrix A with corresponding eigenvalue λ and with

$$A = \begin{pmatrix} a+K_1 & -K_1 \\ -K_2 & b+K_2 \end{pmatrix}$$

also known as the Jacobian matrix (Strogatz, 2000a). From this, the eigenvalues can now be deduced.

As seen in Chapter 2, when considering the variables a, b as real or as complex, there was a significant difference in the solution and behaviour of the system. Hence, it can be observed here too that changing these variables from real to complex will give different conditions for stability on K_1 and K_2. These two different cases will be considered next.

3.1 Real Variables

The aim now is to find the eigenvalues, when $a,b \in \mathbb{R}$, since they determine stability, which will thus give conditions on the coupling. Recalling the Jacobian matrix:

$$A = \begin{pmatrix} a+K_1 & -K_1 \\ -K_2 & b+K_2 \end{pmatrix}$$

The eigenvalues of this matrix are given by the characteristic equation $det(A-\lambda I)=0$, where I is the identity matrix. So the characteristic equation here becomes:

$$det \begin{pmatrix} a+K_1-\lambda & -K_1 \\ -K_2 & b+K_2-\lambda \end{pmatrix} = 0$$

This yields a long equation with many unknowns making it difficult to compute the eigenvalues. However, the actual eigenvalue is not necessarily needed, as all that is required to determine is the real part of the eigenvalue for stability. So how to resolve if all the eigenvalues have negative real part? This can be done by checking some conditions, known as the Routh-Hurwitz criterion.

Routh-Hurwitz Criterion:

For a two-dimensional linear system, the equilibrium is stable (real parts of the eigenvalue are negative) if:

$$\begin{aligned} Trace(A) &< 0 \\ Determinant(A) &> 0 \end{aligned}$$

where A represents the Jacobian matrix. The trace and determinant of this matrix determine the eigenvalues and subject to the conditions above, they ascertain stability (Li and

Wang, 1998). Furthermore, it is important to note that the sum of the eigenvalues equals the trace of the matrix (also found by the sum of the entries on the main diagonal), and the product of the eigenvalues equals the determinant (also found by using $ad - bc$, where a, b, c, d represent the entries in the matrix respectively).

To understand these conditions a bit better, we look at if, $a, b \in \mathbb{R}$, then the eigenvalues will be real or complex conjugates. Also, if we assume that the system is stable, then these eigenvalues will have negative real parts. Therefore, we can see how this satisfies the criteria, for both real and complex conjugates eigenvalues. Suppose the eigenvalues are $\lambda_1 = p$ and $\lambda_2 = q$, where $p, q < 0$ and $p, q \in \mathbb{R}$. Then:

- The trace would give: $Trace = \lambda_1 + \lambda_2 = p + q < 0$. Since, the sum of two negative real numbers is always negative, this verifies the Routh-Hurwitz criteria.
- The determinant would give: $Determinant = \lambda_1 \lambda_2 = pq > 0$. Since, the product of two negative real numbers is positive, this verifies the criteria for real eigenvalues.

Suppose now that the eigenvalues are complex conjugates $\lambda_{1/2} = p \pm iq$, where $p, q \in \mathbb{R}$, with negative real parts (since the system is stable). Then:

- The trace is: $Trace = \lambda_1 + \lambda_2 = p + iq + p - iq = 2p < 0$. Since two times a negative real number will always be negative, hence verifying this condition.
- The determinant is: $Determinant = \lambda_1 \lambda_2 = (p + iq)(p - iq) = p^2 + q^2 > 0$. Since both values are squared, the determinant is always positive, thus verifying this condition for eigenvalues that are complex conjugates.

The above looked at understanding the conditions of the Routh-Hurwitz criterion a bit better for eigenvalues that are real or complex conjugates and how they are verified. Eigenvalues that are complex will be explored later in Chapter 3.2. Thus, using the Routh-Hurwitz criterion, the conditions of stability on our system can be determined in terms of the coupling as:

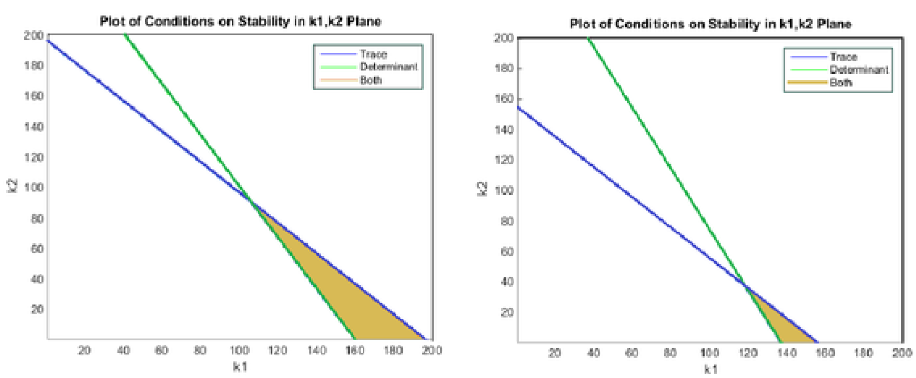
- The Trace gives: $\lambda_1 + \lambda_2 = a + K_1 + b + K_2 < 0$. This gives the condition in terms of the coupling:

$$K_1 + K_2 < -(a+b)$$

- The determinant gives: $\lambda_1 \lambda_2 = ab + aK_2 + bK_1 > 0$. This gives the condition in terms of the coupling:

$$aK_2 + bK_1 < -ab$$

Constraints need to be placed upon the variables a, b where both a, b need to be positive in order to satisfy the inequalities and both need to be non-zero in order for the results to be valid. To understand these conditions on stability, Matlab was used to visualise them by plotting linear inequalities and using the *contour* command to shade, as seen by *Figure 3.1*.



(a) Stability conditions for a, b small (b) Stability conditions for a, b large

Figure 3.1: *Conditions on stability for when a, b are real*

By taking small positive arbitrary values for a, b, the conditions are illustrated as linear inequalities with negative gradients, satisfying the yellow region as shown in the graph. There are two stability boundaries, one caused by the inequality for the trace and the other caused by the determinant. Judging by the graph, stability doesn't occur for negative K_1

values, however by extension, stability seems to occur mostly for negative values for K_2. As shown in the transition from *Figure 3.1a* to *3.1b*, when the values for a, b are increased, the gradients of the inequalities for the trace and determinant become more negative. This in turn shifts the stability region more towards the negative K_2 values. So, with a change in parameter values, the conditions exhibit the same type of structure qualitatively, but with the quantitative values slightly changing.

Interpreting this all practically, it can be seen that when the variables a, b are real, stability doesn't occur for negative values of the coupling coefficient on the first oscillator, K_1. Not only this, but the system is only stable when these variables a, b are positive. Thus, this gives the general conditions on the coupling to ensure the system to come to a synchronous state.

3.2 Complex Variables

Having explored the case when the variables a, b were real, it is important to see what conditions on stability will be obtained when the variables are complex. However, for the complex valued case the situation is a bit more involved. Recall the same system for the oscillators:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a+K_1 & -K_1 \\ -K_2 & b+K_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Since $a, b \in \mathbb{C}$, let:

$$\begin{aligned} a &= a_R + ia_I \\ b &= b_R + ib_I \end{aligned}$$

Where a_R, b_R and a_I, b_I denote the real parts and imaginary parts of a, b respectively.

Suppose now that the eigenvalues will be computed using the above complex values and using $det(A-\lambda I)=0$. The characteristic equation obtained will be more complicated. It will give an inconvenient situation, as it will be thus fairly difficult to infer when the real part of the eigenvalue is negative. However, as seen in the previous section, to determine stability the exact eigenvalues are not necessarily required. All that is required is to determine whether the real parts of the eigenvalues are negative, ensuring the system is stable. Therefore, the aim now is to take a simpler approach and draw out the necessary information required of the eigenvalues. This is done by resorting to a perturbation expansion.

3.2.1 Perturbation Scheme

We now take a radical approximation and apply a small perturbation to the variables. Assuming that the real parts a_R, b_R and the coupling coefficients K_1, K_2 are small, formulated with $\varepsilon \ll 1$, the variables become:

$$\begin{aligned} a &= \varepsilon a_R + i a_I \\ b &= \varepsilon b_R + i b_I \\ K_1 &= \varepsilon K_1 \\ K_2 &= \varepsilon K_2 \end{aligned}$$

Now, the aim is to find the eigenvalues using this perturbation scheme. Exploring the most simplest case of $\varepsilon = 0$, this in turn eliminates the coupling terms, so there is an uncoupled system like in Chapter 2. It also eliminates the real part of the variables a, b, leaving only the imaginary parts. The matrix A will now be a diagonal matrix, with ia_I and ib_I on the main diagonal. In this case, the eigenvalues can be read directly from the matrix giving the two eigenvalues as:

$$\begin{aligned}\lambda_1 &= ia_I \\ \lambda_2 &= ib_I\end{aligned}$$

So, one eigenvalue is the imaginary part of the first variable, a and the second eigenvalue is the imaginary part of the second variable, b. Having purely imaginary eigenvalues, this tells us nothing about stability and the system is neither stable nor unstable.

Since the leading orders of these eigenvalues give no information on stability, the system is further perturbed by adding a small contribution to the eigenvalue:

$$\begin{aligned}\lambda_1 &= ia_I + \varepsilon\lambda_1^a \\ \lambda_2 &= ib_I + \varepsilon\lambda_1^b\end{aligned}$$

Here the perturbation parameter is very small, $\varepsilon \ll 1$ and $\varepsilon\lambda_1^a$ and $\varepsilon\lambda_1^b$ are the small first order corrections to the leading part of the eigenvalue. Considering now when ε is non-zero but very small, the aim is to find out what λ_1^a and λ_1^b is, in terms of the other perturbed variables. This is in order to determine the change and thus find the real part of the eigenvalue corrections, which is the approximation to getting the real part of the actual eigenvalues. This will then help find the conditions for stability. These conditions will be given by $Re(\lambda_1^a) < 0$ and $Re(\lambda_1^b) < 0$.

Moreover, so we have eigenvalues with a small correction and now we will substitute this into the characteristic equation, in order to find λ_1^a. As aforementioned, the characteristic equation is given by $det(A - \lambda I) = 0$, or more simply put: (Strogatz, 2000a)

$$\lambda^2 - \lambda Tr(A) + Det(A) = 0$$

Recalling from Chapter 3.1, the trace and determinant of the given matrix, now in terms of the perturbation scheme, will give:

$$\begin{aligned} Tr(A) &= a + K_1 + b + K_2 \\ &= \varepsilon a_R + ia_I + \varepsilon K_1 + \varepsilon b_R + ib_I + \varepsilon K_2 \\ Det(A) &= ab + aK_2 + bK_1 \\ &= \varepsilon^2 a_R b_R + \varepsilon i a_R b_I + \varepsilon i a_I b_R + i^2 a_I b_I + \varepsilon^2 a_R K_2 + \varepsilon i a_I K_2 + \varepsilon^2 b_R K_1 + \varepsilon i b_I K_1 \end{aligned}$$

This is all put into the characteristic equation, along with the first eigenvalue of $\lambda_1 = ia_I + \varepsilon\lambda_1^a$. The hope is that, since ε is very small, all the important information is captured only in the first two terms of the eigenvalue of order $O(\varepsilon)$ and the higher order terms of ε only represent tiny corrections, thus can be ignored. Therefore, by substituting this all in, ignoring higher order ε terms and simplifying, we obtain:

$$0 + \varepsilon(ia_I\lambda_1^a - ib_I\lambda_1^a - ia_R a_I - ia_I K_1 + ib_I K_1 + ia_R b_I) + O(\varepsilon^2) = 0$$

Taking $O(\varepsilon) = 0$ and factoring out $(a_I - b_I)$

$$\lambda_1^a i(a_I - b_I) = i(a_I - b_I)a_R + i(a_I - b_I)K_1$$

Dividing through by the imaginary unit and the factor $(a_I - b_I)$ gives

$$\lambda_1^a = a_R + K_1$$

Recall that the condition on stability is found from the real part of the eigenvalue being negative. The real part of the eignevalue correction is simply $a_R + K_1$. Therefore, the following compact result is found for the condition on the coupling, from the first eigenvalue:

$$K_1 < -a_R$$

The next step would be to now obtain the second condition on stability using the second eigenvalue of $\lambda_2 = ib_I + \varepsilon\lambda_1^b$. Following the same steps as above, first derive the characteristic equation, then substitute and simplify by ignoring higher order ε terms and then finally collecting $O(\varepsilon)$ terms, we obtain:

$$\lambda_1^b i(b_I - a_I) = i(b_I - a_I)b_R + i(b_I - a_I)K_2$$

Which simplifies to:

$$\lambda_1^b = b_R + K_2$$

Again, to obtain the condition on stability, the real part is negative:

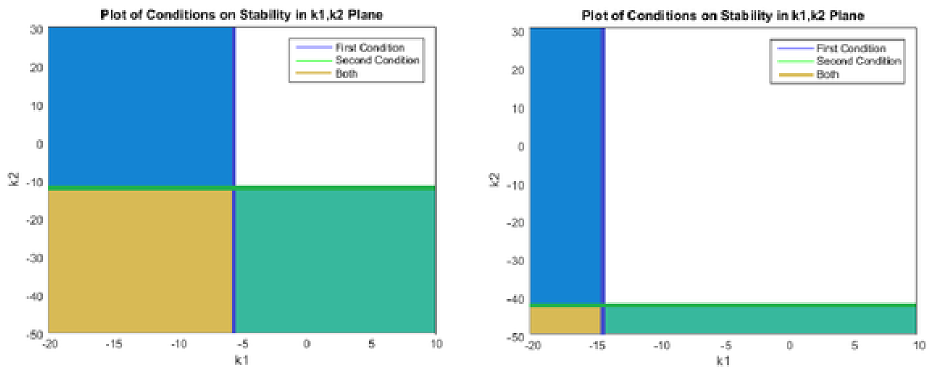
$$K_2 < -b_R$$

Overall, we obtain our two conditions on stability for the coupled oscillators with complex variables a, b as:

$$\begin{aligned} K_1 &< -a_R \\ K_2 &< -b_R \end{aligned}$$

As before, we have two inequalities that place conditions on the coupling coefficients for stability. Notably, it appears that after a detailed analysis, by using perturbative methods, we derive stability conditions that appear remarkably simple; both of these linear conditions depend only on the value of the real part of the variables a, b. It is also very important to place the constraint of $a_R, b_R > 0$, as that is the only way to ensure that the inequality is satisfied. As well as this, it is important that a_R, b_R are both non-zero and distinct and also that a_I, b_I the two imaginary parts or the two frequencies need to also be non-zero and distinct, $a_I \neq b$ in order for all the calculation to make sense and for the result to be valid.

As done in Chapter 3.1, Matlab was used to understand these conditions better, by plotting linear inequalities and using the *contour* command to shade, as visualised in *Figure 3.2* below. As illustrated in the graphs, an overall rectangular structure of the region satisfying



(a) Stability conditions for a_R, b_R small

(b) Stability conditions for a_R, b_R large

Figure 3.2: *Conditions on stability for when a, b is complex*

stability is obtained in the $K_1 - K_2$ plane. The blue region comes from satisfying the first condition of $K_1 < -a_R$, whereas the green region comes from satisfying the second of $K_2 < -b_R$ and thus the yellow region comes from satisfying both inequalities and where stability lies. *Figure 3.2* displays two graphs, both showing the stability conditions for when the variables are complex, but with an increase in a_R and b_R, as we move from *Figure 3.2a* to *3.2b*. This gives an impression of what happens when we change the parameters; qualitatively both regions look the same as both display a rectangular region for stability, but quantitatively the region shifts. With an increase in both parameters, the region shifts more down towards the negative values for both K_1 and K_2. Since a constraint placed on the conditions is a_R, b_R have to both be positive in order to satisfy $(Re(\lambda_1^{a/b}) < 0$, we can see that the conditions satisfy negative values for K_1, K_2. Therefore, this system will reach a synchronised state when the coupling for both oscillators is negative, when a, b are complex.

Generally, when comparing the conditions on stability for the real and complex case, we can see both give linear, continuous regions. In the complex case, the conditions imply that both K_1 and K_2 take only negative values, even on increasing the parameters, whereas in the real case, when a, b is small, K_1 can only take positive values and K_2 takes a small range of positive values but on increasing the parameters, K_2 takes only negative values. On increasing the parameters in the conditions, both regions for both cases shifts lower and become more negative, which has an impact on the coupling. In the case of the model considered in this chapter, for both real and complex variables, the system comes to a synchronous state for a significantly many coupling values. The regions are continuous, so the stability domain is fairly large. This, however is not always the case, as will be shown in the next chapter.

3.3 Contribution of back-coupling

The study of synchrony began with the model explored in Chapter 2, which involved no coupling; that is, no coupling coefficients were involved in either equation. In a practical sense, there was nothing connecting the two oscillators. This system was unstable and did not synchronise. This was accounted for as we moved to Chapter 3.1 and 3.2 where a coupling scheme was added in the form of the back-coupling terms given in the first oscillator by $K_1(y-x)$ and in the second by $K_2(x-y)$. From here, we saw that the system became stable with different conditions on the coupling at different cases for the variables a, b.

But how important is this back-coupling for stability? In other words, how important is the contribution of the x term in the first equation and the y term in the second? The matter is explored by considering the following system, with no back coupling:

$$\begin{aligned}\dot{x} &= ax - K_1 y\\ \dot{y} &= by - K_2 x\end{aligned}$$

This system is a simpler model than previously seen and is still a coupled system, as both oscillators are connected by the coupling coefficients K_1 and K_2. The coupling is not done by the difference, like in the previous case, but by just one variable. But is this kind of coupling sufficient for synchrony? To explore, as done before, this is written in matrix form:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} a & -K_1\\ -K_2 & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

By imploring the same reasoning as done earlier in this chapter, stability of this system is analysed by considering the real part of the eigenvalues. Taking a similar stance to that in Chapter 3.1, using the Routh-Hurwitz criterion, we can determine when the eigenvalues have negative real part and thus becomes stable. In this case, the Jacobian matrix is given by

$$A = \begin{pmatrix} a & -K_1\\ -K_2 & b\end{pmatrix}$$

Recall, by the Routh-Hurwitz criterion, the system is stable when $Trace(A) < 0$ and $Det(A) > 0$. For this system, we get:

$$Trace(A) = \lambda_1 + \lambda_2 = a + b$$

As aforementioned, the trace is given by the sum of the eigenvalues. In this case, the sum of the eigenvalues is just the sum of the two coefficients a and b. However, it is important to note that both individual systems always starts off as unstable, because if they began as stable, there would be no need to stabilise and bring it to a synchronous state! Therefore, for the real case, $a+b>0$, so the sum of the two eigenvalues is always positive, as the system is unstable. In this case, either both eigenvalues has to be positive or at least one of them needs to positive. Because of this, the system can never be stable as stability requires the eigenvalues to have negative real part.

Analysis of the trace is sufficient to prove that the Routh-Hurwitz criteria is not fulfilled here. Therefore, such a system is never stable. It is also interesting to note that the coupling has no effect on the eigenvalue, meaning whatever happens with the coupling coefficients K_1, K_2 this system will always remain unstable. It is due to this that the contribution of the back-coupling is vital. But how does a system described in this section become stable? The situation can be rectified by adding a time delay, which will be the main focus of the next chapter.

Chapter 4

Time Delay

As seen in the previous Chapter 3.3, a system with no back-coupling remains unstable, implying that such a system can never be stable. However, this is not always the case as it can still provide stability when a time delay is added. We study the model equations:

$$\begin{aligned}\dot{x} &= ax(t) - K_1(y(t) - y(t-\tau_2)) \\ \dot{y} &= by(t) - K_2(x(t) - x(t-\tau_1))\end{aligned}$$

These model equations look like something explored in previous chapters, but now with an added time delay in the coupling. The physical implications this has can be visually interpreted as shown in *Figure 4.1* below:

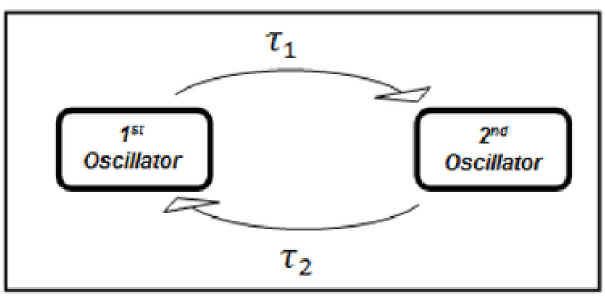


Figure 4.1: *Physical Representation of time-delay*

The time delay exhibited from the first oscillator to the second is given by τ_1 and the time delay exhibited from the second oscillator to the first is given by τ_2. These model equations look similar to what was considered in Chapter 3.3. However, in that case that system was unlikely to stabilise. This chapter aims to investigate whether such a system can stabilise despite not being able to in the previous section, but now with an added time delay.

To investigate this system, as done before, we reduce it to an eigenvalue problem. We seek solutions of the form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

After substituting this into the time-delayed system, we obtain:

$$\begin{aligned} \lambda e^{\lambda t} x_0 &= a x_0 e^{\lambda t} - K_1 (y_0 e^{\lambda t} - y_0 e^{\lambda t} e^{-\lambda \tau_2}) \\ \lambda e^{\lambda t} y_0 &= b y_o e^{\lambda t} - K_2 (x_0 e^{\lambda t} - x_0 e^{\lambda t} e^{-\lambda \tau_1}) \end{aligned}$$

By cancelling out the time dependant exponential terms, we obtain a system where the time delay contribution stays, written in as a two-dimensional matrix form:

$$\lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a & -K_1(1 - e^{-\lambda \tau_2}) \\ -K_2(1 - e^{-\lambda \tau_1}) & b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Now, it is important to analyse the eigenvalues of this system, by finding the characteristic equation, from the Jacobian matrix A, which is given by $\lambda^2 - \lambda Tr(A) + Det(A) = 0$. Therefore, in this case, we have the following characteristic equation:

$$\lambda^2 - \lambda(a+b) + ab - K_1 K_2 (1 - e^{-\lambda \tau_2})(1 - e^{-\lambda \tau_1}) = 0$$

This characteristic equation is not the standard quadratic equation form that we expect and know how to solve. This yields a transcendental equation, which is a more difficult case. Therefore, imploring a similar approach to that of Chapter 3.2, a perturbation expansion will be used as a radical approximation to finding the eigenvalues, in order to then find conditions on the coupling to determine stability.

It is important to note that in previous chapters, the case of when a, b were real and complex were considered and its implications on stability analysed. It can be seen that when $a, b \in \mathbb{R}$, then the Routh-Hurwitz criterion can be applied as done before. However, in this case the trace of the matrix is $Tr(A) = a + b$ which is the situation that was discussed in Chapter 3.3, where there was no back-coupling and this led to an unstable system. Therefore in this case, we will only deal with complex variables for a, b otherwise stability cannot be obtained. This will be considered when applying the perturbation expansion to the system.

4.1 Perturbation Scheme

Perturbation techniques are now utilised in order to find accurate approximations to the solutions of the transcendental characteristic equation, as direct solutions will prove tricky. Also, since we only require conditions on stability, it is sufficient to analyse the approximate solutions in order to understand when the system will be stable. The approach taken in Chapter 3.2 will provide the floor for the analysis here and a similar position will be taken. Considering complex variables, let a and b be given by:

$$\begin{aligned} a &= a_R + ia_I \\ b &= b_R + ib_I \end{aligned}$$

Now, taking approximations on the real parts a_R, b_R of the variables, we assume them to be very small, introducing the perturbation parameter ε, where $\varepsilon \ll 1$. So the variables now become:

$$\begin{aligned} a &= \varepsilon a_R + i a_I \\ b &= \varepsilon b_R + i b_I \end{aligned}$$

Since there is a product of the coupling involved in the characteristic equation, we take a sensible scaling of assuming the coupling K_1, K_2 to be very small, but not as small as in the previous case. So the perturbation expansion is set up in the following way:

$$\begin{aligned} K_1 &= \sqrt{\varepsilon} K_1 \\ K_2 &= \sqrt{\varepsilon} K_2 \end{aligned}$$

This is to ensure that we obtain ε of order 1 in the perturbation expansion. If it was considered the same to that in the previous perturbation expansion, we would get ε of order 2, which would eliminate that term since it would be even smaller.

Moving on, this perturbation scheme is put into our system of equations and then considering when $\varepsilon = 0$ gives the following system of equations:

$$\lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} i a_I & 0 \\ 0 & i b_I \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Here, the coupling vanishes, as does the real part of the variables a, b and we get a diagonal matrix (the entries on the outside of the main diagonal are 0). In this case, the eigenvalues can be read from the matrix off the main diagonal as $\lambda_1 = i a_I$ and $\lambda_2 = i b_I$. This will now be further perturbed and a small eigenvalue correction term will be added, again with the perturbation parameter being very small, $\varepsilon \ll 1$. This gives the following perturbed eigenvalues:

$$\begin{aligned}\lambda_1 &= ia_I + \varepsilon\lambda_1^a \\ \lambda_2 &= ib_I + \varepsilon\lambda_1^b\end{aligned}$$

As done previously, the conditions on stability will now be found from the correction terms λ_1^a and λ_1^b using the perturbation expansion at ε. The perturbed variables a, b, K_1 and K_2 are all substituted into the characteristic equation along with the first eigenvalue and its correction term, $\lambda_1 = ia_I + \varepsilon\lambda_1^a$. After cancelling terms out accordingly and collecting terms of order ε, it is found:

$$0+\varepsilon(ia_I\lambda_1^a - ib_I\lambda_1^a - ia_Ia_R + ia_Rb_I - K_1K_2(1-e^{-i\tau_2 a_I}e^{-\varepsilon\tau_2\lambda_1^a})(1-e^{-i\tau_1 a_I}e^{-\varepsilon\tau_1\lambda_1^a}) + O(\varepsilon^2) = 0$$

The terms of order ε^2 are quadratically small terms in ε and so are negligible. Not only this, but since $\varepsilon \ll 1$, we can also neglect e^{ε} terms too. This gives:

$$ia_I\lambda_1^a - ib_I\lambda_1^a - ia_Ia_R + ia_Rb_I - K_1K_2(1-e^{-i\tau_2 a_I})(1-e^{-i\tau_1 a_I}) = 0$$

The eigenvalue correction is what we are looking for, as the real part of that will determine stability, so on re-writing we get:

$$\lambda_1^a i(a_I - b_I) = ia_Ia_R - ia_Rb_I + K_1K_2(1-e^{-i\tau_2 a_I})(1-e^{-i\tau_1 a_I})$$

The system is only stable when the real part of the eigenvalue is negative. However, going by the above equation, it is difficult to infer clearly where the real or the imaginary part is. Therefore, we can simplify this. Let:

$$(1-e^{-i\tau_2 a_I}) = e^{-i\frac{\tau_2 a_I}{2}}[e^{i\frac{\tau_2 a_I}{2}} - e^{-i\frac{\tau_2 a_I}{2}}]$$

Using the complex identity of $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$, the above becomes:

$$(1-e^{-i\tau_2 a_I}) = 2ie^{-i\frac{\tau_2 a_I}{2}}\sin(\frac{\tau_2 a_I}{2})$$

It's now more clear to see where the real and imaginary part lie. This is done for the second term similar to this, to get:

$$\lambda_1^a i(a_I - b_I) = ia_I a_R - ia_R b_I + K_1 K_2 (4i^2 \sin(\frac{\tau_2 a_I}{2})\sin(\frac{\tau_1 a_I}{2}) e^{-i\frac{\tau_2 a_I}{2}} e^{-i\frac{\tau_1 a_I}{2}})$$

Dividing through by the factor $i(a_I - b_I)$ gives:

$$\lambda_1^a = a_R - \frac{4K_1K_2(\sin(\frac{\tau_2 a_I}{2})\sin(\frac{\tau_1 a_I}{2})e^{-i\frac{\tau_2 a_I}{2}}e^{-i\frac{\tau_1 a_I}{2}})}{i(a_I - b_I)}$$

From here we have a real part and then a complex contribution. It is important to determine the real and imaginary part of this complex term, as we can take out factors:

$$Re\left(\frac{e^{-\frac{\tau_2 a_I}{2i}}e^{-\frac{\tau_1 a_I}{2i}}}{i(a_I - b_I)}\right) = \frac{-ie^{-i(\frac{\tau_2 a_I}{2}+\frac{\tau_1 a_I}{2})}}{(a_I - b_I)}$$

Using Euler's formula for complex numbers $e^{i\theta} = \cos\theta + i\sin\theta$ and using the fact that $\sin(\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$, the above is now considered in terms of trigonometric functions:

$$-ie^{-i(\frac{\tau_2 a_I}{2}+\frac{\tau_1 a_I}{2})} = -i\cos\left(\frac{\tau_2 a_I}{2}+\frac{\tau_1 a_I}{2}\right) + i^2\sin\left(\frac{\tau_2 a_I}{2}+\frac{\tau_1 a_I}{2}\right)$$

This gives the overall real part of this complex contribution as:

$$Re\left(\frac{e^{-\frac{\tau_2 a_I}{2i}}e^{-\frac{\tau_1 a_I}{2i}}}{i(a_I - b_I)}\right) = \frac{-\sin\left(\frac{\tau_2 a_I}{2}+\frac{\tau_1 a_I}{2}\right)}{(a_I - b_I)}$$

This is then put with the real part of λ_1^a, to give the real part of the eigenvalue correction as

$$Re(\lambda_1^a) = a_R + \frac{4K_1K_2(\sin(\frac{\tau_2 a_I}{2})\sin(\frac{\tau_1 a_I}{2})\sin\left(\frac{\tau_2 a_I}{2} + \frac{\tau_1 a_I}{2}\right)}{(a_I - b_I)}$$

As per the condition on stability, this real part needs to be negative, giving the following condition, from the first eigenvalue correction on the coupling:

$$K_1K_2 < \frac{-a_R(a_I - b_I)}{4(\sin(\frac{\tau_2 a_I}{2})\sin(\frac{\tau_1 a_I}{2})\sin\left(\frac{\tau_2 a_I}{2} + \frac{\tau_1 a_I}{2}\right)}$$

Analysis into this condition will be further explored in the following section. However, this is not until the second condition for stability is obtained using the second eigenvalue of $\lambda_2 = ib_I + \varepsilon\lambda_1^b$. Applying the exact same approach as the first eigenvalue, where this eigenvalue along with the other perturbed variables are substituting into the characteristic equation and first order ε terms are collected . The second condition on stability can be derived as:

$$K_1K_2 < \frac{-b_R(b_I - a_I)}{4(\sin(\frac{\tau_2 b_I}{2})\sin(\frac{\tau_1 b_I}{2})\sin\left(\frac{\tau_2 b_I}{2} + \frac{\tau_1 b_I}{2}\right)}$$

These conditions are now further looked at and the different variations of these conditions that can inhibit or enhance stability.

4.2 Analysis into Stability Conditions

Analysis into the two conditions that need to be satisfied in order to gain stability, will now be carried out. These conditions involve a product of the coupling indicating boundaries which will be hyperbolic, which is a general feature of such a coupling scheme. This contrasts greatly to the conditions found in the previous chapter for a coupled system without time delay, which were linear. The region of stability here will depend solely on quite a few parameters that are involved in the conditions, that is the real and imaginary parts of the variables a, b and the time delays. Notably, there is a trigonometric sine function included in the conditions as a product. The sine function is a vital and influential part of these conditions. An interesting property of this function is that it is periodic with period 2π, and so when considering time delay values of τ and $\tau+2\pi$, the same behaviour can be observed . As well as this, the sine function is an odd function. These attributes will prove significant in the subsequent analysis. The aim now is to observe which combination of parameters enable stability. It is fair to say that, since the conditions depend on so many parameters, there are many different combinations of these parameters that will make the system stable. A few of these cases are now studied.

4.2.1 When the conditions diverge

The conditions on stability involve the trigonometric sine function as a product in the denominator, which enables us to understand more clearly where these conditions diverge and thus no stability occurs. It is clear that $\sin(x) = 0$, $\{x = \pi k, k\varepsilon\mathbb{Z}\}$. Therefore, there will be no stability when $\tau_1 a_I = 2\pi k$,$\tau_2 a_I = 2\pi k$ and $\tau_1 a_I + \tau_2 a_I = 2\pi k$ where $k \in \mathbb{Z}$. This also applies to the second condition involving b_I, whereby replacing a_I with b_I will give those constraints on stability. As well as this, it is important to place the constraint of both imaginary parts being distinct, $a_I \neq b_I$. It is notable that not only is the time delay crucial in determining stability, but so are the frequencies a_I, b_I. This is a key result that is evident even upon considering other cases.

4.2.2 Effect of small time delay

The case of when the time delay is small and its implications on the stability region will now be analysed. This will be done by taking different parameter values and to ensure objective observations are made, the parameters for a_R, b_R will remain constant. Small time delay will be looked at, with varying frequencies a_I, b_I. This is because the frequency acts on the time delay and depending on the sign of a_I, b_I, the inequality for the conditions change. So it will be interesting to consider what happens to the stability region when changing the frequencies, for small time delay.

Let's consider taking the following arbitrary values for the parameters, when the frequencies have the same sign, namely $a_I, b_I > 0$ of $a_R = 1$, $b_R = 1$, $a_I = 1$, $b_I = 2$ and taking arbitrary small time delay of $\tau_1 = \frac{\pi}{6}$ and $\tau_2 = \frac{\pi}{9}$. These are values randomly selected, satisfying the criteria we have set out for in this analysis. Using Matlab, a function file was created to obtain the numerical conditions upon $K_1$and K_2 to give the following:

$$\begin{aligned} K_1K_2 &< \frac{1}{0.076} \\ K_1K_2 &< \frac{-1}{0.524} \end{aligned}$$

These stability conditions were plotted in Matlab to give:

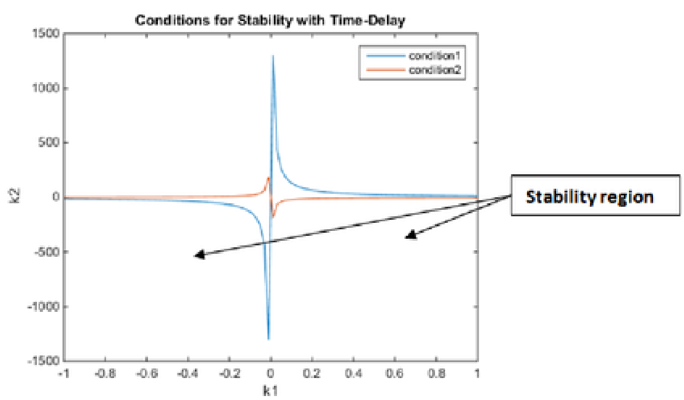


Figure 4.2: *Stability Conditions for small time delay with $a_I = 1, b_I = 2$*

The two conditions can be visualised in *Figure 4.2.* The blue hyperbola represents the first condition involving a_I which gives positive hyperbolic boundaries and the red hyperbola represents the other condition involving b_I, given by negative hyperbolic boundaries. Since the time delay is small and the frequencies are both positive, the sine function in the denominator gives positive values. Both conditions are simultaneously satisfied in the white region below the blue and red hyperbola, indicating the stability region as shown above. This gives a fairly large, continuous domain for stability.

Let's now consider the case when the frequencies are of opposite signs, which leads to the conditions having opposite inequality signs. Keeping all other parameter values the same, the frequencies are altered, taking $a_I = 1$and $b_I = -2$, the conditions now become:

$$\begin{aligned} K_1 K_2 &< \frac{-3}{0.076} \\ K_1 K_2 &> \frac{3}{-0.524} \end{aligned}$$

These conditions can be visualised as below:

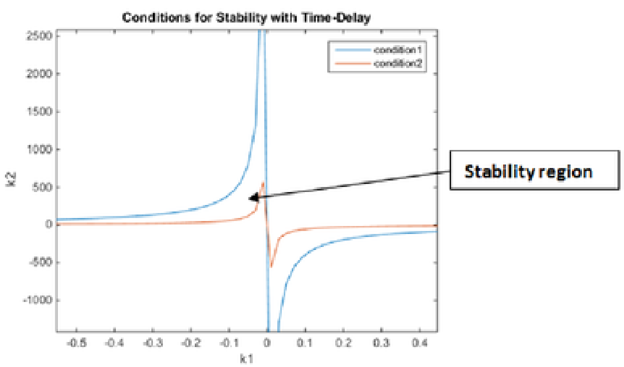


Figure 4.3: *Stability Conditions for small time delay with* $a_I = 1, b_I = -2$

From *Figure 4.3*, we observe a similar graph to *Figure 4.2*, but now the two inequalities will satisfy the region between the two hyperbolas in the second quadrant. Let us first explore what is happening with the conditions numerically when there is this change in frequencies. When $b_I = -2 < 0$, the second condition will now give a negative denominator due to the nature of the odd sine function. This in turn, results in the sign of the inequality changing, graphically giving the stability region to be now between the two hyperbolas. Not only this, but there is a significant change in the condition boundaries; both conditions display negative hyperbolic boundaries. Predicatably, the behaviour of the other cases for the frequencies can be inferred. If we consider when both frequencies are negative, $a_I < 0, b_I < 0$, the stability region will be similar to that found in *Figure 4.2*, except that it will now be reflected to become the region above the two hyperbolas. This reflective property will be evident also when we consider the other case of varying frequencies for $a_I < 0$ and $b_I > 0$. As opposed to what was shown in *Figure 4.3*, the stability region will now be given as between the two hyperbolas but now reflected and given in the fourth quadrant, instead of the second.

These results show that with this change in frequencies, there is now suddenly a drastic change in the size of the stability region for small time delay. For frequencies that were of opposite signs, a smaller and more bounded stability domain was found, which is in huge contrast to what was found when the frequencies were of the same sign and positive. This indicates just how important the frequency acting on the time delay is for stability.

4.2.3 Effect of large time delay

The effect of small time delay with varying frequencies showed a significant difference. But will this same effect be visible for larger time delay values? As aforementioned, the sine function is periodic with period 2π. Therefore, it's important to note that a time delay τ considered, will be the same as considering $\tau + 2\pi$. Looking at small time delay, values for τ were taken to be close to 0. In this case, larger values for the time delay will be chosen mod 2π. Again, in order to make objective comparisons, the values of a_R, b_R will remain constant and the frequencies will take the same values too. Let's consider the following arbitrary values for the parameters in our stability conditions: $a_R = 1$, $b_R = 1$, $a_I = 1$, $b_I = 2$ and taking random larger time delay values close to 2π that satisfy the constraints, we consider, $\tau_1 = \frac{11\pi}{6}$ and $\tau_2 = \frac{17\pi}{9}$. Studying this first case of both frequencies being positive and carrying out a similar analysis to that of the previous section, the stability condition can be visualised in *Figure 4.4*.

The results in *Figure 4.4* are interesting and display very similar results to that of smaller

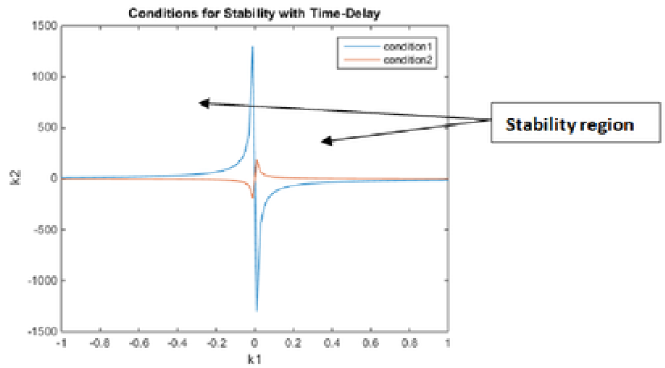


Figure 4.4: *Stability Conditions for large time delay with* $a_I = 1, b_I = 2$

time delay with positive frequencies. However, in this case, the hyperbolic boundaries have been reflected, as has the stability region. Qualitatively, similar behaviour is observed but quantitatively stability occurs for different coupling values.

Let's now observe the stability conditions for when the frequencies are of different signs, as before with $a_I = 1, b_I = -2$. Taking a similar stance to that in the previous section, analysis of the conditions leads to the following stability region, shown in *Figure 4.5*.

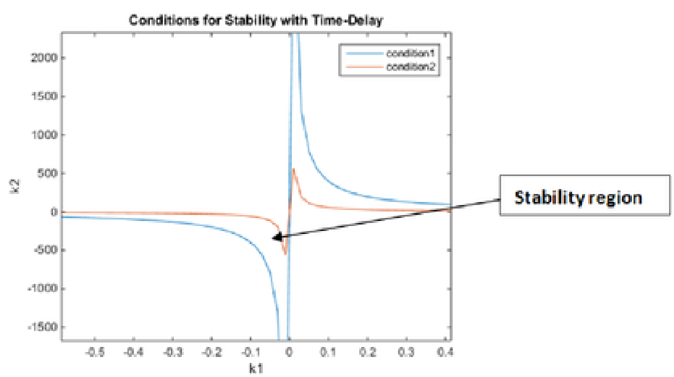


Figure 4.5: *Stability Conditions for large time delay with* $a_I = 1, b_I = -2$

Again, the behaviour observed is similar to that of small time delay with frequencies $a_I = 1, b_I = -2$ in the sense that the stability region is found to be between the two hyperbolas, however here it is reflected to be in the third quadrant. As well as this, the difference in condition boundaries can be seen, where there is now only positive hyperbolic boundaries. As shown previously, the other cases of the frequencies will behave the same here, giving reflective regions of stability. It is also important to note that the stability domain will never be at the origin, although it may asymptotically reach close to the origin, it will never be included.

Overall, looking at the conditions for stability of coupled systems with time delay, interesting properties are displayed, unlike what was seen in earlier chapters. In this case, hyperbolic boundaries for the conditions are observed and what made these conditions more intriguing was that a trigonometric function was present. The periodicity and oscillatory behaviour of the sine function was apparent in the change in behaviour of stability when the parameters associated with this function were changed, namely the frequencies. Furthermore, since the sine graph exhibits reflective behaviour, this gave similar reflective behaviour for change in frequencies and the stability domain.

A key finding of this chapter was that time delay is in fact crucial to make this type of coupling work and enable stability. There were some time delays that led to the conditions diverging and synchronisation did not take place in those instances. Further analysis led to concluding that, how long the time delay was didn't in fact make much difference as to whether synchronisation occurred, just the presence of the time delay was important. Long and short time delays overall gave similar stability regions, but for different coupling values, showing that delay length isn't significant for synchronisation. What is, however, is the frequency acting upon the time delay. This shows just how important the imaginary part of the complex variables a, b was and just how crucial it was to consider complex variables overall. The optimal situation in this case would be either short or long time delays but with only positive or only negative frequencies, as these gave the largest domains for stability and thus are optimal for synchronisation of these coupled units.

Chapter 5

Conclusion

In conclusion, this thesis aimed to study the synchronisation properties and to understand the conditions that enable two oscillators come to a synchronous state. It was found that, upon considering different models of oscillators, from uncoupled, to coupled, to then a time delay coupling model, different conditions for a stable, synchronous system was obtained. The study of synchrony began with first studying the equations of motion that have been the basis of modelling the two oscillators throughout the research. The research first looked at studying these equations of motion in an uncoupled case, where both oscillators were not connected and thus analysed independently. With no coupling between the two systems, the behaviour was investigated when the variables changed from real to complex, where interesting patterns could be seen in this transition of variables. The dynamical behaviour went from simple monotonic behaviour with both stable and unstable equilibria when the variables were real, to producing oscillatory behaviour when complex variables were considered. This pattern in the change of behaviour was evident throughout the study.

The study then moved on to look at a model of two equations that were now coupled and conditions on the coupling were sought for which the system would become stable.

Practically, this could be seen as a common support on which both oscillators lie and interact though and so the aim was to find conditions for this interaction, or coupling. When analysing a system of coupled equations and looking at real and complex variables in the equations of motion, we derived different conditions on the coupling for stability. The regions of stability depended greatly on the input variables a, b which were set into the equations of motion. The coupling has a direct impact on synchronisation, even a weak interaction can synchronize two clocks or oscillators (Pikovsky et al. 2003). This influence of the type of coupling was evident to some extent in this study but depended heavily on the model being considered. Such was the case when another coupled system was analysed, however due to the type of coupling, synchronisation didn't occur. Therefore, it's fair to say that the type of coupling in a model has a direct impact on synchronisation.

Furthermore, this understanding was further enhanced when a time delay was included in a model that was coupled but did not synchronise. It was shown that the time delay was in fact crucial in bringing stability to the system and without it, the system would fail to synchronise. Investigations into studying how the time delay affected stability showed that there were certain values of time-delay that diverged the system and did not work. However, altering the time delay had significant effects on the stability regions. Longer time delays compared to shorter time delays did not give any noteworthy results, although the change in frequencies having an influence on this time delay, did. It was interesting to see how the stability region changed with the change in frequency, giving a significant result.

A consistent result found in this research was that the synchronisation of self-sustained oscillators depend on the model being considered of its oscillators. In this study, two very simple equations of motion were initially considered which then logically progressed throughout the study to analysing a system with time delay. According to our model, time delay was found to be crucial. However, this result is not necessarily consistent with other researchers in this field who have studied synchronisation in different ways. In some cases its argued that the delay in coupling interferes with the appearance of synchronisation, branded as 'death by delay', whereas in others adding a delay into the interaction leads to the occurrence of synchronisation (Klinshov, Nekorkin, 2013).

Analysis into different models, gave different results. In their study of synchronisation, Klinshov and Nekorkin (2013) analysed many different models, mostly based on the well known Kuramoto model. Briefly, this model focuses on the frequency of N-coupled oscillators and is based on how oscillators with a common frequency will synchronise (Strogatz, 2000b). After looking at different models and the influence of time delay, Klinshov and Nekorkin (2013) found that the answer to whether time delay promotes or obstructs the occurrence of synchronisation is an ambiguous one and depends on the coupling and on the variables in the system. This is a fairly synonymous judgment to what was found in this study; synchronisation depends on the model by which the oscillators are governed by, where in this case time delay actually led to synchronisation.

This thesis simply scratched the surface of what is a largely expanding area of research into synchronisation. It is interesting to see how synchronisation changes with different models, such as the Kuramoto model and could be something for one to consider further looking at. This thesis only focuses on a particular model and then analysed its stability by reducing it to an eigenvalue problem and using perturbation techniques. It would be interesting to take different approaches to find stability, such as the study of Lyapunov exponents or even the Master Stability function and thus analyse its implications on synchronisation.

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