Stability Analysis of Linear Autonomous Retarded Differential-Difference Equations with Constant Coefficients

Ashley Barnes

140880236

Queen Mary
University of London

MTHM038

2014 / 2015

School of Mathematical Sciences

Supervisor: Dr Wolfram Just

Abstract

Two methods for the stability analysis of linear autonomous retarded differential-difference equations are applied to some fundamental first and second order homogeneous scalar equations. The first method uses Pontryagin's theorems on the zeros of exponential polynomials, and gives stability criteria in the form of inequalities that impose upper and lower bounds on the equation's parameters. Using this method, stability criteria for the Hayes equation, a first order equation, and for the general form of damped delayed harmonic oscillators, where the damping takes only positive values, are obtained. The stability criteria, for each equation, are then shown in the form of a stability chart, showing the stable domains in the parameter space of each equation. The second method is the D-subdivision method, and allows us to more directly compute the stability charts of linear autonomous retarded differential-difference equations. This method is used to produce stability charts for three equations. The first is the Hayes equation, the second is an important second order equation as found in [8] that is often encountered in control theory and in the theory of balancing, and finally the stability charts of the general form of damped delayed harmonic oscillators, where the damping can take negative as well as positive values, are given.

Contents

Chapter 1

Introduction

In many of the branches of science and technology, we often encounter dynamical systems in which the current rate of change of state is independent of the past states, and is determined only by the present state of the system. Dynamical systems of this form can often be modelled by differential equations involving the state, and the rate of change of state of the system. However it is not uncommon to encounter dynamical systems with time-delay, where the current rate of change of state depends not only on the present state but also on past states of the system. Volterra was one of the first to introduce a dynamical system that included past dependence, in his work on predator-prey models in 1928, in which he argued that the growth of the predator population depends not only on the present quantity of prey but also on past quantities, for example due to gestation periods. The development of control theory in the 1940s also showed how time-delay is commonly found in dynamical systems with feedback control, due to the finite speed of information transmission and data processing.

For systems with time-delay, standard differential equations are often only a first approximation to the true nature of the system, and in order to model the system more precisely, equations that include the past states of the system are required. These equations are described by functional differential equations (FDEs), which are equations involving the function $x(t)$ of one scalar argument t called time, and its derivatives for several values of the argument t. If the value of the highest derivative, called the order of the FDE, at time t depends only on the values of lower derivatives at preceding times, then the equation is called a retarded functional differential equation (RFDE). The simplest RFDEs are those in which the time-delay takes only discrete values, which are called retarded differential-difference equations (RDDEs).

One of the main differences between dynamical systems with and without time-delay, is that time-delay produces infinite dimensional dynamics, as opposed to the finite dimensional dynamics of delay-free systems. Consequently stability analysis is often more complicated for systems with time-delay.

In this project the focus will be on the stability analysis of some fundamental homogeneous linear autonomous scalar RDDEs with constant coefficients. In Chapter 2 we show how to derive the so called characteristic functions and characteristic equations for RDDEs of this form, and give an important stability condition as found in [6]. This condition states that if all the roots of the characteristic equation of a linear autonomous RDDE lie to the left of the imaginary axis, then the RDDE is said to be asymptotically stable. There are various analytical and numerical methods available to determine whether all the characteristic roots of a linear autonomous RDDE lie to the left of the imaginary axis. However in this project only two methods will be used.

Chapter 3 will be devoted to an analytical method, which is used to investigate the stability of two equations. The first is a homogeneous linear autonomous first order scalar RDDE with constant coefficients, sometimes called the Hayes equation[1]. The second is the general form of a damped delayed harmonic oscillator[2], where the damping takes only positive values, which corresponds to a second order homogeneous linear autonomous scalar RDDE with constant coefficients. The reason this method can be used to analyse the stability of these equations, is due to the fact that for both of these equations, the characteristic function can be reduced to an exponential polynomial. Hence a result of Pontryagin as found in [7], that gives a necessary and sufficient condition for all the zeros of an exponential polynomial to lie to the left of the imaginary axis, can be used. This method, for each equation, yields stability criteria in the form of inequalities that impose upper and lower bounds on the equation's parameters, which are then shown

[1] The stability criteria for this equation were first shown by Hayes in [5].

[2] Recall the general form of a damped harmonic oscillator is given by $m\ddot{x}(t)+c\dot{x}(t)+kx = 0$, where m is the mass, c is the damping coefficient and k is the spring constant.

via a stability chart in the parameter space of the equation.

The second method, called the D-subdivision method in [6] is given in Chapter 4, and is used to produce stability charts for three equations. The first is the Hayes equation, which is used to illustrate the techniques used in the method. The second is a second order homogeneous linear autonomous scalar RDDE of the form as shown in [8], that is widely seen in control theory and in the theory of balancing. Finally the third equation is the general form of the damped delayed harmonic oscillator given in the first method, although now the damping can take negative as well as positive values. The stability charts for each of these RDDEs, are constructed by using the characteristic equation to derive the so called D-curves. These D-curves subdivide the coefficient space of the RDDE into domains in which the number of unstable characteristic roots, i.e. roots with positive real part, is constant. The determination of the number of unstable characteristic roots in each domain of the subdivided coefficient space, is then done via the calculation of the root-crossing direction along the D-curves, as shown in [9]. This then allows us to determine the stability domains of the RDDE, which are the domains with zero unstable characteristic roots.

Chapter 2

The Characteristic Equation and Stability

The general form of a differential-difference equation (DDE) of differential order n and difference order m, that is both homogeneous and autonomous, is given by

$$F\left[x(t), x(t-\tau_1), \ldots, x(t-\tau_m), \ldots, x^{(n)}(t), x^{(n)}(t-\tau_1), \ldots, x^{(n)}(t-\tau_m)\right] = 0 \ , \tag{2.1}$$

where F is a given real function of $(m+1)(n+1)$ real variables and where $\tau_1, \tau_2, \ldots, \tau_m$ are positive constants called the delays. If the highest derivative in (2.1) at time t depends only on the values of lower derivatives at preceding times then (2.1) is said to be a retarded differential-difference equation (RDDE). For example, the general form of first order homogeneous linear autonomous scalar DDEs with constant coefficients is given by

$$a_1\dot{x}(t) + b_1\dot{x}(t-\tau) + a_0x(t) + b_0x(t-\tau) = 0 \ , \tag{2.2}$$

where $x(t) \in \mathbb{R}$, a_0, a_1, b_0, b_1 are real constants, and τ is a positive constant. If we have that $a_1 \neq 0$ and $b_1 = 0$ then equation (2.2) is said to be a RDDE. The characteristic function of an equation of the form (2.2) is derived by substituting the trial exponential solution $x(t) = Ae^{\lambda t}$, $A, \lambda \in \mathbb{C}$ into (2.2) which gives

$$\left(a_1\lambda + b_1\lambda e^{-\lambda\tau} + a_0 + b_0e^{-\lambda\tau}\right) Ae^{\lambda t} = 0 \ . \tag{2.3}$$

Thus $x(t) = Ae^{\lambda t}$ is a solution of (2.2) for all t, if and only if λ is a zero of the transcendental function

$$D(\lambda) = a_1\lambda + b_1\lambda e^{-\lambda\tau} + a_0 + b_0e^{-\lambda\tau} \ , \tag{2.4}$$

which is called the characteristic function of (2.2). The equation $D(\lambda) = 0$ is called the characteristic equation of (2.2), the roots of which are called the characteristic roots of (2.2).

Let us now look at the more general form of homogeneous linear autonomous scalar DDEs, of differential order n and difference order m, with constant coefficients, given by

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}x^{(j)}(t-\tau_i) = 0 \ , \tag{2.5}$$

where $x(t) \in \mathbb{R}$, the a_{ij} and τ_i are constants, and where $0 = \tau_0 < \tau_1 < \cdots < \tau_m$. If $a_{0n} \neq 0$ and $a_{in} = 0$ for $i = 1, \ldots, m$ then (2.5) is said to be a RDDE. The characteristic function for equations of the form (2.5) can be obtained, as in the first order case, by substituting the trial exponential solution $x(t) = Ae^{\lambda t}$, $A, \lambda \in \mathbb{C}$ into (2.5). Hence the characteristic function of (2.5) is given by

$$D(\lambda) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\lambda^{j}e^{-\lambda\tau_i} \ . \tag{2.6}$$

The importance in deriving, for a DDE, the corresponding characteristic function is due to the role the characteristic roots often play in the stability analysis of the DDE. For example, it is shown in [6] that for a RDDE, if all the characteristic roots have negative real parts then the trivial solution $x(t) \equiv 0$ of the RDDE is asymptotically stable. In addition to this, it is also shown in [6] that in the case of a linear autonomous RDDE, if the trivial solution is asymptotically stable then we can say that the RDDE itself is asymptotically stable.

In order to simplify later analysis in determining whether all the characteristic roots of a DDE have negative real parts, it is often convenient, for equations of the form (2.2), to multiply the characteristic equation (2.4) by $e^{\lambda\tau}$ to get

$$D_*(\lambda) = a_1\lambda e^{\lambda\tau} + b_1\lambda + a_0e^{\lambda\tau} + b_0 = 0 \ , \tag{2.7}$$

and for DDEs of the form (2.5) to multiply the characteristic equation by $e^{\lambda\tau_m}$ to get

$$D_*(\lambda) = e^{\lambda\tau_m} \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} \lambda^j e^{-\lambda\tau_i} = 0 \ . \tag{2.8}$$

It is shown in [1] that, for equations of the form (2.2) and (2.5), the roots of $D(\lambda) = 0$ are the same as the roots of $D_*(\lambda) = 0$. Therefore determining whether all the roots of $D(\lambda) = 0$ lie to the left of the imaginary axis is equivalent to determining whether all the roots of $D_*(\lambda) = 0$ lie to the left of the imaginary axis. Thus with this in mind, for equations of the form (2.2) and (2.5), the equation $D_*(\lambda) = 0$ shall also be called the characteristic equation.

Chapter 3

The Pontryagin Method

The analytical method presented in this chapter uses the fact that the characteristic function of a homogeneous linear autonomous RDDE, with constant coefficients and commensurable delays, can be reduced to an exponential polynomial, that is a polynomial in z and e^z. Consequently a theorem found in [7], which gives a necessary and sufficient condition for the zeros of an exponential polynomial to lie to the left of the imaginary axis, can be used to determine the stability of the RDDE. For completeness the relevant definitions and theorems needed for stability analysis, as found in [7][1], will be included, without proof, in Section 3.1. Then by using the techniques as shown in Chapter 13 of [1], in [2], and in the appendix of [4], the stability of two RDDEs will be investigated.

3.1 Definitions and Theorems

Definition 1. *Let $h(z,w)$ be a polynomial in the two variables z and w with coefficients that may be complex*

$$h(z,w) = \sum_{m,n} a_{mn} z^m w^n \ , \tag{3.1}$$

where m,n are non-negative integers. The term $a_{rs}z^r w^s$ is called the principal term of the polynomial (3.1) if $a_{rs} \neq 0$ and for every other term of the polynomial $a_{mn}z^m w^n$ with $a_{mn} \neq 0$ we either have $r > m$ and $s > n$, or $r > m$ and $s = n$, or we have $r = m$ and $s > n$.

The importance of this principal term in stability analysis is shown in the next theorem.

[1]These definitions and theorems are also found in the appendix of [4] and in Chapter 13 of [1].

Theorem 1. *If the polynomial $h(z, w)$ in* (3.1) *has no principal term then the function $H(z) = h(z, e^z)$ has an unbounded number of zeros with arbitrarily large positive real part.*

Definition 2. *Let $f(z, u, v)$ be a polynomial in the variables z, u and v which we can write in the form*

$$f(z, u, v) = \sum_{m,n} z^m \phi_m^{(n)}(u, v) \ , \tag{3.2}$$

where $\phi_m^{(n)}(u, v)$ is a polynomial of degree n that is homogeneous in u and v. The principal term of the polynomial (3.2) *is defined as the term $z^r \phi_r^{(s)}(u, v)$ such that for all other terms of the polynomial $z^m \phi_m^{(n)}(u, v)$, we either have $r > m$ and $s > n$, or $r > m$ and $s = n$, or we have $r = m$ and $s > n$.*

Similar to Theorem 1 for polynomials of the form (3.1), we have the following theorem for polynomials of the form (3.2).

Theorem 2. *If the polynomial* (3.2) *does not have a principal term, then the function defined as $F(z) = f(z, cos(z), sin(z))$ has an unbounded number of roots that are not real.*

Before the next theorem is given, we first introduce the following notation. Suppose we have a polynomial of the form (3.2) with principal term $z^r \phi_r^{(s)}(u, v)$, we let $\phi^{*(s)}(u, v)$ denote the coefficient of z^r in (3.2), and we define $\Phi^{*(s)}(z) = \phi^{*(s)}(cos(z), sin(z))$. This notation is used in the next theorem which is one of the two main theorems we will use for stability analysis, and gives conditions such that a polynomial of the form (3.2) has only real roots.

Theorem 3. *Let $f(z, u, v)$ be a polynomial of the form* (3.2) *with principal term $z^r \phi_r^{(s)}(u, v)$. If ϵ is such that $\Phi^{*(s)}(\epsilon + iy) \neq 0$, for every $y \in \mathbb{R}$, then in the strip $-2k\pi + \epsilon \leq x \leq 2k\pi + \epsilon$, $z = x + iy$, the function $F(z) = f(z, cos(z), sin(z))$ will have, for all sufficiently large integers k exactly $4ks + r$ zeros. Therefore the function $F(z) = f(z, cos(z), sin(z))$ will have only real roots if and only if, for sufficiently large integers k, $F(z)$ has exactly $4ks + r$ roots in the strip $-2k\pi + \epsilon \leq x \leq 2k\pi + \epsilon$.*

Theorem 3 is used in conjuction with the next theorem, which represents the second of the two main theorems we shall use to investigate stability.

Theorem 4. *Let* $H(z) = h(z, e^z)$, *where* $h(z, w)$ *is a polynomial with a principal term. Suppose the function* $H(iy)$, $y \in \mathbb{R}$, *is decomposed into its real and imaginary parts, that is we get* $H(iy) = F(y) + iG(y)$. *If all the zeros of the function* $H(z)$ *lie to the left of the imaginary axis, then the zeros of* $F(y)$, $G(y)$ *are real, alternating and*

$$G'(y)F(y) - G(y)F'(y) > 0 \ , \tag{3.3}$$

for $y \in \mathbb{R}$. *Conversely, all the zeros of* $H(z)$ *will lie to the left of the imaginary axis provided that either of the following conditions is satisfied*

(a) All the zeros of the functions $F(y)$, $G(y)$ *are real, alternate and the inequality* (3.3) *is satisfied for at least one value of* y.

(b) All the zeros of the function $F(y)$ *are real, and for each zero the inequality* (3.3) *is satisfied, that is* $F'(y)G(y) < 0$ *at all the zeros of* $F(y)$.

(c) All the zeros of the function $G(y)$ *are real and for each zero inequality* (3.3) *is satisfied, that is* $G'(y)F(y) > 0$ *at the zeros of* $G(y)$.

3.2 The Hayes Equation

One of the simplest examples of a system with time-delay is the Hayes equation, which is a homogeneous linear autonomous first order scalar RDDE given by

$$\dot{x}(t) - a_0 x(t) - b_0 x(t-\tau) = 0 \ , \tag{3.4}$$

where $a_0, b_0 \in \mathbb{R}$, and τ is a positive constant. Substituting the trial exponential solution into (3.4), and multiplying by $e^{\lambda\tau}$ yields the characteristic function

$$D_*(\lambda) = (\lambda - a_0)e^{\lambda\tau} - b_0 \ , \tag{3.5}$$

and introducing the non-dimensional quantity $z = \lambda\tau$ into (3.5) gives the exponential polynomial

$$H_1(z) = (z-p)e^z - q \ , \tag{3.6}$$

where $p = a_0\tau$ and $q = b_0\tau$. Since the exponential polynomial (3.6) has principal term ze^z, Theorem 4 can be applied to determine when the zeros of (3.6) all lie to the left of the imaginary axis. Substituting $z = iy$, $y \in \mathbb{R}$, into (3.6) gives

$$H_1(iy) = -ysin(y) - pcos(y) - q + i[ycos(y) - psin(y)] \ , \tag{3.7}$$

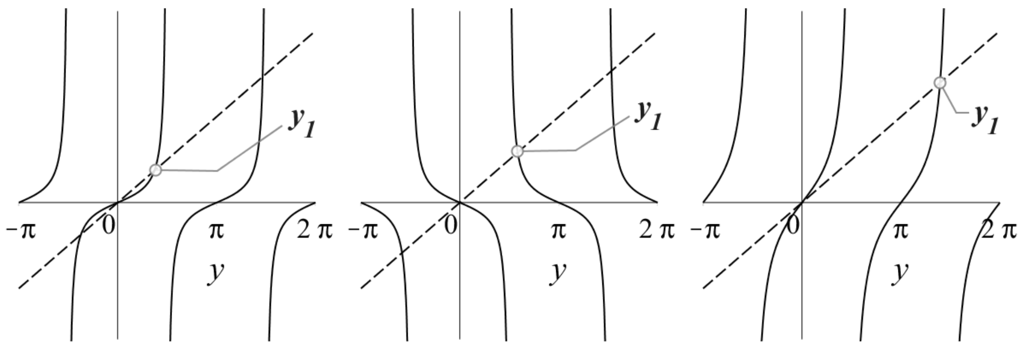
and so

$$F_1(y) = -ysin(y) - pcos(y) - q \ , \tag{3.8}$$

$$G_1(y) = ycos(y) - psin(y) \ . \tag{3.9}$$

As the function $G_1(y)$ is of a slightly simpler form than $F_1(y)$, we will try to determine the values of p and q such that condition (c) of Theorem 4 is satisfied. Firstly it should be noted that the principal term of (3.9) is given by $ycos(y)$, hence $\Phi^{*(1)}(y) = cos(y)$ and so we can take $\epsilon = 0$ in Theorem 3. This is because if $\epsilon = 0$, then for all $\beta \in \mathbb{R}$,

$$\Phi^{*(1)}(\epsilon + i\beta) = \Phi^{*(1)}(i\beta) = cos(i\beta) = \frac{e^{-\beta} + e^{\beta}}{2} \neq 0 \ . \tag{3.10}$$



(a) $p = 0.5$ **(b)** $p = -0.5$ **(c)** $p = 1.5$

Figure 3.1: Plot showing the first positive root of (3.11) for different values of p.

Therefore in order for all the roots of $G_1(y) = 0$ to be real, there must be exactly $4k+1$ roots of $G_1(y) = 0$ in the interval $[-2k\pi, 2k\pi]$ for sufficiently large integers k. To determine the number of roots of $G_1(y) = 0$ in this interval, we first observe that $y_0 = 0$ is a root, and that $G_1(y) = 0$ can be rewritten as

$$y = p\ tan(y)\ . \tag{3.11}$$

The analysis of the number of roots of (3.11) in the interval $[-2k\pi, 2k\pi]$ is split into the three cases: $p = 1$, $p > 1$ and $p < 1$. Firstly if $p = 1$ then we have $G_1'(y_0) = 0$, and so the inequality (3.3) is not satisified at the root at the origin, therefore for stability we must have that $p \neq 1$.

If $p > 1$, then due to the shape of the tangent curve, as shown in Figure 3.1c, there is no root of $G_1(y) = 0$ in the interval $(0, \pi)$, whereas in every other interval $(j\pi, (j+1)\pi)$, where j is a positive integer, there is exactly one root. Therefore for every positive integer k, in the interval $(0, 2k\pi]$ there are only $2k-1$ roots, and as both the left and right hand side of (3.11) are odd in y, there must also be $2k-1$ roots in the interval $[-2k\pi, 0)$. Thus for $p > 1$, including the root at the origin, there are only $4k-1$ roots of $G_1(y) = 0$ in the interval $[-2k\pi, 2k\pi]$ no matter how large the integer k is chosen to be.

Whereas if $p < 1$, then for every integer n there is exactly one root in the interval $(n\pi, (n+1)\pi)$. Hence, for all non-negative integers k, there are exactly $2k$ roots in the interval $(0, 2k\pi]$, and as both sides of (3.11) are odd, there must also be $2k$ roots in the interval $[-2k\pi, 0)$. Thus for $p < 1$, for every non-negative integer k, including the root at the origin, there are exactly $4k + 1$ roots of $G_1(y) = 0$ in the interval $[-2k\pi, 2k\pi]$. Therefore in order for all the roots of $G_1(y) = 0$ to be real, we must have that $p < 1$.

Let $p < 1$, then all the roots of $G_1(y) = 0$ are real, and according to Theorem 4, in order for all the zeros of $H_1(z)$ to lie to the left of the imaginary axis, the inequality $F_1(y)G_1'(y) > 0$ must be satisfied at all the roots of $G_1(y) = 0$. To simplify the analysis of this inequality, we first label the roots of $G_1(y) = 0$. Firstly the root at the origin is labelled y_0, and to the root in the interval $((j-1)\pi, j\pi)$, where j is a positive integer, we attach the label y_j. As the zeros of $G_1(y)$ come in pairs of the form $\pm y$, as can be seen from (3.11), the negative roots of $G_1(y) = 0$ are labelled as $y_{-j} = -y_j$, for $j = 1, 2, \ldots$.

Since we have that

$$F_1(y) = -ysin(y) - p\ cos(y) - q\ , \tag{3.12}$$

$$G_1'(y) = (1-p)cos(y) - ysin(y)\ , \tag{3.13}$$

which are both even functions in y, and as the zeros of $G_1(y) = 0$ come in pairs of the form $\pm y$, the requirement that $F_1(y)G_1'(y) > 0$ at all roots reduces to $F_1(y)G_1'(y) > 0$ at the non-negative roots of $G_1(y) = 0$. Firstly at $y_0 = 0$,

$$F_1(0)G_1'(0) = -(p+q)(1-p)\ , \tag{3.14}$$

and as $p < 1$ we must have that $p < -q$ in order for $F_1(y_0)G_1'(y_0) > 0$ to be satisfied. At the positive roots of $G_1(y) = 0$, $F_1(y)$ and $G_1'(y)$ can be rewritten using (3.11) and the fact that

$$sin(y_j) = \frac{(-1)^{j-1}y_j}{\sqrt{p^2 + y_j^2}}\ , \tag{3.15}$$

at the j^{th} positive root of $G_1(y) = 0$, and so we have that

$$F_1(y_j)G_1'(y_j) = \left(\frac{1}{\sqrt{p^2+y_j^2}}\right)(y_j^2+p^2-p)\left(\sqrt{p^2+y_j^2}+(-1)^{j-1}q\right) \ , \tag{3.16}$$

at the j^{th} positive root of $G_1(y) = 0$. The second expression in (3.16) can be rewritten as

$$y^2+p^2-p = \left(\frac{y}{sin^2(y)}\right)\left(y-\frac{1}{2}sin(2y)\right) \ , \tag{3.17}$$

which is positive at any nonzero root of $G_1(y) = 0$. Therefore in order for $F_1(y)G_1'(y) > 0$ to be satisfied at the positive roots we require that, for every positive integer j,

$$\sqrt{p^2+y_j^2}+(-1)^{j-1}q > 0 \ . \tag{3.18}$$

At the odd labelled positive roots, (3.18) simplifies to $\sqrt{p^2+y^2}+q > 0$, and at the even labelled positive roots, (3.18) simplifies to $\sqrt{p^2+y^2}-q > 0$. Since both $\sqrt{p^2+y^2}+q$ and $\sqrt{p^2+y^2}-q$ are increasing quantities in y, in order for (3.18) to be satisfied at all positive roots we just need that $\sqrt{p^2+y_1^2}+q > 0$ and $\sqrt{p^2+y_2^2}-q > 0$. However as $p < -q$, we have that $\sqrt{p^2+y_2^2}-q > |p|+p \geq 0$, and so (3.18) is satisfied at all the even labelled positive roots of $G_1(y) = 0$. Therefore in order for $F_1(y)G_1'(y) > 0$ to be satisfied at the non-negative roots of $G_1(y) = 0$, and thus for all the roots of $G_1(y) = 0$, we require that $p < -q < \sqrt{p^2+y_1^2}$. It should be noted that if $p = 0$, then $G_1(y) = ycos(y)$ and so $y_1 = \frac{\pi}{2}$, and the above condition becomes $0 < -q < \frac{\pi}{2}$.

We have therefore shown, by loosely following the proof in [1], the result as found in [5], that the zeros of $H_1(z) = (z-p)e^z - q$ all lie to the left of the imaginary axis, and thus equation (3.4) is asymptotically stable if and only if

(i) $p < 1$, and

(ii) $p < -q < \sqrt{p^2+y_1^2}$,

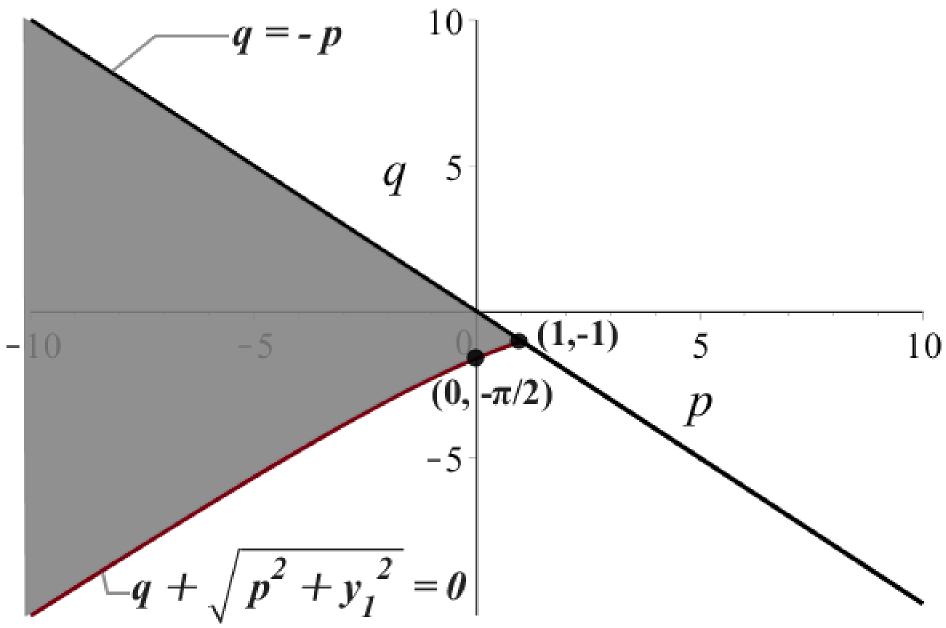


Figure 3.2: The stability chart for equation (3.4) in the parameter space (p, q).

where $p = a_0\tau$ and $q = b_0\tau$, and where y_1 denotes the root of $y = p\ tan(y)$ such that $0 < y < \pi$ if $p \neq 0$ and $y_1 = \frac{\pi}{2}$ if $p = 0$. The resulting stability chart for equation (3.4) is shown in Figure 3.2, with the stability domain shaded grey.

3.3 Damped Delayed Harmonic Oscillators

Damped harmonic oscillators subjected to time delayed feedback control, where the damping only takes positive values, are governed by second order homogeneous linear autonomous scalar RDDEs of the form

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = Kx(t-\tau) \ , \tag{3.19}$$

where $a_1 > 0$, $a_0, K \in \mathbb{R}$, and $\tau > 0$. The characteristic function of (3.19) is given by

$$D_*(\lambda) = \left[\lambda^2 + a_1\lambda + a_0\right] e^{\lambda\tau} - K \ , \tag{3.20}$$

and substituting the non-dimensional quantity $z = \lambda\tau$ into (3.20) yields the exponential polynomial

$$H_2(z) = (z^2 + az + b)e^z - c \ , \tag{3.21}$$

where $a = a_1\tau$, $b = a_0\tau^2$ and $c = K\tau^2$, and so $a > 0$, as $a_1 > 0$, and $b, c \in \mathbb{R}$. As (3.21) has the principal term z^2e^z, Theorem 4 can be applied to determine when the zeros of (3.21) all lie to the left of the imaginary axis. Substituting $z = iy$, $y \in \mathbb{R}$ into the exponential polynomial (3.21) gives

$$H_2(iy) = (b-y^2)cos(y) - aysin(y) - c + i[(b-y^2)sin(y) + aycos(y)] \ , \tag{3.22}$$

and so

$$F_2(y) = (b - y^2)cos(y) - aysin(y) - c \ , \tag{3.23}$$

$$G_2(y) = (b - y^2)sin(y) + aycos(y) \ . \tag{3.24}$$

As in the case of the Hayes equation, since the function $G_2(y)$ is of a slightly simpler form than $F_2(y)$, we shall try to use part (c) of Theorem 4 to determine the appropriate conditions such that all the zeros of (3.21) lie to the left of the imaginary axis.

Firstly, as the principal term of (3.24) is $-y^2sin(y)$, we have that $\Phi^{*(1)}(y) = -sin(y)$, and so we can take $\epsilon = \frac{\pi}{2}$ in Theorem 3. This is because for all

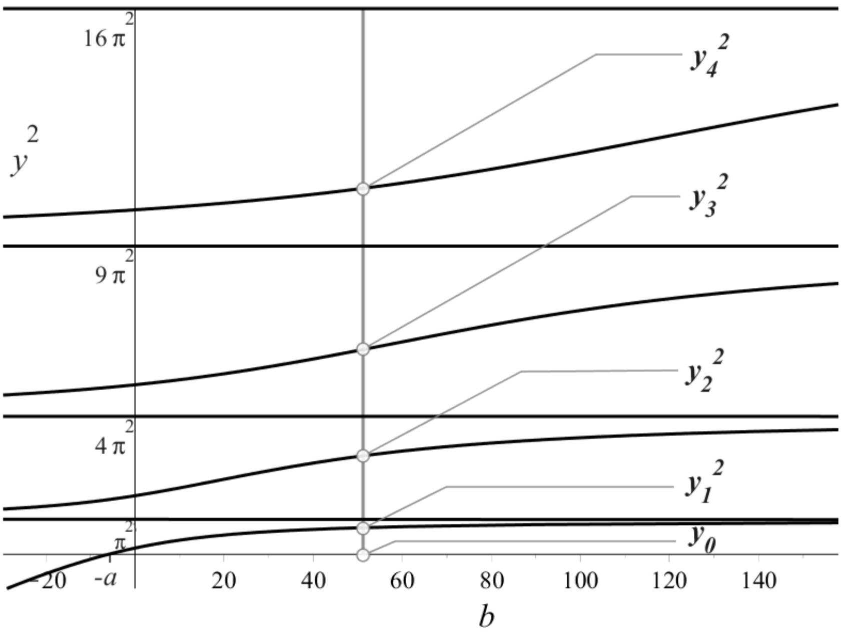


Figure 3.3: Distribution of the zeros of (3.26) for $a > 0$.

$\beta \in \mathbb{R}$, we have $\Phi^{*(1)}(\epsilon + i\beta) = -\left[\frac{e^{-\beta}+e^{\beta}}{2}\right] \neq 0$. Therefore in order for all the roots of $G_2(y) = 0$ to be real, there must be exactly $4k + 2$ roots of $G_2(y) = 0$ in the interval $[-2k\pi + \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}]$ for sufficiently large integers k. To determine the number of roots in this interval, we first note that $y = 0$ is a root of $G_2(y) = 0$, and that $G_2(y) = 0$ can be rewritten at the non-zero roots as

$$\frac{y^2 - b}{ay} = cot(y) \ , \tag{3.25}$$

or alternatively as

$$b = y^2 - aycot(y) \ . \tag{3.26}$$

Using Figure 3.3, which shows the distribution of the zeros of $G_2(y)$ in the (b, y^2) plane for $a > 0$, we can see that in order to determine when there are $4k+2$ roots, for sufficiently large integers k, in the interval $[-2k\pi+\frac{\pi}{2}, 2k\pi+\frac{\pi}{2}]$, the analysis should be divided into the three cases: $b > 0$, $-a < b \leq 0$ and $b \leq -a$.

Case 1: $b > 0$

We first count the roots of (3.25) on $(0, 2k\pi + \frac{\pi}{2}]$, and as $y > 0$ on this interval, we have that the function given by the left hand side of (3.25) is concave, and increases continuously from $-\infty$ to ∞ as y increases from 0 to ∞. We take k large enough so that the left hand side of (3.25) is positive for $y \geq 2(k-1)\pi$.

Since the cotangent curve has one branch on each interval $(0, \pi)$, $(\pi, 2\pi)$, $\dots$, and as we know that the left hand side of (3.25) is concave, we have that there is exactly one root in each interval $(0, \pi)$, $(\pi, 2\pi)$,$\dots$, $((2k-1)\pi, 2k\pi)$, and so there are exactly $2k$ roots in the interval $(0, 2k\pi)$. Also since we have that the left hand side is positive for $y \geq 2(k-1)\pi$, and due to the additional branch of the cotangent curve, there is an additional root of $G_2(y) = 0$ in the interval $(2k\pi, 2k\pi + \frac{\pi}{2})$. Hence there are $2k+1$ roots of (3.25) in $(0, 2k\pi + \frac{\pi}{2}]$.

As we have that there are $2k + 1$ roots of (3.25) in $(0, 2k\pi + \frac{\pi}{2}]$, and since there is a root at the origin, in order for Theorem 3 to be satisfied there must be exactly $2k$ roots in the interval $[-2k\pi + \frac{\pi}{2}, 0)$. Using the fact that the left hand side of (3.25) is positive for $y \geq 2(k-1)\pi$, we have that the largest root of $G_2(y) = 0$ in the interval $(0, 2k\pi]$ is in $(0, 2k\pi - \frac{\pi}{2}]$. This is because we took k large enough so that the left hand side of (3.25) is positive on the interval $(2k\pi - \frac{\pi}{2}, 2k\pi]$, and due to the fact that the cotangent curve is negative on this interval, thus there can't be a root of (3.25) in $(2k\pi - \frac{\pi}{2}, 2k\pi]$. As we know that there are $2k$ roots in $(0, 2k\pi]$, there must be $2k$ roots in $(0, 2k\pi - \frac{\pi}{2}]$, and as both sides of (3.25) are odd, there must also be $2k$ roots in $[-2k\pi + \frac{\pi}{2}, 0)$.

Therefore for sufficiently large integers k, $G_2(y) = 0$ has 4k+2 roots in the interval $[-2k\pi + \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}]$, and so by Theorem 3, for $b > 0$, all the roots of $G_2(y) = 0$ are real.

Case 2: $-a < b \leq 0$

As before since both the left hand side and right hand side of (3.25) are odd, we only need to count the zeros of $G_2(y)$ on $(0, 2k\pi + \frac{\pi}{2}]$. However since

$b \leq 0$ and we know $y > 0$ on this interval, we have that the left hand side of (3.25) is positive, and is convex for $b < 0$ and increases continuously for $b = 0$. Therefore we can take $k \geq 1$ in Theorem 3. By the same argument as in Case 1, there are exactly $2k+1$ roots in the interval $(0, 2k\pi + \frac{\pi}{2}]$. Also since the left hand side of (3.25) is positive on $(2k\pi - \frac{\pi}{2}, 2k\pi]$ and the cotangent curve is negative on this interval, there must be $2k$ roots in $[-2k\pi + \frac{\pi}{2}, 0)$. Hence there are $4k+2$ roots in $[-2k\pi + \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}]$, and therefore by Theorem 3, for $-a < b \leq 0$, all the roots of $G_2(y) = 0$ are real.

<u>Case 3:</u> $b \leq -a$

As $b < 0$, the left hand side of (3.25) is positive and convex, however as we can see in Figure 3.3, there is no root of $G_2(y) = 0$ in the interval $(0, \pi)$. Hence for every positive integer k, there are at most $2k-1$ roots in $(0, 2k\pi)$, and so there can't be $4k+2$ roots in the interval $[-2k\pi + \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}]$. Therefore, as $G_2(y)$ can not have only real roots, stability can be ruled out for $b \leq -a$.

We now aim to show conditions such that each root of $G_2(y) = 0$ satisfies the inequality (3.3) in Theorem 4 in both the remaining cases: for $b > 0$ and for $-a < b \leq 0$. It should be noted that in both the remaining cases, for each integer n, there is exactly one root of $G_2(y) = 0$ in the interval $(n\pi, (n+1)\pi)$.

In order to simplify the analysis of $F_2(y)G_2'(y)$ we first label the roots of $G_2(y) = 0$. The root at the origin is labelled $y_0 = 0$, and for each positive integer j the root in the interval $((j-1)\pi, j\pi)$ is labelled y_j. It is apparent from Figure 3.3 that the roots of $G_2(y) = 0$, except for the root at the origin, come in pairs of the form $\pm y$. Therefore the negative roots of $G_2(y) = 0$ can be labelled as $y_{-j} = -y_j$ for $j = 1, 2, \dots$.

We have

$$F_2(y) = (b - y^2)cos(y) - aysin(y) - c \ , \tag{3.27}$$

$$G_2'(y) = (a + b - y^2)cos(y) - (2 + a)ysin(y) \ , \tag{3.28}$$

and as both $F_2(y)$ and $G_2'(y)$ are even functions in y, we just require that the inequality $F_2(y)G_2'(y) > 0$ is satisfied at the non-negative roots of $G_2(y) = 0$.

For the root of the origin we have

$$F_2(0)G_2'(0) = (b-c)(a+b) \ , \tag{3.29}$$

and using the relation (3.25), at any non-zero root of $G_2(y) = 0$, (3.27) and (3.28) can be rewritten as

$$F_2(y) = \left(\frac{sin(y)}{ay}\right)\left(-(y^2-b)^2 - y^2a^2 - c\frac{ay}{sin(y)}\right) \ , \tag{3.30}$$

$$G_2'(y) = \left(\frac{sin(y)}{ay}\right)\left(-(y^2-b)^2 - y^2a^2 - a(y^2+b)\right) \ . \tag{3.31}$$

Therefore at the positive roots the inequality $F_2(y)G_2'(y) > 0$ becomes

$$\frac{sin^2(y)}{a^2y^2}\left[-(y^2-b)^2 - y^2a^2 - c\frac{ay}{sin(y)}\right]\left[-(y^2-b)^2 - y^2a^2 - a(y^2+b)\right] > 0 \, . \tag{3.32}$$

The analysis of (3.32) is separated into the two remaining cases.

<u>Case 1:</u> $b > 0$

Firstly from (3.29) it is clear that in order for $F_2(y_0)G_2'(y_0) > 0$ we require $c < b$. Also as the third factor of (3.32) is negative in this case, (3.32) simpifies to

$$\left[-(y^2-b)^2 - y^2a^2 - c\frac{ay}{sin(y)}\right] < 0 \ , \tag{3.33}$$

at the positive roots of $G(y) = 0$. To simplify the analysis of (3.33), we use a technique shown in [2] and define the positive quantity $V(y)$ by

$$V(y) = \sqrt{(y^2-b)^2 + a^2y^2} \ , \tag{3.34}$$

and using (3.25) we get that

$$sin(y_j) = \frac{(-1)^{j-1}ay_j}{V(y_j)} \ , \tag{3.35}$$

at the j^{th} positive zero of $G_2(y)$. After substituting (3.35) into (3.33) we get that in order for $F_2(y)G_2'(y) > 0$ at the positive roots, we require

$$c(-1)^j < V(y_j), \qquad j = 1, 2, \dots \ . \tag{3.36}$$

Therefore at the even labelled positive roots we have inequalities of the form $c < V(y)$, and at the odd labelled positive roots we have inequalities of the form $-V(y) < c$. Recalling that in order for $F_2(y_0)G_2'(y_0) > 0$ we require $c < b$, then we obtain the following sets of inequalities,

$$c < b = V(y_0), \quad c < V(y_2), \quad c < V(y_4), \quad \dots \,, \tag{3.37}$$

$$-V(y_1) < c, \quad -V(y_3) < c, \quad -V(y_5) < c, \quad \dots \,. \tag{3.38}$$

It is apparent that the sets of inequalities (3.37) and (3.38) bound c from above and below respectively. To determine the governing inequalities, which are given by the smallest $V(y)$ in (3.37) and (3.38), we first note that the quantity in the square root in (3.34) can be rewritten as $y^4 + (a^2 - 2b)y^2 + b^2$. Thus if $a^2 \geq 2b$ then $V(y)$ is an increasing function in y, and so the governing inequalities are $c < b = V(y_0)$ and $-V(y_1) < c$. Whereas if $a^2 < 2b$ then, by using differential calculus, we can see that $y^4 + (a^2 - 2b)y^2 + b^2$ is minimised by $y^2 = b - \frac{a^2}{2}$. Therefore if we define y_* to be the even non-negative labelled root whose value squared is closest to $b - \frac{a^2}{2}$ and y_{**} to be the odd positive labelled root whose value squared is closest to $b - \frac{a^2}{2}$, then we have that the governing inequalities are given by $-V(y_{**}) < c$ and $c < V(y_*)$.

Case 2: $-a < b \leq 0$

In this case the third factor in (3.32) can be rewritten using (3.34) and (3.35), to get

$$\left[-(y^2 - b)^2 - y^2a^2 - a(y^2 + b)\right] = -\left[\frac{a^2y^2}{sin^2(y)} + ab + ay^2\right] \,, \tag{3.39}$$

and since $-a < b$ we have

$$-\left[\frac{a^2y^2}{sin^2(y)} + ab + ay^2\right] < -\left[a^2\left(\frac{y^2}{sin^2(y)} - 1\right) + ay^2\right] < 0 \,, \tag{3.40}$$

where the last inequality is valid because for $y \neq 0$ we have that $\frac{y^2}{sin^2(y)} > 1$. This means that just as in Case 1, in order for $F_2(y)G_2'(y) > 0$ at the positive roots we need

$$\left[-(y^2 - b)^2 - y^2a^2 - c\frac{ay}{sin(y)}\right] < 0 \,, \tag{3.41}$$

and as in Case 1 this leads to requiring

$$c(-1)^j < V(y_j), \qquad j = 1, 2, \dots \ . \tag{3.42}$$

However since in this case $b \leq 0$, we have that $a^2 \geq 2b$ and so $V(y)$ is an increasing function in y. Therefore the governing inequalities are given by $c < b$ and $-V(y_1) < c$.

We therefore have the following criteria for stability, as found in [2]. Let $H_2(z) = (z^2 + az + b)e^z - c$, where $a > 0$ and $b, c \in \mathbb{R}$, then all the zeros of $H_2(z)$ will have negative real parts, and thus equation (3.19) is asymptotically stable if and only if

(i) $b > 0$ and $-V(y_{**}) < c < V(y_*)$, or

(ii) $-a < b \leq 0$ and $-V(y_1) < c < b$,

where $a = a_1\tau$, $b = a_0\tau^2$ and $c = K\tau^2$, and where y_* and y_{**} denote the even non-negative and odd positive labelled root respectively whose value squared is closest to $b - \frac{a^2}{2}$, and y_1 denotes the root of $G_2(y) = 0$ in the interval $(0, \pi)$.

In order to plot the above criteria in the form of a stability chart, we first replace the inequalities (3.37) and (3.38) by

$$c = b, \quad c = V(y_2), \quad c = V(y_4), \quad \dots \ , \tag{3.43}$$

$$-V(y_1) = c, \quad -V(y_3) = c, \quad -V(y_5) = c, \quad \dots \ . \tag{3.44}$$

For a given value of a, (3.43) and (3.44) represent a set of curves in the (b, c) plane, and in order to satisfy the stability criteria, we must have that c lies below the lowest curve in the set (3.43) and above the topmost curve in the set (3.44). However since the curves (3.43) and (3.44) cross themselves, the zeros of $G_2(y)$ that yield the governing inequalities change with b. The resulting stability chart for equation (3.19), with the stability domain shaded grey, is shown in Figure 3.4 for $a = 1$, and for other values of a in Figure 3.5.

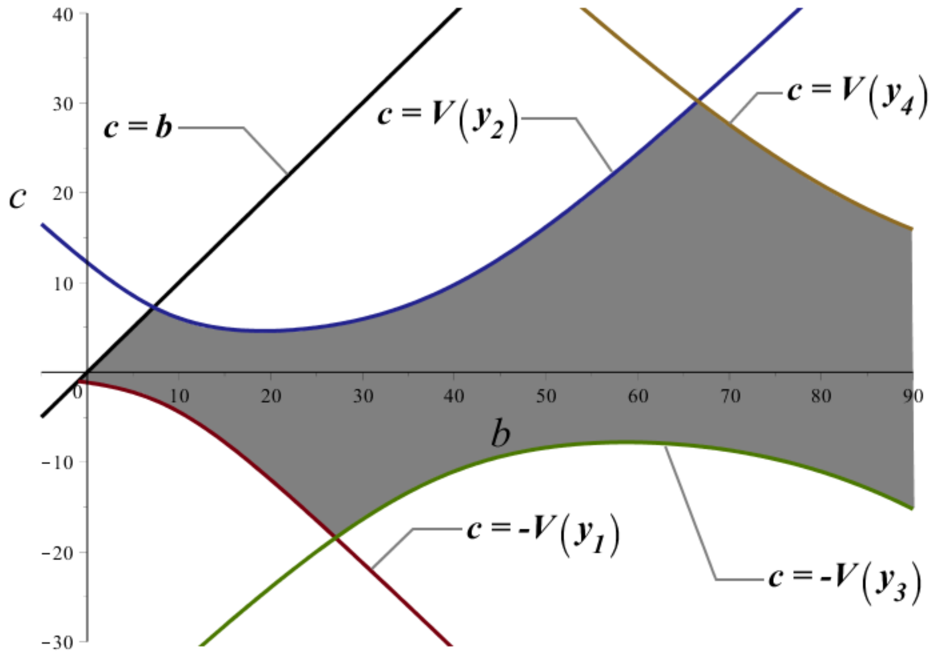


Figure 3.4: The stability chart of equation (3.19), in the (b, c) plane for $a = 1$.

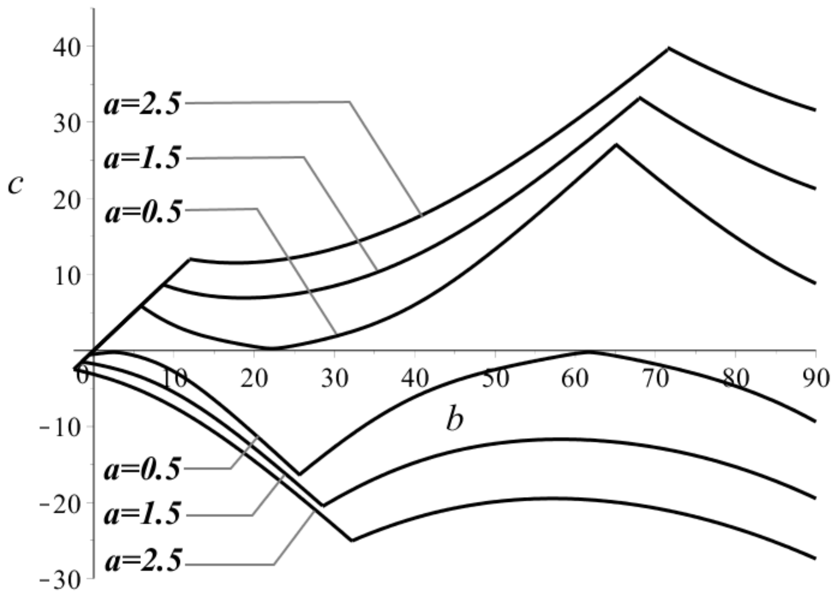


Figure 3.5: Stability charts of equation (3.19) showing the boundary curves for stability domains in the (b, c) plane for different values of a.

Chapter 4

The D-subdivision Method

The stability properties of differential-difference equations are often more easily represented in the form of stability charts that show the stable domains, or alternatively the number of unstable characteristic roots, in the coefficient space of the equation. These stability charts, for linear autonomous RDDEs, can be constructed via the D-subdivision method. This method, as given in [6] and in [9], is outlined in Section 4.1, before being applied to produce stability charts for three homogeneous linear autonomous RDDEs with constant coefficients.

4.1 Method

Suppose we have a homogeneous linear autonomous scalar RDDE of the general form

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}x^{(j)}(t-\tau_i) = 0 \ , \tag{4.1}$$

where $a_{0n} \neq 0$ and $a_{in} = 0$ for $i = 1, \ldots, m$, with the characteristic equation

$$D(\lambda) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\lambda^j e^{-\lambda\tau_i} = 0 \ , \tag{4.2}$$

the left hand side of which is often called a quasi-polynomial. In order to determine conditions under which the characteristic equation (4.2) only has roots with negative real parts, the D-subdivision method can be applied, which uses the fact that the roots of the characteristic equation (4.2) are continuous functions of the system parameters a_{ij} and τ_i.

The method is as follows, we first introduce $\lambda = \gamma \pm i\omega$, $\omega \geq 0$, into the characteristic equation (4.2) and then separate the resulting equation into its real and imaginary parts. Then letting $\gamma = 0$ yields the curves

$$R(\omega) = 0 \ , \quad S(\omega) = 0 \ , \quad \omega \in [0, \infty) \ , \tag{4.3}$$

where ω is the parameter of the curves, and where $R(\omega)$ and $S(\omega)$ denote the real and imaginary parts of $D(i\omega)$ respectively. Thus we can see that the characteristic equation has a pair of imaginary roots of the form $\lambda = \pm i\omega$ if and only if $R(\omega) = 0$ and $S(\omega) = 0$. The curves given by (4.3) are called the D-curves, and they subdivide the coefficient space of (4.1).

Since the roots of the characteristic equation (4.2) are continuous with respect to changes in the system parameters, the number of unstable roots may change only by the passage of some roots through an imaginary axis. Therefore the D-curves subdivide the coefficient space of (4.1) into domains in which the number of unstable roots is constant. Hence if the number of unstable roots is known for at least one point in each domain of D-subdivision, then the number of unstable roots is known for all points in every domain.

In order to determine the number of unstable roots in each domain, there are various methods available, such as using Stepan's formula as shown in [9]. However we shall use the calculation of the root-crossing direction along the D-curves, also shown in [9]. This is defined as the sign of the partial derivative of the real part of the characteristic root with respect to one of the system coefficients along the D-curves (4.3). Hence if the number of unstable roots is known for at least one point in the coefficient space, then by considering the root-crossing direction along the D-curves, the number of unstable roots in all other domains of the coefficient space can be found.

It should be noted that if the characteristic equation of (4.1) depends on two parameters then the D-subdivision method produces a stability chart in the coefficient space of the system, where the stability boundaries are given by the D-curves that bound domains with zero unstable characteristic roots.

4.2 The Hayes Equation

Recall the Hayes equation is given by

$$\dot{x}(t) - a_0 x(t) - b_0 x(t-\tau) = 0 \ , \tag{4.4}$$

where $a_0, b_0 \in \mathbb{R}$, $\tau > 0$, and the characteristic equation is given by

$$D(\lambda) = \lambda - a_0 - b_0 e^{-\lambda\tau} = 0 \ . \tag{4.5}$$

Following the D-subdivision method, we first substitute $\lambda = \gamma \pm i\omega$, $\omega \geq 0$ into (4.5) and then decompose the resulting equation into its real and imaginary parts to get

$$\text{Re} : \gamma - a_0 - b_0 e^{-\gamma\tau} cos(\omega\tau) = 0 \ , \tag{4.6}$$

$$\text{Im} : \omega + b_0 e^{-\gamma\tau} sin(\omega\tau) = 0 \ . \tag{4.7}$$

Letting $\gamma = 0$ in (4.6) and (4.7) gives

$$R(\omega) = -a_0 - b_0 cos(\omega\tau) = 0 \ , \tag{4.8}$$

$$S(\omega) = \omega + b_0 sin(\omega\tau) = 0 \ , \tag{4.9}$$

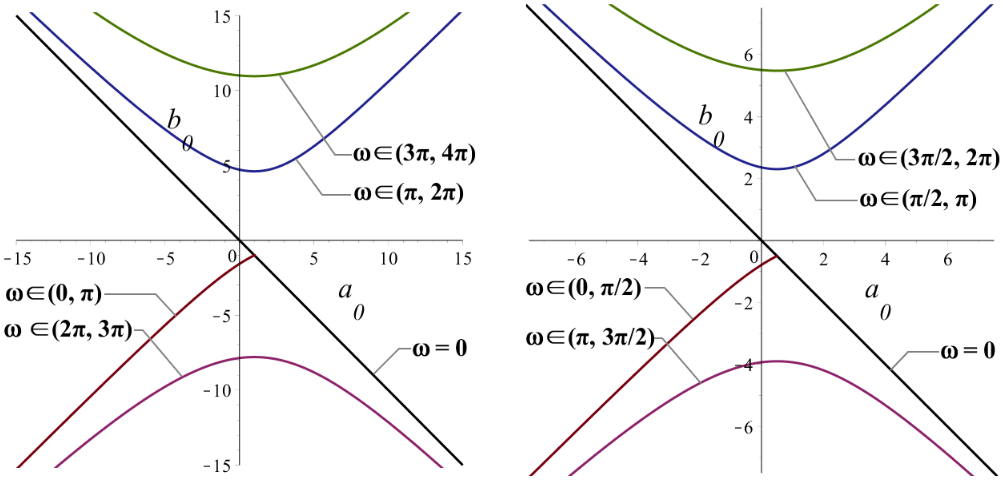
which yields the D-curves as a parametric function of ω in the form

$$\text{If} \quad \omega = 0 \ : \quad a_0 = -b_0 \ , \tag{4.10}$$

$$\text{If} \quad \omega\tau \neq k\pi \ , \quad k \in \mathbb{N} \ : \quad a_0 = \frac{\omega cos(\omega\tau)}{sin(\omega\tau)} \ , \quad b_0 = \frac{-\omega}{sin(\omega\tau)} \ . \tag{4.11}$$

The D-curves (4.10) and (4.11) are shown in Figures 4.1a and 4.1b in the coefficient space (a_0, b_0), for two different values of the delay. As mentioned in Section 4.1, these D-curves subdivide the coefficient space (a_0, b_0) into domains in which the number of unstable characteristic roots is constant. To determine the stability domains of (4.4), we first take the partial derivatives of (4.6) and (4.7) with respect to b_0[1]. This gives

[1] Using the notation γ'_{b_0} and ω'_{b_0}, to denote the partial derivatives of γ and ω with respect to b_0.



(a) $\tau = 1$ **(b)** $\tau = 2$

Figure 4.1: The D-curves (4.10) and (4.11) for different values of τ, for some initial parameter intervals.

$$\gamma^{'}_{b_0}(1 + b_0\tau e^{-\gamma\tau}cos(\omega\tau)) - e^{-\gamma\tau}cos(\omega\tau) + \omega^{'}_{b_0}b_0\tau e^{-\gamma\tau}sin(\omega\tau) = 0 \ , \quad (4.12)$$

$$\omega^{'}_{b_0}(1 + b_0\tau e^{-\gamma\tau}cos(\omega\tau)) + e^{-\gamma\tau}sin(\omega\tau) - \gamma^{'}_{b_0}b_0\tau e^{-\gamma\tau}sin(\omega\tau) = 0 \ , \quad (4.13)$$

and as $\gamma = 0$ along the D-curves, equations (4.12) and (4.13) become

$$\gamma^{'}_{b_0}(1 + b_0\tau cos(\omega\tau)) - cos(\omega\tau) + \omega^{'}_{b_0}b_0\tau sin(\omega\tau) = 0 \ , \quad (4.14)$$

$$\omega^{'}_{b_0}(1 + b_0\tau cos(\omega\tau)) + sin(\omega\tau) - \gamma^{'}_{b_0}b_0\tau sin(\omega\tau) = 0 \ . \quad (4.15)$$

The solution of (4.14) and (4.15) for $\gamma^{'}_{b_0}$ is given by

$$\gamma^{'}_{b_0} = \frac{cos(\omega\tau) + b_0\tau}{(1 + b_0\tau cos(\omega\tau))^2 + (b_0\tau sin(\omega\tau))^2} \ , \quad (4.16)$$

which we can use to calculate the root-crossing direction along the D-curves (4.10) and (4.11).

Firstly as $\omega = 0$ along the D-curve (4.10), equation (4.16) simplifies to $\gamma^{'}_{b_0} = \frac{1}{1+b_0\tau}$, and so we see that if $b_0 > \frac{-1}{\tau}$ then $\gamma^{'}_{b_0}$ is positive. Since this D-curve (4.10) is associated with a real characteristic root crossing the imaginary axis at the origin, if $\gamma^{'}_{b_0}$ is positive then as b_0 is increased the characteristic root crosses the imaginary axis through the origin from left to right.

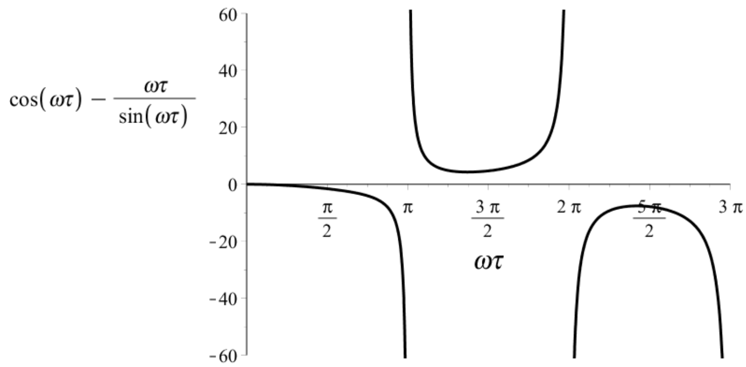


Figure 4.2: Root-crossing direction along the D-curves (4.11).

Hence if the D-curve (4.10) is crossed, by increasing b_0, in a region where $b_0 > \frac{-1}{\tau}$, then we have that a stable characteristic root becomes unstable. Similarly for $b_0 < \frac{-1}{\tau}$ we have $\gamma^{'}_{b_0}$ is negative and so the characteristic root crosses the imaginary axis from right to left, through the origin, as b_0 is increased. Hence if we cross (4.10) by increasing b_0, in a region where $b_0 < \frac{-1}{\tau}$, then an unstable characteristic root becomes stable.

In order to calculate the root-crossing direction along the D-curves (4.11), we first notice that the denominator of (4.16) is positive, and so equation (4.16) gives

$$sgn(\gamma^{'}_{b_0}) = sgn(cos(\omega\tau) + b_0\tau) \ . \tag{4.17}$$

Thus as we have that $b_0 = \frac{-\omega}{sin(\omega\tau)}$ along the D-curves (4.11), the root-crossing direction along these D-curves will be given by the sign of $cos(\omega\tau) - \frac{\omega\tau}{sin(\omega\tau)}$. This leads to two possible cases, which are most readily seen from Figure 4.2.

Firstly along the D-curves (4.11), where $\omega\tau \in ((2k-1)\pi, 2k\pi)$, for $k \in \mathbb{Z}^+$, we have $cos(\omega\tau) - \frac{\omega\tau}{sin(\omega\tau)} > 0$, and so $\gamma^{'}_{b_0}$ is positive. Therefore as the D-curves (4.11), where $\omega\tau \in ((2k-1)\pi, 2k\pi)$, for $k \in \mathbb{Z}^+$, are crossed by increasing b_0, we must have that two stable characteristic roots become unstable. This is because the D-curves (4.11) are associated with a complex conjugate pair of characteristic roots of the form $\lambda = \pm i\omega$.

Similarly along the D-curves (4.11), where $\omega\tau \in (2k\pi, (2k+1)\pi)$, for $k \in \mathbb{N}$, we have $cos(\omega\tau) - \frac{\omega\tau}{sin(\omega\tau)} < 0$ and so $\gamma^{'}_{b_0}$ is negative. Therefore as the D-curves (4.11), where $\omega\tau \in (2k\pi, (2k+1)\pi)$, for $k \in \mathbb{N}$, are crossed by increasing b_0, we have that two unstable characteristic roots become stable.

As the root-crossing direction along the D-curves (4.10) and (4.11) has been found, in order to determine the number of unstable characteristic roots in each domain of the subdivided coefficient space, we just need to know the number of unstable roots at a single point in the coefficient space of (4.4). Since we have that when $b_0 = 0$, (4.4) reduces to an ordinary differential equation (ODE) with one characteristic root, given by $\lambda = a$, the domain in the coefficient space (a_0, b_0) where $a_0 < 0$ and $b_0 = 0$, corresponding to an asymptotically stable ODE, must have zero unstable characteristic roots. Using this information, the number of unstable characteristic roots in every other domain of the coefficient space (a_0, b_0) can be found.

The corresponding stability chart of equation (4.4), showing the number of unstable roots in each domain, is given in Figure 4.3 for $\tau = 1$. In Figure 4.3 the black line represents the D-curve (4.10), and the blue and red lines represent the D-curves (4.11) where we have that $\omega\tau \in ((2k-1)\pi, 2k\pi)$, $k \in \mathbb{Z}^+$, and $\omega\tau \in (2k\pi, (2k+1)\pi)$, $k \in \mathbb{N}$, respectively. The shaded domain in Figure 4.3 represents the domain where there are zero unstable characteristic roots. Hence equation (4.4) is asymptotically stable in this shaded domain.

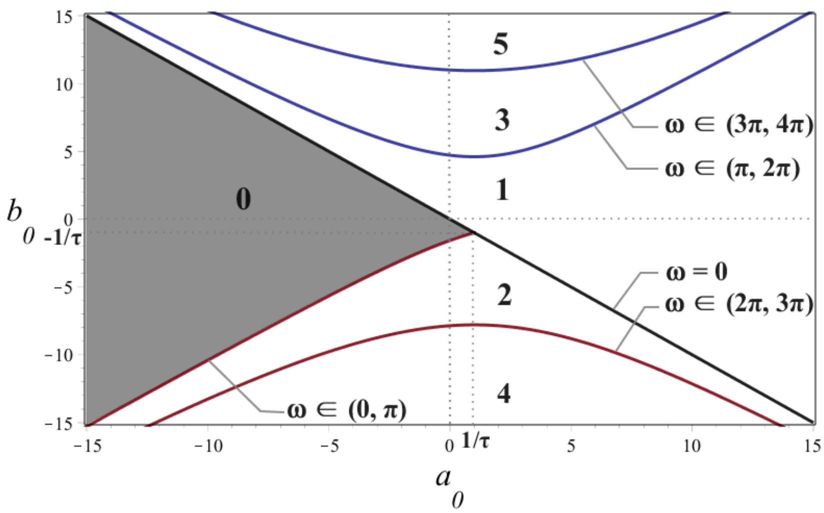


Figure 4.3: The stability chart of equation (4.4) for $\tau = 1$.

4.3 An Undamped Delayed Oscillator

In the paper [8], Stepan shows that many of the simple mechanical models of balancing with time-delay, that use the Newtonian equations of the inverted pendulum, can be reduced to a second order homogeneous linear autonomous scalar RDDE having the standard linearised form

$$\ddot{x}(t) + b_1\dot{x}(t-\tau) + a_0 x(t) + b_0 x(t-\tau) = 0 \ , \tag{4.18}$$

where $a_0 < 0$, $b_0, b_1 \in \mathbb{R}$ and $\tau > 0$, at the upward position of the pendulum. Since the trivial solution $x(t) \equiv 0$ represents the upward position of the pendulum, the importance of its stability is clear. This equation also represents the general form of an undamped harmonic oscillator subjected to time-delayed proportional derivative control, where a_0 represents the negative stiffness, b_0 the proportional gain, b_1 the derivative gain, and τ is the feedback delay.

The characteristic function of (4.18) is given by

$$D(\lambda) = \lambda^2 + a_0 + b_1\lambda e^{-\lambda\tau} + b_0 e^{-\lambda\tau} \ , \tag{4.19}$$

and substituting $\lambda = \gamma \pm i\omega$, $\omega \geq 0$ into the characteristic equation and separating the resulting equation into real and imaginary parts yields

$$Re : \gamma^2 - \omega^2 + a_0 + b_0 e^{-\gamma\tau} cos(\omega\tau) + b_1\gamma e^{-\gamma\tau} cos(\omega\tau) + b_1\omega e^{-\gamma\tau} sin(\omega\tau) = 0 \ , \tag{4.20}$$

$$Im : 2\gamma\omega - b_0 e^{-\gamma\tau} sin(\omega\tau) + b_1\omega e^{-\gamma\tau} cos(\omega\tau) - b_0\gamma e^{-\gamma\tau} sin(\omega\tau) = 0 \ . \tag{4.21}$$

Letting $\gamma = 0$ in (4.20) and (4.21) gives

$$R(\omega) = a_0 - \omega^2 + b_0 cos(\omega\tau) + b_1\omega sin(\omega\tau) = 0 \ , \tag{4.22}$$

$$S(\omega) = b_1\omega cos(\omega\tau) - b_0 sin(\omega\tau) = 0 \ , \tag{4.23}$$

which yields the D-curves as a parametric function of ω in the form

$$\text{If} \quad \omega = 0 \ : \quad b_0 = -a_0 \ , \tag{4.24}$$

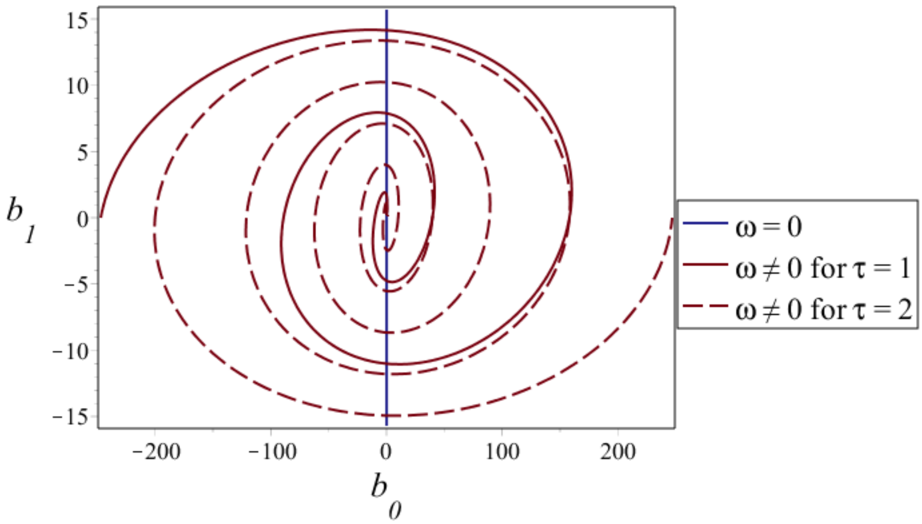


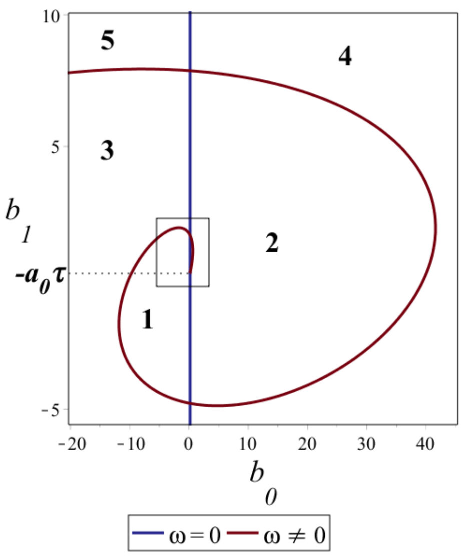
Figure 4.4: The D-curves (4.24) and (4.25) for $a_0 = -0.2$ and different values of τ.

$$\text{If} \quad \omega \neq 0 \ : \quad b_0 = (\omega^2 - a_0)cos(\omega\tau) \ , \quad b_1 = \frac{\omega^2 - a_0}{\omega} sin(\omega\tau) \ . \tag{4.25}$$

As shown in Figure 4.4, for fixed a_0, the straight line (4.24) and the spiralling curve (4.25) divide the coefficient space (b_0, b_1) into infinitely many domains. In order to determine the stability domains in the parameter space (b_0, b_1), we first calculate the root-crossing direction along the D-curve (4.24) by taking the partial derivatives of (4.20) and (4.21) with respect to b_0, and since $\gamma = 0$, $\omega = 0$ and $b_0 = -a_0$ along this D-curve, we get

$$\gamma'_{b_0} = \frac{-1}{a_0\tau + b_1} \ . \tag{4.26}$$

Hence for $b_1 < -a_0\tau$ we have γ'_{b_0} is positive, and for $b_1 > -a_0\tau$ we have γ'_{b_0} is negative. Since the D-curve (4.24) is associated with a real characteristic root crossing the imaginary axis through the origin, if this D-curve is crossed by increasing b_0, in a region where $b_1 < -a_0\tau$, then the associated characteristic root crosses the imaginary axis from left to right through the origin, and so a stable characteristic root becomes unstable. Similarly if the D-curve (4.24) is crossed by increasing b_0, in a region where $b_1 > -a_0\tau$, then the characteristic



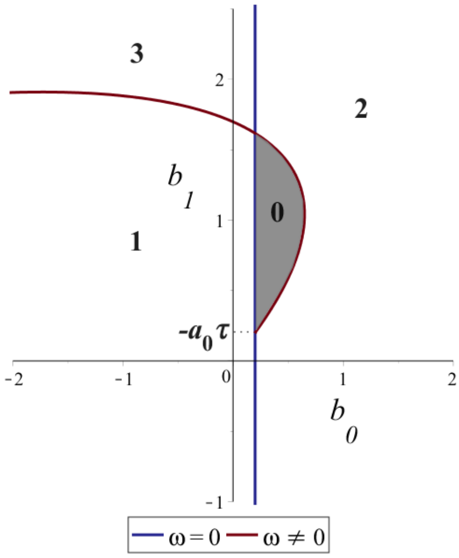
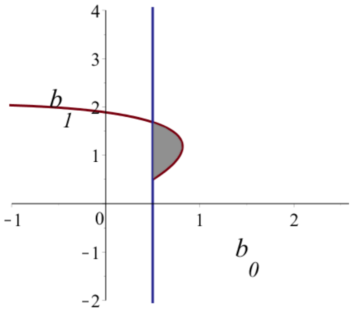


Figure 4.5: The stability chart of equation (4.18) showing the numbers of unstable roots in the coefficient space (b_0, b_1) for $a_0 = -0.2$ and $\tau = 1$, where the rightmost figure is the marked area in the left figure on a larger scale.

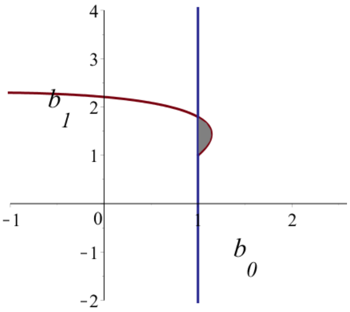
root associated with (4.24) crosses the imaginary axis from right to left, and so an unstable characteristic root becomes stable.

Since we know that when $b_0 = 0$ and $b_1 = 0$ in (4.18), the resulting ordinary differential equation has one unstable characteristic root, using the calculation of the root-crossing direction along the D-curve (4.24), the number of unstable characteristic roots in every other domain of the (b_0, b_1) coefficient space can be determined. The resulting stability chart, showing the number of unstable roots in each domain, is given in Figure 4.5 for $a_0 = -0.2$ and $\tau = 1$. The shaded domain represents the domain where there are zero unstable roots, i.e. the domain in which equation (4.18) is asymptotically stable.

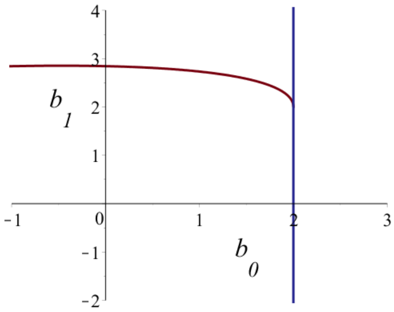
The stability domains of equation (4.18) with $\tau = 1$ are shown in Figures 4.6a to 4.6c for different values of a_0, and it is apparent that as a_0 decreases the stability domain shrinks, until it disappears completely at a critical value of a_0. Similarly for fixed a_0, as shown in Figure 4.6d, as the delay τ increases, the stability domain shrinks, until it disappears completely



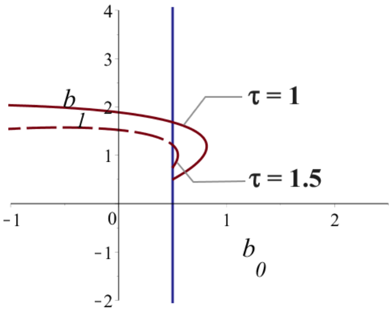
(a) $a_0 = -0.5$, $\tau = 1$.



(b) $a_0 = -1$, $\tau = 1$.



(c) $a_0 = -2$, $\tau = 1$.



(d) $a_0 = -0.5$.

Figure 4.6: The stability domains for equation (4.18) for different parameter values.

at a critical value of the delay τ. These critical values can be determined, as shown in [9], by analysing the tangent of the parametric curve (4.25) at $\omega = 0$, which can be found by applying L'Hôpital's rule, this analysis yields

$$\lim_{\omega\to 0} \frac{\mathrm{d}b_1}{\mathrm{d}b_0} = \lim_{\omega\to 0} \frac{\frac{\mathrm{d}b_1}{\mathrm{d}\omega}}{\frac{\mathrm{d}b_0}{\mathrm{d}\omega}} = \frac{12\tau + 2a_0\tau^3}{12 + 6a_0\tau^2}\ . \tag{4.27}$$

Therefore in order for the tangent to be vertical we need $12 + 6a_0\tau^2 = 0$, and so the critical value of a_0 is given by $a_0^{crit} = \frac{-2}{\tau^2}$. This can also be rewritten in terms of the delay, so for fixed a_0 the critical delay is given by $\tau^{crit} = \sqrt{\frac{-2}{a_0}}$. Hence if $a_0 < a_0^{crit}$ or $\tau > \tau^{crit}$ then equation (4.18) is unstable for all values of b_0 and b_1.

4.4 Damped Delayed Harmonic Oscillators

Let us now look at the more general form of damped delayed oscillators like (3.19), that are governed by the second order homogeneous linear autonomous scalar RDDE

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = Kx(t-\tau) \ , \tag{4.28}$$

where $a_1, a_0, K \in \mathbb{R}$ and $\tau > 0$. The characteristic equation of (4.28) is given by

$$D(\lambda) = \lambda^2 + a_1\lambda + a_0 - Ke^{-\lambda\tau} = 0 \ , \tag{4.29}$$

and substituting $\lambda = \gamma \pm i\omega$, $\omega \geq 0$ into (4.29) and separating the resulting equation into real and imaginary parts yields

$$\text{Re} \ : \gamma^2 + a_1\gamma + a_0 - \omega^2 - Ke^{-\gamma\tau}cos(\omega\tau) = 0 \ , \tag{4.30}$$

$$\text{Im} \ : 2\gamma\omega + a_1\omega + Ke^{-\gamma\tau}sin(\omega\tau) = 0 \ . \tag{4.31}$$

In the case $\gamma = 0$, we get

$$R(\omega) = a_0 - \omega^2 - Kcos(\omega\tau) = 0 \ , \tag{4.32}$$

$$S(\omega) = a_1\omega + Ksin(\omega\tau) = 0 \ , \tag{4.33}$$

which can be rewritten to give the D-curves in the two cases: $a_1 = 0$ and $a_1 \neq 0$. For $a_1 = 0$, the D-curves are given by

$$\text{If} \quad \omega\tau = k\pi \ : \quad K = (-1)^k\left(a_0 - \left(\tfrac{k\pi}{\tau}\right)^2\right) \ , \tag{4.34}$$

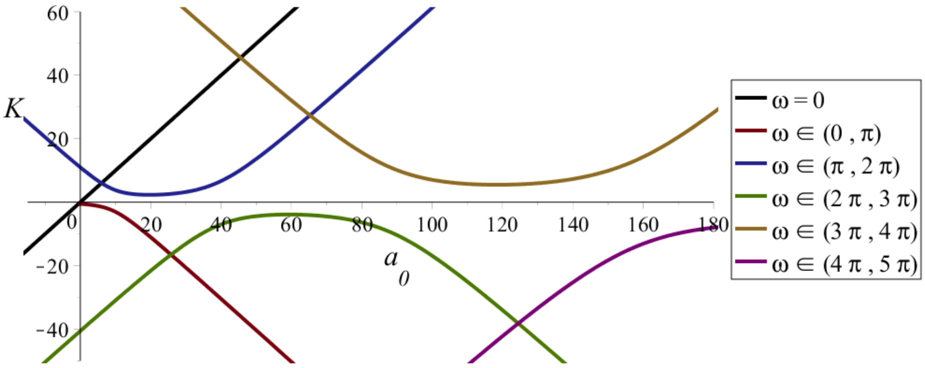
$$\text{If} \quad \omega\tau \neq k\pi \ : \quad a_0 = \omega^2 \ , \quad K = 0 \ , \tag{4.35}$$

which are straight lines in the (a_0, K) plane, and for $a_1 \neq 0$ we have

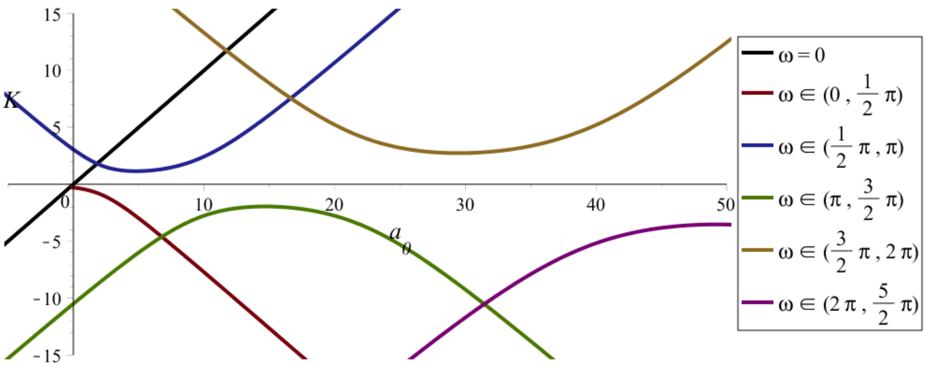
$$\text{If} \quad \omega = 0 \ : \quad a_0 = K \ , \tag{4.36}$$

$$\text{If } \omega\tau \neq k\pi, \ k \in \mathbb{N} \ : \quad a_0 = \omega^2 - \frac{a_1\omega cos(\omega\tau)}{sin(\omega\tau)} \ , \quad K = \frac{-a_1\omega}{sin(\omega\tau)} \ . \tag{4.37}$$

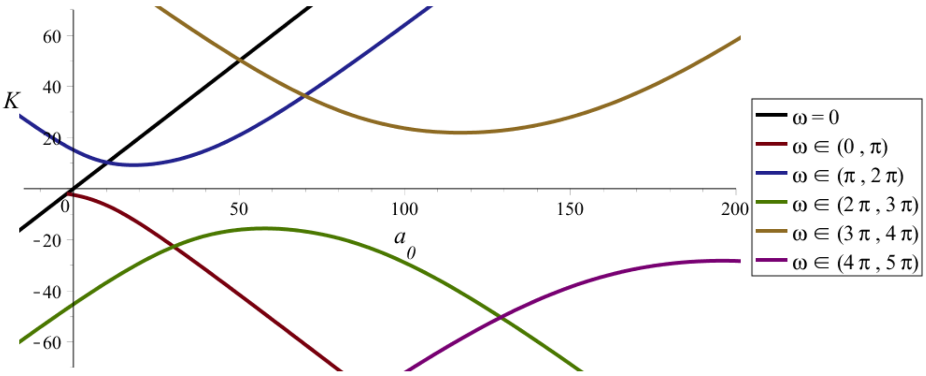
The D-curves (4.36) and (4.37) are shown in Figures 4.7 and 4.8 for different values of a_1 and τ.



(a) $a_1 = 0.5$ and $\tau = 1$.

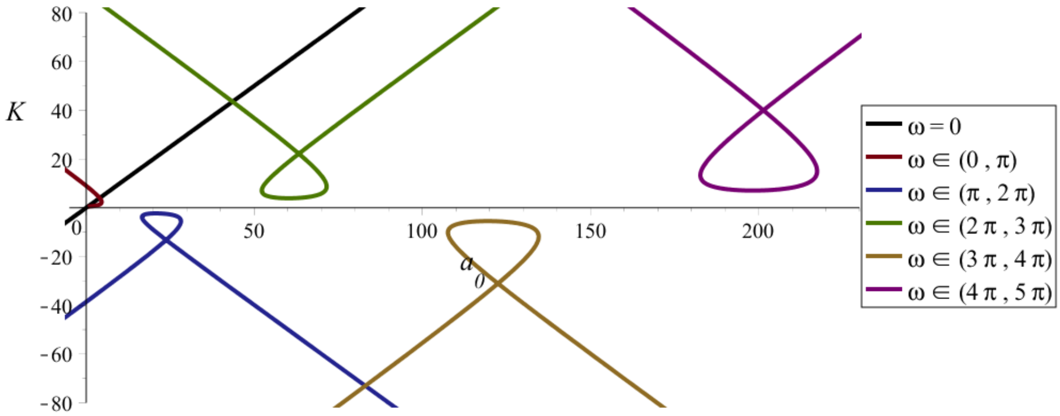


(b) $a_1 = 0.5$ and $\tau = 2$.

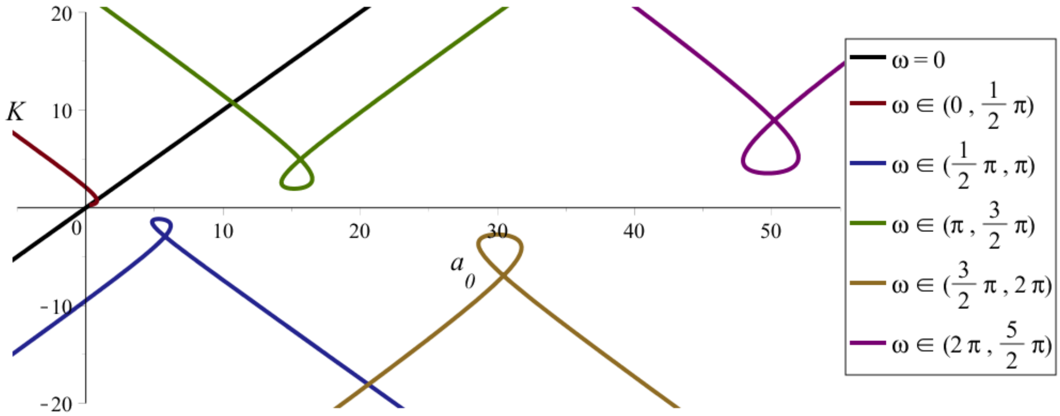


(c) $a_1 = 2$ and $\tau = 1$.

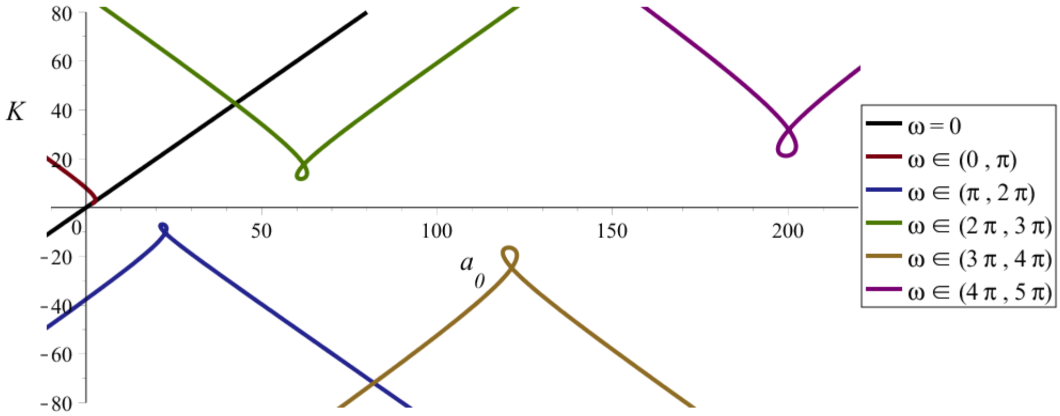
Figure 4.7: The D-curves (4.36) and (4.37) for different values of $a_1 > 0$ and τ.



(a) $a_1 = -0.5$ and $\tau = 1$.



(b) $a_1 = -0.5$ and $\tau = 2$.



(c) $a_1 = -1.5$ and $\tau = 1$.

Figure 4.8: The D-curves (4.36) and (4.37) for different values of $a_1 < 0$ and τ.

In order to determine the stability domains of (4.28), we first calculate the root-crossing direction along the D-curves, and so we take the partial derivatives of (4.30) and (4.31) with respect to a_0, and as $\gamma = 0$ along the D-curves, this yields

$$\gamma^{'}_{a_0}\left[a_1 + K\tau cos(\omega\tau)\right] + \omega^{'}_{a_0}\left[K\tau sin(\omega\tau) - 2\omega\right] + 1 = 0 \ , \tag{4.38}$$

$$\omega^{'}_{a_0}\left[a_1 + K\tau cos(\omega\tau)\right] - \gamma^{'}_{a_0}\left[K\tau sin(\omega\tau) - 2\omega\right] = 0 \ , \tag{4.39}$$

and the solution of (4.38) and (4.39) for $\gamma^{'}_{a_0}$ is given by

$$\gamma^{'}_{a_0} = \frac{-\left(a_1 + K\tau cos(\omega\tau)\right)}{(a_1 + K\tau cos(\omega\tau))^2 + (K\tau sin(\omega\tau) - 2\omega)^2} \ . \tag{4.40}$$

The analysis of (4.40) is split into the two cases: $a_1 = 0$ and $a_1 \neq 0$.

<u>Case 1:</u> $a_1 = 0$

In this case it is adequate to just calculate the root-crossing direction along the D-curves (4.34) where we have $\omega\tau = k\pi$. Along these D-curves, as the denominator of (4.40) is positive, the root-crossing direction is given by the sign of $\left(\frac{k\pi}{\tau}\right)^2 - a_0$. Hence if $a_0 < (\frac{k\pi}{\tau})^2$ then $\gamma^{'}_{a_0}$ is positive and if $a_0 > (\frac{k\pi}{\tau})^2$ then $\gamma^{'}_{a_0}$ is negative. Since the D-curve (4.34) with $k = 0$ is associated with a real characteristic root crossing the imaginary axis through the origin, if this D-curve is crossed by increasing a_0, in a region where $a_0 < 0$ then a stable characteristic root becomes unstable, and if this D-curve is crossed by increasing a_0 in a region where $a_0 > 0$ then an unstable characteristic root becomes stable. Furthermore, as the D-curves (4.34) where $k > 0$, are associated with a complex conjugate pair of characteristic roots, if they are crossed by increasing a_0, in a region where $a_0 < (\frac{k\pi}{\tau})^2$, then two stable characteristic roots become unstable, and if they are crossed by increasing a_0 in a region where $a_0 > (\frac{k\pi}{\tau})^2$, then two unstable characteristic roots become stable.

<u>Case 2:</u> $a_1 \neq 0$

As $\omega = 0$ along the D-curve (4.36), equation (4.40) reduces to $\gamma^{'}_{a_0} = \frac{-1}{a_1 + K\tau}$. Hence if $K < \frac{-a_1}{\tau}$ then $\gamma^{'}_{a_0}$ is positive and if $K > \frac{-a_1}{\tau}$ then $\gamma^{'}_{a_0}$ is negative.

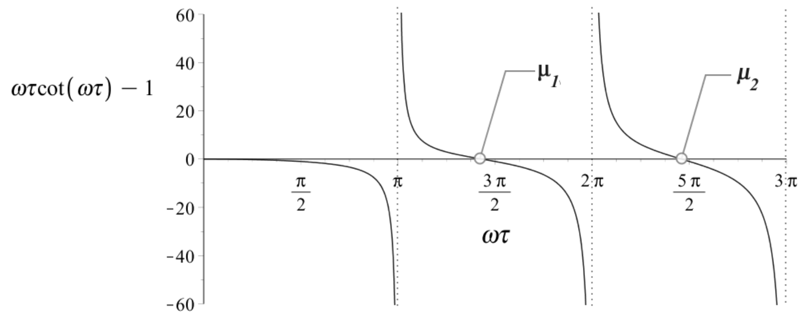


Figure 4.9: Plot showing the values of $\omega\tau cot(\omega\tau) - 1$ for $\omega\tau \in (0, 3\pi)$.

Since the D-curve (4.36) is associated with a real characteristic root crossing the imaginary axis through the origin, if it is crossed by increasing a_0, in a region where $K < \frac{-a_1}{\tau}$ then a stable characteristic root becomes unstable. Similarly if (4.36) is crossed by increasing a_0, in a region where $K > \frac{-a_1}{\tau}$, then an unstable characteristic root becomes stable.

In order to determine the root-crossing direction along the D-curves (4.37), we again use the fact that the denominator of (4.40) is positive, and so (4.40) gives

$$sgn(\gamma_{a_0}^{'}) = sgn(-\left[a_1 + K\tau cos(\omega\tau)\right]) \ . \tag{4.41}$$

However $K = \frac{-a_1\omega}{sin(\omega\tau)}$ along the D-curves (4.37), and so (4.41) becomes

$$sgn(\gamma_{a_0}^{'}) = sgn(a_1 \left[\omega\tau cot(\omega\tau) - 1\right]) \ . \tag{4.42}$$

Using Figure 4.9, we can see that $\omega\tau cot(\omega\tau) - 1 < 0$ for $\omega\tau \in (0, \pi)$, whereas in every other π-interval $(\pi, 2\pi)$,$(2\pi, 3\pi)$, $\ldots$, the sign of $\omega\tau cot(\omega\tau) - 1$ changes. In order to be able to determine the sign of $\gamma_{a_0}^{'}$ we first label the zeros of $\omega\tau cot(\omega\tau) - 1$ in the following way, to the zero in the interval $\omega\tau \in (k\pi, (k + 1)\pi)$, for $k \in \mathbb{Z}^+$, we attach the label μ_k. Therefore for $\omega\tau \in (k\pi, \mu_k)$, $k \in \mathbb{Z}^+$, we have that $\omega\tau cot(\omega\tau) - 1 > 0$, and for $\omega\tau \in (\mu_k, (k + 1)\pi)$, $k \in \mathbb{Z}^+$, we have $\omega\tau cot(\omega\tau) - 1 < 0$.

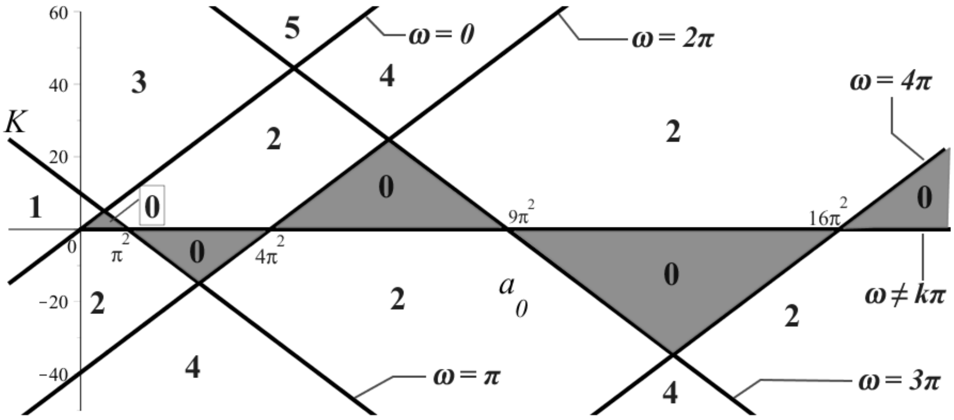
Hence for $a_1 > 0$, if $\omega\tau \in (0, \pi)$ or $\omega\tau \in (\mu_k, (k + 1)\pi)$, $k \in \mathbb{Z}^+$, then $\gamma_{a_0}^{'}$ is negative, and if $\omega\tau \in (k\pi, \mu_k)$, $k \in \mathbb{Z}^+$, then $\gamma_{a_0}^{'}$ is positive. As the D-curves

(4.37) are associated with a complex conjugate pair of characteristic roots of the form $\lambda = \pm i\omega$ we have the following result. If the D-curves (4.37), where $\omega\tau \in (k\pi, \mu_k)$ for $k \in \mathbb{Z}^+$, are crossed by increasing a_0, then two stable roots become unstable. Similarly if the D-curves (4.37), where either $\omega\tau \in (0, \pi)$ or $\omega\tau \in (\mu_k, (k+1)\pi)$, $k \in \mathbb{Z}^+$, are crossed by increasing a_0, then two unstable roots become stable.

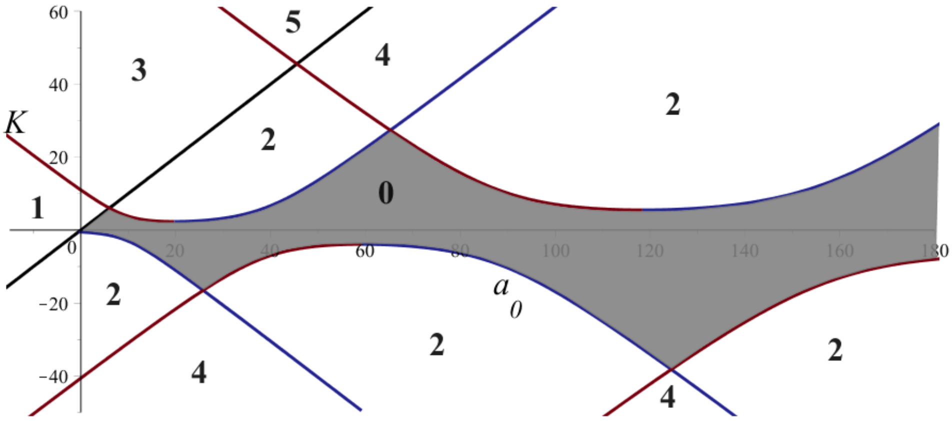
Whereas for $a_1 < 0$ we have that if $\omega\tau \in (k\pi, \mu_k)$, $k \in \mathbb{Z}^+$, then γ'_{a_0} is negative, and if $\omega\tau \in (0, \pi)$ or $\omega\tau \in (\mu_k, (k+1)\pi)$, $k \in \mathbb{Z}^+$, then γ'_{a_0} is positive. Thus if the D-curves (4.37), where $\omega\tau \in (k\pi, \mu_k)$ for $k \in \mathbb{Z}^+$, are crossed by increasing a_0, then two unstable roots become stable, and if the D-curves (4.37), where either $\omega\tau \in (0, \pi)$ or $\omega\tau \in (\mu_k, (k+1)\pi)$, $k \in \mathbb{Z}^+$, are crossed by increasing a_0, then two stable roots become unstable.

Thus the root-crossing direction has been found, in the case of $a_1 = 0$ along the D-curves (4.34), and in the case of $a_1 \neq 0$ along the D-curves (4.36) and (4.37). In order to establish the stability domains of (4.28), all that needs to be determined is the number of unstable roots at one point in the coefficient space (a_0, K). As $K = 0$, $a_0 < 0$ in (4.28), corresponds to an ODE with one unstable root, using the calculation of the root-crossing direction along the D-curves (4.34), (4.36) and (4.37) as found above, the number of unstable roots in every other domain of the coefficient space (a_0, K) can be found.

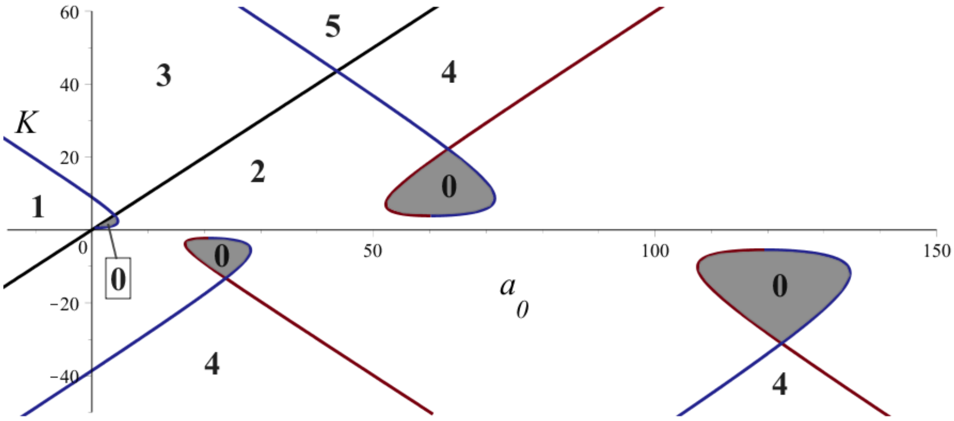
The resulting stability charts, showing the number of unstable roots in each domain of the coefficient space (a_0, K), are shown in Figures 4.10a to 4.10c for $a_1 = 0$, $a_1 = 0.5$, $a_1 = -0.5$ and $\tau = 1$. It should be noted that in Figures 4.10b and 4.10c, the black lines represent the D-curve (4.36), and the burgundy and navy lines represent the D-curves (4.37) where $\omega\tau cot(\omega\tau) - 1 > 0$ and $\omega\tau cot(\omega\tau) - 1 < 0$ respectively. In Figures 4.10a to 4.10c, the domains in which there are zero unstable roots, and hence the domains where equation (4.28) is asymptotically stable are shaded grey. As can be seen from Figure 4.10b, the stability domains for $a_1 = 0$ consist of an infinite number of triangles adjoining the axis $K = 0$, whereas for $a_1 \neq 0$ the stability domains take form in or around these triangles.



(a) $a_1 = 0$ and $\tau = 1$.



(b) $a_1 = 0.5$ and $\tau = 1$.



(c) $a_1 = -0.5$ and $\tau = 1$.

Figure 4.10: The stability charts of equation (4.28) for different parameter values, showing the numbers of unstable roots in the domains of the coefficient space (a_0, K).

Chapter 5

Summary

In this project two different methods for the stability analysis of linear autonomous RDDEs are applied to some fundamental first and second order homogeneous scalar equations. The first method uses Pontryagin's results on the zeros of exponential polynomials, and is used to investigate the stability of two equations. The first is the Hayes equation which, by loosely following the proofs in [1] and the appendix of [4], yields the stability criteria as found in [5]. The method is then applied to the general form of a damped delayed harmonic oscillator, where the damping takes only positive values, the characteristic equation of which reduces to an exponential polynomial of the form (3.21). The stability analysis, using this method, of a similar exponential polynomial $H(z) = (z^2 + az + b)e^z + c$ for $a > 0$, $b \geq 0$ and $c \in \mathbb{R}$ can be found in [1], however this analysis is shown to be defective in [2]. The analysis of the exponential polynomial (3.21), for $a > 0$ and $b, c \in \mathbb{R}$, as found in this project uses techniques from both [1] and [2], to show the stability criteria found in [2].

The second method is called the D-subdivision method, and is used to produce stability charts for three homogeneous linear autonomous scalar RDDEs. To first illustrate the method, we use the Hayes equation, and this obtains a stability chart in the coefficient space (a_0, b_0) equivalent to the stability chart obtained via the first method. The second equation is an important second order equation, of the form as found in [8], which yields the stability chart as shown in [8] and [9]. Finally the third equation is the general form of the damped delayed harmonic oscillator as given in the first method, although the damping can now take negative as well as positive values, and we obtain the stability charts as shown in [3].

The Pontryagin method can also be used to analyse the stability of damped delayed harmonic oscillators with negative damping, as shown in [2], however trying to determine the appropriate stability criteria is complicated. The analysis using the D-subdivision is not trivial either due to the intrinsic difficulties in trying to determine the number of unstable roots in each domain of D-subdivision. It is also shown in [2] that the Pontryagin method can be used to analyse polynomials of the form $H(z) = (z^2 + az + b)e^z + cz^n$, for $n = 1, 2$, and for systems with more than one delay term. However this method can't be used in general for equations of the form (2.5) with incommensurable delays. Instead for equations of this form, another analytical method can be used, as shown in [6], that uses a theorem of Chebotarev, however this method is not efficient in practice due to the infinitely many inequalities that have to be considered.

In this project the focus has been on some basic scalar RDDEs, and so the stability investigations can be approached analytically. However there are also various numerical methods available for the stability analysis of linear autonomous RDDEs with constant coefficients, such as the integral criterion of stability, that uses the argument principle from complex analysis, or the DDE-biftool package for Matlab, which can approximate the rightmost characteristic root of the DDE. It should be noted that for the stability analysis of more complicated equations, like RFDEs with distributed or time-dependent delays, only numerical methods are viable. The effects of these different types of delays on the stability domains of the equations given in this project, would represent an interesting area to study further.

Bibliography

[1] Bellman, R. and Cooke, K. (1963). *Differential-difference equations.* New York: Academic Press.

[2] Bhatt, S.J. and Hsu, C.S. (1966). 'Stability criteria for second-order dynamical systems with time lag'. *J. Appl. Mech.* **33**(1), pp.113-118.

[3] Bhatt, S.J. and Hsu, C.S. (1966). 'Stability charts for second-order dynamical systems with time lag'. *J. Appl. Mech.* **33**(1), pp.119-124.

[4] Hale, J.K. and Lunel, S.M.V. (1993). *Introduction to functional differential equations.* New York: Springer-Verlag.

[5] Hayes, N.D. (1950). 'Roots of the transcendental equation associated with a certain differential-difference equation'. *J. London Math. Soc.* **s1-25**(3), pp.226-232.

[6] Kolmanovskii, V.B. and Nosov, V.R. (1986). *Stability of functional differential equations.* London: Academic Press.

[7] Pontryagin, L. S. (1955). 'On the zeros of some elementary transcendental functions'. *Amer. Math. Soc. Transl.* **1**(2), pp.95-110.

[8] Stepan, G. (2009). 'Delay effects in the human sensory system during balancing'. *Phil. Trans. R. Soc. A.* **367**(1891), pp.1195-1212.

[9] Stepan, G. and Insperger, T. (2011). *Semi-Discretization for Time-Delay Systems.* New York: Springer-Verlag.