Queen Mary University of London

MTH4101/4201 Calculus II

Lecture Notes

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1 Infinite sequences and series

1.1 Sequences [Thomas' Calculus, Section 9.1]

A **sequence** is a list of numbers in a given *order*:

$$a_1, a_2, a_3, \ldots, a_n, \ldots .$$

Each of the a_1, a_2, etc. represents a number; these are the *terms* of the sequence. For example

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

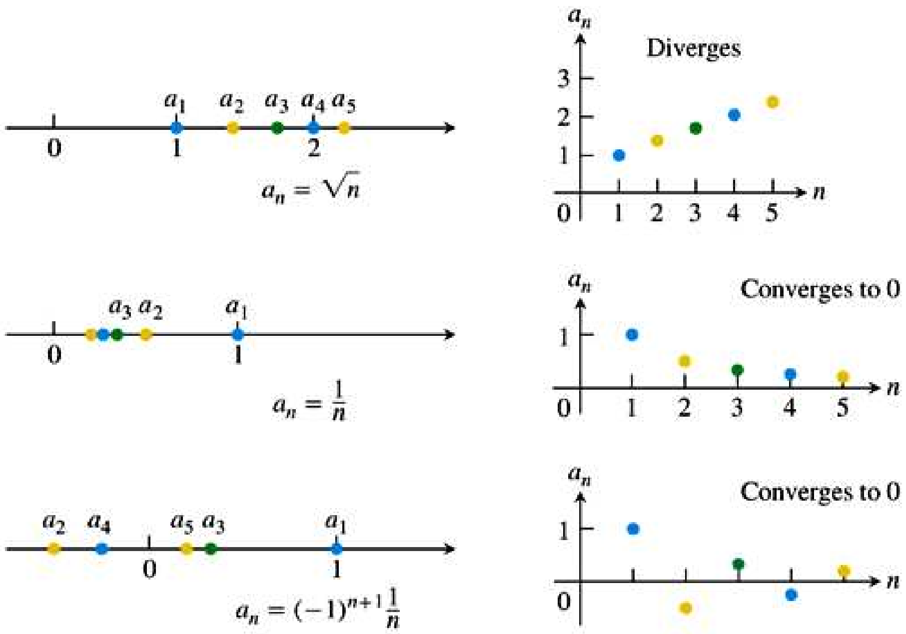
has first term $a_1 = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the *index* of a_n and denotes where a_n occurs in the list.
We can consider the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ as a function that sends 1 to a_1, 2 to a_2, etc. and in general sends the positive integer n to the nth term a_n.

DEFINITION Infinite Sequence
An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

Sequences can be described by *rules* or by *listing terms*. For example,

$$\begin{aligned}
a_n &= \sqrt{n} & \{a_n\} &= \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\} = \{\sqrt{n}\}_{n=1}^{\infty} \\
b_n &= (-1)^{n+1}(1/n) & \{b_n\} &= \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\} \\
c_n &= (n-1)/n & \{c_n\} &= \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n-1}{n}, \ldots\right\} \\
d_n &= (-1)^{n+1} & \{d_n\} &= \{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}
\end{aligned}$$

Sequences can be illustrated graphically either as points on a real axis or as the graph of a function defining the sequence:



Consider the following sequences:

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$	terms approach 0 as n gets large
$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1-\frac{1}{n}, \ldots\right\}$	terms approach 1 as n gets large
$\left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots, \sqrt{n}, \ldots\right\}$	terms get larger than any number as n increases
$\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$	terms oscillate between 1 and -1, never converging to a single value

This leads to the definition of **convergence**, **divergence** and a **limit**:

DEFINITIONS Converges, Diverges, Limit

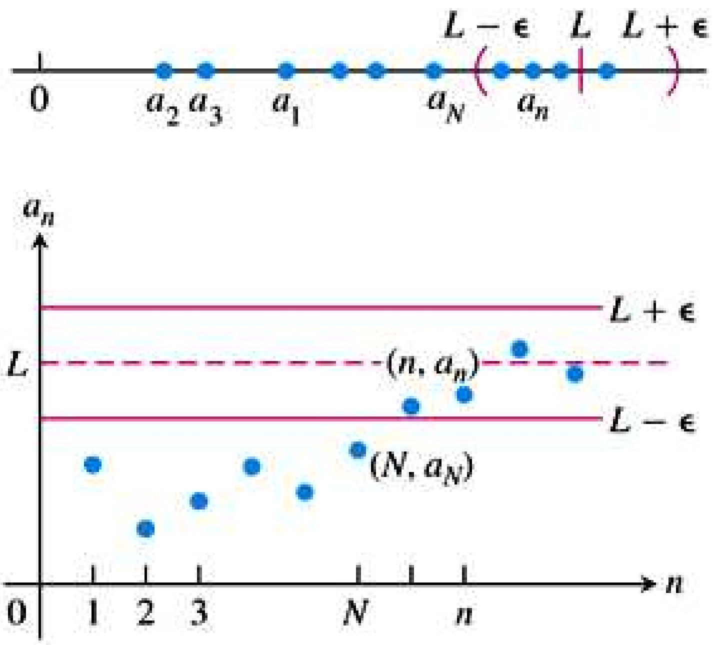
The sequence $\{a_n\}$ **converges** to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence

The concept of a limit is illustrated in the following figure:



Here $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$.

We will now consider two examples of the application of the definitions.

Example:
We want to prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Let $\epsilon > 0$ be given. We need to find an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon\,.$$

This condition will be satisfied provided $1/n < \epsilon$, which means $n > 1/\epsilon$. Therefore if N is any integer greater than (or equal to) $1/\epsilon$, the implication will hold for all $n > N$. Hence $\lim_{n\to\infty}(1/n) = 0$. For example, suppose we take $\epsilon = 0.01$ then the condition is just $n > 100$.

Example:
We want to prove that the sequence

$$\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\} \quad \text{diverges.}$$

proof by contradiction: Assume that the sequence converges to some number L. Choose $\epsilon = \frac{1}{2}$ in the definition of the limit and so all terms a_n of the sequence with n larger than some N must lie within $\epsilon = \frac{1}{2}$ of L:

$$n > N \quad \Rightarrow \quad |a_n - L| < \frac{1}{2}\,.$$

Since 1 is in every other term of the sequence, 1 must lie within ϵ of L. Hence

$$|1 - L| = |L - 1| < \frac{1}{2} \quad \text{or} \quad \frac{1}{2} < L < \frac{3}{2}\,.$$

Then -1 is also in every other term and so we must have

$$|L - (-1)| < \frac{1}{2} \quad \text{or} \quad -\frac{3}{2} < L < -\frac{1}{2}\,.$$

However, this is a *contradiction*: Both conditions cannot be satisfied simultaneously. Therefore no such limit exists and so the sequence diverges.
There is a second type of divergence:

DEFINITION Diverges to Infinity

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty\,.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty\,.$$

Example:

$$\lim_{n\to\infty} \sqrt{n} = \infty \quad \text{(proof?)}$$

note: The sequence $\{1, -2, 3, -4, 5, \dots\}$ *also* diverges, but not to ∞ or $-\infty$.

Sequences are functions with domain restricted to $n \in \mathbb{N}$, hence:

THEOREM 1
Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
4. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (Any number k)
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

We can use these rules to help us calculate limits of sequences.

Example:
Find $\lim\limits_{n\to\infty} \frac{n-1}{n}$.

$$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1\,.$$

Example:
Find $\lim\limits_{n\to\infty} \frac{5}{n^2}$.

$$\lim_{n\to\infty} \frac{5}{n^2} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0\,.$$

The **Sandwich Theorem for Sequences** provides another method for finding the limits of sequences:

THEOREM 2 The Sandwich Theorem for Sequences
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Note that if $|b_n| \leq c_n$ and $c_n \to 0$ as $n \to \infty$, then $b_n \to 0$ also, because $-c_n \leq b_n \leq c_n$.

Example:
Find $\lim\limits_{n\to\infty} \frac{\sin n}{n}$.
By the properties of the sine function we have $-1 \leq \sin n \leq 1$ for all n. Therefore

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

because of $\lim_{n\to\infty}(-1/n) = \lim_{n\to\infty}(1/n) = 0$ and the use of the Sandwich Theorem.

Example:
Find $\lim_{n\to\infty} \dfrac{1}{2^n}$.
$1/2^n$ must always lie between 0 and $1/n$ (e.g. $\frac{1}{2} < 1, \frac{1}{4} < \frac{1}{2}, \frac{1}{8} < \frac{1}{3}, \frac{1}{16} < \frac{1}{4}, \ldots$). Therefore

$$0 \le \frac{1}{2^n} \le \frac{1}{n} \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{1}{2^n} = 0\,.$$

The limits of sequences can also be determined by using the following theorem:

> **THEOREM 3 The Continuous Function Theorem for Sequences**
> Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example:
Determine the limit of the sequence $\{2^{1/n}\}$ as $n \to \infty$.
We already know that the sequence $\{\frac{1}{n}\}$ converges to 0 as $n \to \infty$. Let $a_n = 1/n$, $f(x) = 2^x$ and $L = 0$ in the continuous function theorem for sequences. This gives

$$2^{1/n} = f(1/n) \to f(L) = 2^0 = 1 \quad \text{as} \quad n \to \infty\,.$$

Hence the sequence $\{2^{1/n}\}$ converges to 1.

We can also make use of l'Hôpital's Rule to find the limits of sequences. To do so we need to make use of the following theorem:

> **THEOREM 4**
> Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then
> $$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L.$$

Example:
Show that $\lim_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = 0$.

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n}{(1/2)n^{-1/2}}$$

(using l'Hôpital's Rule *by treating* n *as a continuous real variable*)

$$= \lim_{n\to\infty} 2 \cdot \frac{n^{1/2}}{n} = 2 \lim_{n\to\infty} \frac{1}{n^{1/2}} = 0\,.$$

Example:
Does the sequence whose nth term is $a_n = ((n+1)/(n-1))^n$ converge? If so, find $\lim_{n\to\infty} a_n$.

If we just take the straightforward limit we get the indeterminate form 1^∞. Typically with questions of this type we take the logarithm. This gives:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\,\ln\left(\frac{n+1}{n-1}\right).$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\ln a_n &= \lim_{n\to\infty} n\ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}\\ &= \lim_{n\to\infty}\frac{\ln(n+1)-\ln(n-1)}{1/n}\\ &= \lim_{n\to\infty}\frac{-2/(n^2-1)}{-1/n^2} \quad \text{(using l'Hôpital's Rule)}\\ &= \lim_{n\to\infty}\frac{2n^2}{n^2-1} = 2\,.\end{aligned}$$

Let $b_n = \ln a_n$ Then $\lim_{n\to\infty} b_n = 2$ and since $f(x) = e^x$ is continuous we have by the continuous function theorem for sequences

$$a_n = e^{\ln a_n} = e^{b_n} \to e^2 \quad \text{as} \quad n\to\infty\,.$$

Therefore the sequence $\{a_n\}$ converges to e^2.

End of Week 1

The following Theorem summarizes some common results for the limits of sequences:

THEOREM 5
The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty}\frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty}\sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$
5. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$
6. $\lim_{n\to\infty}\frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n\to\infty$.

The first result can be proved using l'Hôpital's rule. The second and third results can be proved using logarithms and applying the previous theorems. Proofs of the remaining results are given in Appendix 5 of Thomas' Calculus.

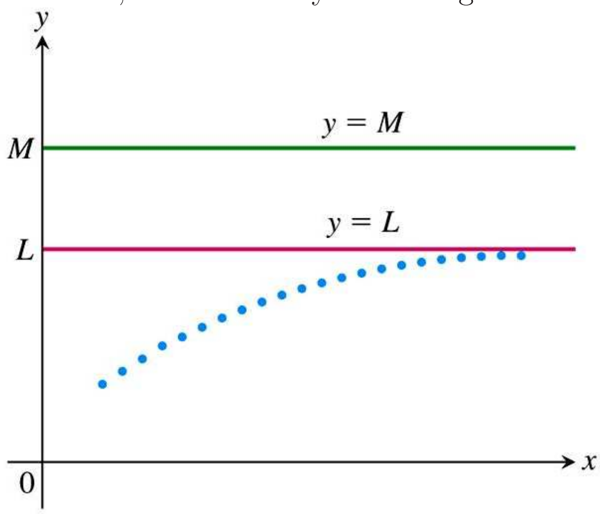
Example:
Show that $\lim_{n\to\infty} \sqrt[n]{n^2} = 1$.

$$\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} \left(n^{1/n}\right)^2 = (1)^2 = 1\,.$$

For *bounded, monotonic* sequences there is the following theorem:

THEOREM 6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

For example, look at a bounded, monotonically increasing function:



Example:

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1\,.$$

1.2 Series

1.2.1 Infinite series and some examples [Thomas' Calculus, Section 9.2]

An **infinite series** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots\,.$$

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots\,.$$

DEFINITIONS Infinite Series, *n*th Term, Partial Sum, Converges, Sum

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Example:
A *geometric series* has the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

where a and r are fixed real numbers and $a \neq 0$. The quantity r is called the *ratio* of the geometric series and can be positive or negative.
In the special case where $r = 1$ the nth partial sum is

$$s_n = a + a \cdot 1 + a \cdot 1^2 + \cdots + a \cdot 1^{n-1} = na$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$ depending on the sign of a. If $r = -1$ the series diverges because either $s_n = a$ or $s_n = 0$ depending on the value of n.
Now consider the case of a geometric series with $|r| \neq 1$. We have

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ s_n - rs_n &= a - ar^n \quad \text{or} \quad s_n(1-r) = a(1-r^n) \\ \Rightarrow s_n &= \frac{a(1-r^n)}{1-r} \quad (r \neq 1). \end{aligned}$$

Therefore, if $|r| < 1$ then $r^n \to 0$ as $n \to \infty$ and hence $s_n \to a/(1-r)$. If $|r| > 1$ then $|r^n| \to \infty$ and the series diverges. So we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for} \quad |r| < 1$$

and the geometric series converges.
For example,

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}\cdots=\sum_{n=1}^{\infty}\frac{1}{9}\left(\frac{1}{3}\right)^{n-1}=\frac{(1/9)}{1-(1/3)}=\frac{1}{6}\qquad(a=1/9,\ r=1/3)$$

and

$$5-\frac{5}{4}+\frac{5}{16}-\frac{5}{64}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n 5}{4^n}=\frac{5}{1+(1/4)}=4\qquad(a=5,\ r=-1/4)\,.$$

Example:
Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\,.$$

Note that we can use partial fractions to write

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}\,.$$

Hence the sum of the first k terms is

$$\sum_{n=1}^{k}\frac{1}{n(n+1)}=\sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

and so the kth partial sum is

$$\begin{aligned} s_k &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{k}-\frac{1}{k+1}\right)\\ &= \frac{1}{1}+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\cdots+\left(-\frac{1}{k}+\frac{1}{k}\right)-\frac{1}{k+1} \end{aligned}$$

Hence $s_k\to 1$ as $k\to\infty$ and so the series converges giving

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1\,.$$

Suppose the series $\sum_{n=1}^{\infty}a_n$ converges to a sum S and the nth partial sum of the series is $s_n=a_1+a_2+\cdots+a_n$. When n is large, both s_n and s_{n-1} are close to S and therefore their difference a_n is close to zero. Using the Difference Rule for sequences we have

$$a_n=s_n-s_{n-1}\quad\to\quad S-S=0\quad\text{as}\quad n\to\infty\,.$$

Hence:

THEOREM 7

If $\sum_{n=1}^{\infty}a_n$ converges, then $a_n\to 0$.

This, in turn, leads to

> **The nth-Term Test for Divergence**
>
> $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Examples:

$$\sum_{n=1}^{\infty} n^2 \quad \text{diverges because} \quad n^2 \to \infty$$

$$\sum_{n=1}^{\infty} \frac{n+1}{n} \quad \text{diverges because} \quad \frac{n+1}{n} \to 1$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \quad \text{diverges because} \quad \lim_{n\to\infty} (-1)^{n+1} \quad \text{does not exist}$$

$$\sum_{n=1}^{\infty} \frac{-n}{2n+5} \quad \text{diverges because} \quad \lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0\,.$$

Note that the *converse of the above theorem is false*: If $a_n \to 0$ this does **not** imply that the series $\sum_{n=1}^{\infty} a_n$ converges.

Example:
Consider the unusual case of a series where $a_n \to 0$ but the series itself diverges:

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n} + \cdots$$

where there are two terms of $1/2$, four terms of $1/4$, ..., 2^n terms of $1/2^n$, etc. In this case each grouping of terms adds up to 1 so the partial sums must increase without bound and so the series diverges, even though the terms of the series form a sequence that converges to 0.

If we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series:

> **THEOREM 8**
> If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then
>
> 1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
> 2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
> 3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (Any number k).

Example:
Find $\sum_{n=1}^{\infty}(3^{n-1}-1)/6^{n-1}$.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3^{n-1}-1}{6^{n-1}} &= \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}-\frac{1}{6^{n-1}}\right)=\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}-\sum_{n=1}^{\infty}\frac{1}{6^{n-1}} \\ &= \frac{1}{1-(1/2)}-\frac{1}{1-(1/6)} \quad \text{(two geometric series)} \\ &= 2-\frac{6}{5}=\frac{4}{5}\,.\end{aligned}$$

We can add a finite number of terms or delete a finite number of terms without altering the convergence or divergence of a series but if the series is convergent this will usually alter the sum. Consider the series

$$\sum_{n=1}^{\infty}a_n = a_1+a_2+\cdots+a_{k-1}+\sum_{n=k}^{\infty}a_n\,.$$

If $\sum_{n=1}^{\infty}a_n$ converges, then $\sum_{n=k}^{\infty}a_n$ converges for any $k>1$. Conversely, if $\sum_{n=k}^{\infty}a_n$ converges for any $k>1$, then $\sum_{n=1}^{\infty}a_n$ converges.
Note that re-indexing a series (e.g. changing the starting value of the index) does not alter its convergence, provided the order of the terms is preserved.
For example, raise the starting value of the index h units:

$$n=k-h: \quad \sum_{n=1}^{\infty}a_n=\sum_{k=1+h}^{\infty}a_{k-h}=a_1+a_2+a_3+\cdots\,.$$

Lower the starting value of the index h units:

$$n=k+h: \quad \sum_{n=1}^{\infty}a_n=\sum_{k=1-h}^{\infty}a_{k+h}=a_1+a_2+a_3+\cdots\,.$$

1.2.2 The Integral Test [Thomas' Calculus, Section 9.3]

For a given series $\sum a_n$ we want to know: (1) Does it converge? (2) If it converges, what is its sum?
A corollary of the Monotonic Sequence Theorem is that the series $\sum_{n=1}^{\infty}a_n$ of *non-negative* terms converges *if and only if* (why?) its partial sums are bounded from above.

Example:
Consider the **harmonic series**:

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots\,.$$

This series is actually divergent even though the nth term $1/n\to 0$ as $n\to\infty$, cf. the *n-th term test* seen before. However, the series has no upper bound for its partial sums. We can see this by writing the series as

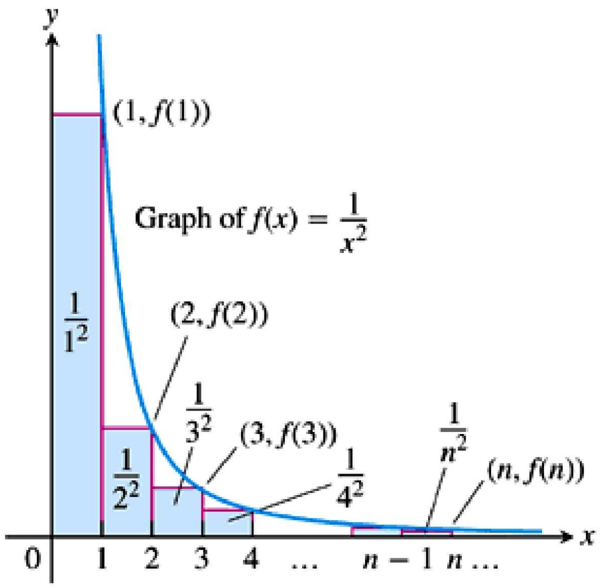
$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)+\cdots\,.$$

Now $\frac{1}{3}+\frac{1}{4} > \frac{2}{4} = \frac{1}{2}$, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{4}{8} = \frac{1}{2}$, $\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16} > \frac{8}{16} = \frac{1}{2}$ and so on. Therefore the sum of the 2^n terms ending with $1/2^{n+1}$ is $> 2^n/2^{n+1} = 1/2$. Therefore the sequence of partial sums is not bounded from above, and so the harmonic series diverges.

Now consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Does it converge or diverge? To answer this question we will consider a new approach involving the use of integration. What we need to do is to compare the series $\sum_{n=1}^{\infty} 1/n^2$ with the integral $\int_1^\infty 1/x^2 \, \mathrm{d}x$.



$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &= f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 \cdots + f(n)\cdot 1 \\ &< f(1) + \int_1^n \frac{1}{x^2}\,\mathrm{d}x \quad \text{lower sum} \\ &< 1 + \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x \end{aligned}$$

Therefore

$$s_n < 1 + \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1 + \left[-\frac{1}{x}\right]_1^\infty = 2\,.$$

Thus $s_n < 2$ for all n, the partial sums are bounded from above (by 2) and therefore (why?) the series converges. Note that the series and the integral need not have the same value in the convergent case.
The approach we have just taken leads us to

> **THEOREM 9 The Integral Test**
> Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

End of Week 2

The Integral Test can be used to show that the p-*series* $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.[1]

Example:
Show that the series $\sum_{n=1}^{\infty} 1/(n^2+1)$ converges by the integral test.
The function $f(x) = 1/(x^2+1)$ is positive, continuous and decreasing for $x \geq 1$. Also

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x &= \lim_{b\to\infty} [\arctan x]_1^b = \lim_{b\to\infty} [\arctan b - \arctan 1] \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

and so the series converges (but we do not know its sum).

1.2.3 Absolute convergence and the Ratio Test [Thomas' Calculus, Sections 9.5 and 9.6]

For a series with both positive and negative terms it is sometimes useful to consider the **absolute values** of its terms:

> **DEFINITION** A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Example:
The series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots = \sum_{n=0}^{\infty} 5\left(\frac{-1}{4}\right)^n$$

is a geometric series that converges absolutely, because

$$\sum_{n=0}^{\infty} \left|5\left(\frac{-1}{4}\right)^n\right| = \sum_{n=0}^{\infty} 5\left(\frac{1}{4}\right)^n = 5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \ldots$$

converges with $|r| = r = 1/4 < 1$ (to 20/3). Note that the original series also converges, as $|r| = 1/4 < 1$ (but to 4).
This exemplifies the following theorem:[2]

[1] See the Thomas' Calculus Section 9.3, p.555 for a proof.
[2] See Section 9.5, p.565 for a short, clever proof.

THEOREM 12—The Absolute Convergence Test
If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

This theorem enables us to apply tests that rely on series of positive terms, such as the integral test, more generally.

Example:

For

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$$

the corresponding series of absolute values reads

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots .$$

By using the Integral Test we have shown before that the latter series converges. The former thus converges absolutely, and according to the above theorem it therefore converges.

DEFINITION A series that converges but does not converge absolutely **converges conditionally**.

Example:

As we show below, the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. However, it does not converge absolutely, because we have seen that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

does not converge.

We can prove that the alternating harmonic series converges by applying the following theorem (also called Leibniz Test):

THEOREM 15—The Alternating Series Test
The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Example:
The above alternating harmonic series satisfies all of the above three requirements with $N = 1$ and hence converges.

Getting back to the geometric series $\sum a_n = \sum ar^n$, we know that it converges for the ratio $|r| = |a_{n+1}/a_n| < 1$. This result is generalised by the following theorem:

THEOREM 13—The Ratio Test
Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

A proof of the above results is given in the textbook.
The two series we looked at in the last section are good examples of cases where $\rho = 1$ and the test is inconclusive:

$$\sum \frac{1}{n} \quad : \quad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1 \quad (n \to \infty)$$

$$\sum \frac{1}{n^2} \quad : \quad \frac{a_{n+1}}{a_n} = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1^2 = 1 \quad (n \to \infty)\,.$$

In each case $\rho = 1$ (i.e. the test is inconclusive) and yet we know that $\sum 1/n$ diverges whereas $\sum 1/n^2$ converges.

Example:
Use the Ratio Test to investigate the convergence of the following series:

$$\text{(a)} \sum_{n=1}^{\infty}\frac{2^n+5}{3^n}\,, \quad \text{(b)} \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}\,, \quad \text{(c)} \sum_{n=1}^{\infty}\frac{n!}{n^n}\,.$$

(a)

$$\begin{aligned}
a_n &= \frac{2^n+5}{3^n}; \qquad a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}; \\
\frac{a_{n+1}}{a_n} &= \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3}\cdot\frac{2^{n+1}+5}{2^n+5} = \frac{1}{3}\left(\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}\right) \\
&\to \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3} < 1 \text{ as } n\to\infty \text{ and the series converges.}
\end{aligned}$$

(b)

$$\begin{aligned}
a_n &= \frac{(2n)!}{(n!)^2}; \qquad a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2}; \\
\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{(n+1)!(n+1)!}\cdot\frac{n!\,n!}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\
&= \frac{4n+2}{n+1} = \frac{4+2/n}{1+1/n} \to 4 > 1 \text{ and the series diverges.}
\end{aligned}$$

(c)

$$\begin{aligned}
a_n &= \frac{n!}{n^n}; \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}; \\
\frac{a_{n+1}}{a_n} &= \frac{(n+1)!\,n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n^n}{(n+1)^n(n+1)} \\
&= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \to \frac{1}{e} < 1 \\
&\text{and the series converges.}
\end{aligned}$$

As we can see, the Ratio Test is often useful when the terms of a series contain factorials involving n or expressions raised to the power involving n.

1.3 Power series

1.3.1 Power series and convergence [Thomas' Calculus, Section 9.7]

A **power series** is like an "infinite polynomial", i.e., it is an infinite series in powers of some variable, usually x:

DEFINITIONS Power Series, Center, Coefficients

A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

If they converge, such series can be *added, subtracted, multiplied, differentiated and integrated* to give new power series.

Example:
Consider the power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots .$$

This matches the form of (2) in the former definition with $a = 2$, $c_n = (-1/2)^n$. It is a geometric series with the first term 1 and ratio $r = -(x-2)/2$. The series converges for $|(x-2)/2| < 1$ or $0 < x < 4$. The sum is

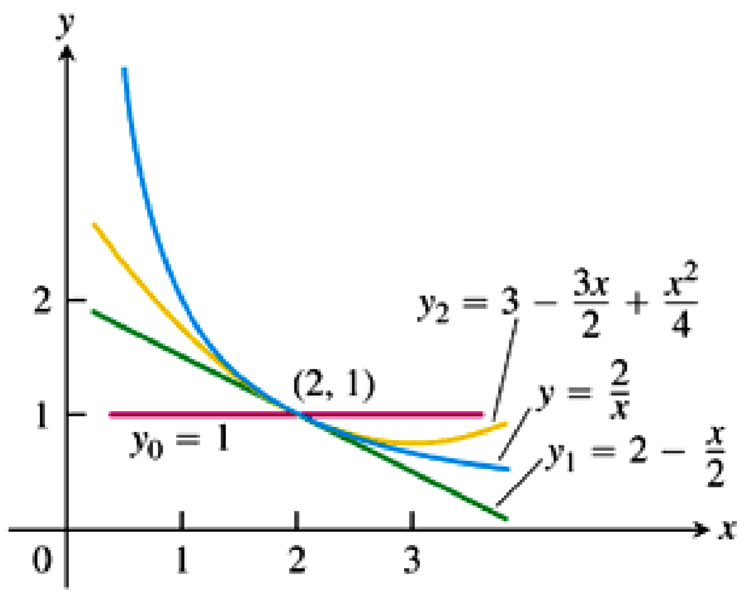
$$\frac{1}{1-r} = \frac{1}{1+(x-2)/2} = \frac{2}{x} .$$

Hence

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^2 (x-2)^n + \cdots , \quad 0 < x < 4 .$$

We can consider the series as a sequence of partial sums which are polynomials $P_n(x)$ that approximate $2/x$:

$$\begin{aligned} f(x) = \frac{2}{x}; \quad P_0(x) &= 1 = y_0 \\ P_1(x) &= 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2} = y_1 \\ P_2(x) &= 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4} = y_2 \\ &\vdots \\ &\text{etc.} \end{aligned}$$



The convergence and divergence of a power series is clarified by the following theorem:

THEOREM 18 The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

COROLLARY TO THEOREM 18

The convergence of the series $\sum c_n (x - a)^n$ is described by one of the following three possibilities:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

Here R is called the **radius of convergence** and the interval of radius R centred at $x = a$ is called the **interval of convergence**.

Example:
Find the values of x for which the series

$$\sum_{n=0}^{\infty} (2x)^n$$

converges absolutely, specifying both the radius and interval of convergence.

This is a geometric series with first term $a = 1$ and ratio $r = 2x$. It **converges absolutely** for $|r| < 1$, that is, $|2x| < 1$ or $-1/2 < x < 1/2$, and diverges elsewhere. Hence, the **radius of convergence** is $R = 1/2$ and the **interval of convergences** $-1/2 < x < 1/2$.

To summarise, we can test a power series for convergence using several methods:

1. Use a test such as the *ratio test* to find the interval $|x - a| < R$ where the series converges absolutely.

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint using a test such as the integral test or the alternating series test.

3. If R is finite, the series diverges for $|x - a| > R$.

Example:
Use the ratio test to determine the convergence of

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots .$$

We have

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1}\frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2 .$$

Therefore the series converges absolutely for $x^2 < 1$ and diverges for $x^2 > 1$. At $x = 1$ the series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ which converges by the alternating series test. The series also converges at $x = -1$, as can be shown by the alternating series test.

End of Week 3

1.3.2 Taylor and Maclaurin series [Thomas' Calculus, Section 9.8]

Assume that the function $f(x)$ can be represented as a power series,

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + \cdots ,$$

which converges for $a - R < x < a + R$ with $R > 0$. Can we calculate the coefficients a_n in terms of $f(x)$?
It can be shown[3] that $f(x)$ has *derivatives of all orders* inside this interval by *differentiating the power series term by term*:

$$\begin{aligned}
f'(x) &= a_1 + 2a_2(x-a) + \cdots + na_n(x-a)^{n-1} + \cdots \\
f''(x) &= 1\cdot 2a_2 + 2\cdot 3a_3(x-a) + \cdots + n(n-1)a_n(x-a)^{n-2} + \cdots \\
&\vdots \\
f^{(n)}(x) &= n!\,a_n + \text{ a sum of terms with } (x-a) \text{ as a factor.}
\end{aligned}$$

Therefore

$$f'(a) = a_1\,,\ f''(a) = 1\cdot 2a_2\,,\ f'''(a) = 1\cdot 2\cdot 3a_3\,,\ \ldots\,, f^{(n)}(a) = n!\,a_n\,.$$

This gives us a formula for the coefficients in the power series:

$$a_n = \frac{f^{(n)}(a)}{n!}\,.$$

It also suggests that *if* f has a power series representation *then* it must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

[3] This is a theorem, which can be proved. Likewise, it can be proved that $f(x)$ can be *integrated term by term*; see Thomas' Calculus, end of Section 9.7. for details.

leading us to the following definition:

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

Example:

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

$$\begin{aligned}
f(x) &= x^{-1}; \quad f(2) = 2^{-1} = \frac{1}{2} \\
f'(x) &= -x^{-2}; \quad f'(2) = -\frac{1}{2^2} \\
f''(x) &= 2!\,x^{-3}; \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3} \\
&\vdots \\
f^{(n)}(x) &= (-1)^n n!\,x^{-(n+1)}; \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}\,.
\end{aligned}$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots\,.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$, or $0 < x < 4$ with sum

$$S = \frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}\,.$$

Related to the Taylor *series* is the Taylor *polynomial* of order n:

DEFINITION Taylor Polynomial of Order n

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

There is a similar definition for Maclaurin polynomials.

Example:
Find the Taylor polynomials of order 0, 2 and 4 for the function $f(x) = \cos x$ at $a = 0$.
We have

$$f(x) = \cos x\,, \quad f'(x) = -\sin x\,, \quad f''(x) = -\cos x\,, \quad f'''(x) = \sin x\,, \quad f^{(4)}(x) = \cos x$$

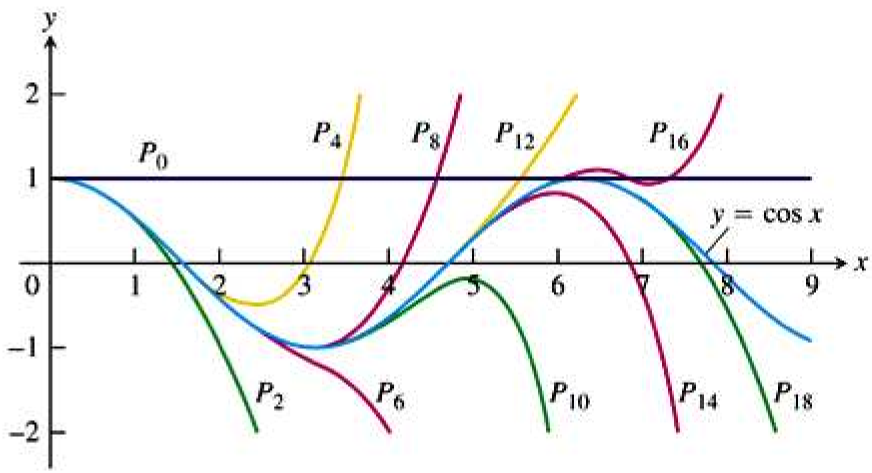
and

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1\,.$$

By using the previous definition, the first three Taylor polynomials of $f(x) = \cos x$ about $a = 0$ are

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= 1 - \frac{x^2}{2!} \\ P_4(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\,. \end{aligned}$$

The following figure shows how successive Taylor polynomials provide better and better approximations to the function as $n \to \infty$:



Below we give the Taylor series expansions for a variety of functions about $a = 0$ and $a = 1$. These can all be derived using the methods in this section.

Taylor series about $a = 0$:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots . \end{aligned}$$

Taylor series about $a = 1$:

$$\begin{aligned} \ln x &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \\ \sqrt{x} &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \cdots . \end{aligned}$$

1.3.3 Hyperbolic functions [Thomas' Calculus, Section 7.7]

This is a Supplement to Calculus I that I was asked to include, as this material was not covered in the last semester. While you need to know what hyperbolic functions are - see above ! -, this part is not directly examinable for Calculus II.
You are strongly encouraged to read through Section 7.7 in the textbook, as here I only cover essential parts of it.

Every function f can be decomposed into

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even function}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd function}}$$

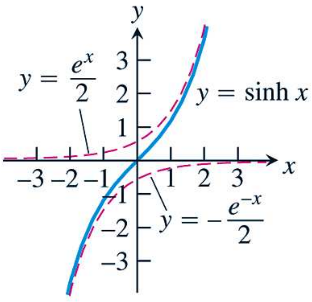
For $f(x) = e^x$ we have

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{=\cosh x} + \underbrace{\frac{e^x - e^{-x}}{2}}_{=\sinh x},$$

called **hyperbolic sine** and **hyperbolic cosine**.

Define tanh, coth, sech and csch in analogy to trigonometric functions:

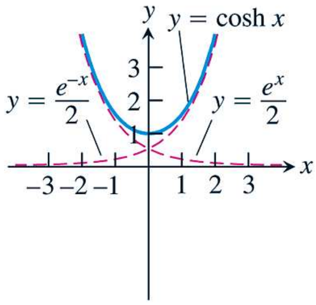
TABLE 7.5 The six basic hyperbolic functions



(a)

Hyperbolic sine:

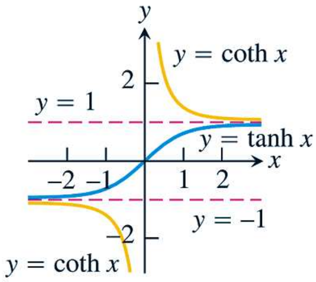
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



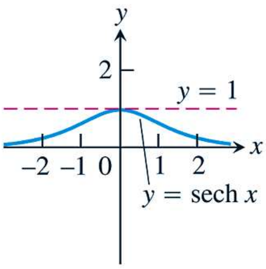
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

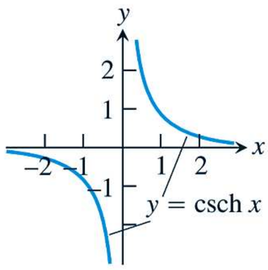
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Compare the following with trigonometric functions:

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

How do we differentiate hyperbolic functions?

example:

$$\begin{aligned}\frac{d}{dx}\sinh x &= \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \\ \frac{d}{dx}\cosh x &= \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x\end{aligned}$$

Inverse hyperbolic functions are defined in analogy to trigonometric functions:

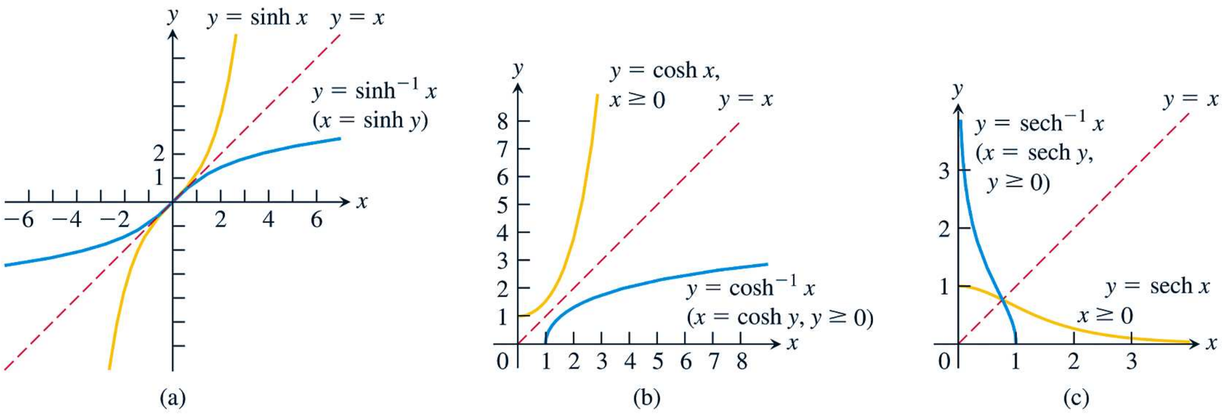


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

1.3.4 Convergence of Taylor Series and error estimates [Thomas' Calculus, Section 9.9]

There are still two unanswered questions about Taylor series:

1. **When** does a Taylor series **converge** to the function that generated it?
2. **How accurately** do a function's Taylor polynomials **approximate** the function on a given interval?

To answer these questions we need to make use of **Taylor's Formula**:

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$\begin{aligned}f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),\end{aligned} \tag{1}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

The quantity $R_n(x)$ in this formula is called the **remainder of order** n or the **error term** for the approximation of f by $P_n(x)$ over I. If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series *converges* to f on I and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Taylor's formula is a special case of *Taylor's Theorem*, which in addition requires differentiability at the end points I. This theorem can in turn be understood as a generalization of the Mean Value Theorem (set $n = 0$ in the above formula).
Finally we can use the **Remainder Estimation Theorem** to provide an estimate of the error:

THEOREM 23 The Remainder Estimation Theorem
If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this condition holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

End of Week 4

The usefulness of this theorem is demonstrated by the following example:

Example:
Show that the Taylor series for $\sin x$ at $a = 0$ converges to $\sin x$ for all x.
The Taylor series for $\sin x$ at $a = 0$ was

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \ldots ,$$

see the list of Taylor series on p.23. According to Taylor's Formula we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x) .$$

Applying the Remainder Estimation Theorem with $M = 1$ gives

$$|R_{2k+1}(x)| \leq 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!} \to 0 \text{ as } k \to \infty \text{ for all } x.$$

(cf. the list of sequences and their limits discussed in Week 1) Therefore $R_{2k+1}(x) \to 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x.

note:
1. Analogous results of convergence for all x about $x = 0$ hold for e^x and $\cos x$, see the textbook.
2. Since every Taylor series is a power series, they can be *added, subtracted and multiplied* on the intersection of their intervals of convergence.

2 Partial derivatives

2.1 Functions of two variables, their limits and derivatives

2.1.1 Functions of Several Variables [Thomas' Calculus, Section 13.1]

Reminder: What is a function?
In *Calculus 1* and in *Numbers, Sets and Functions* you have learned the following:

Definition:
A **function** from a set D (domain) to a set Y (range) is a rule that assigns a *unique* (single) value $y \in Y$ to each $x \in D$.

So far you have dealt with functions of a *single* variable, such as

$$f : \mathbb{R} \to \mathbb{R} \quad , \quad x \mapsto y = f(x)$$

with, for example, $f(x) = x^2$.
Functions of *several variables* are defined in complete analogy to functions of one variable in terms of uniqueness, domain, codomain, range, etc. (without involving complex numbers):

> **DEFINITIONS** Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number
>
> $$w = f(x_1, x_2, \ldots, x_n)$$
>
> to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

In the following we will focus on functions of two variables.

Examples:

$$V = V(r,h) = \pi r^2 h \quad \text{(volume of cylinder, radius } r\text{, height } h\text{)}$$
$$M = M(r,\rho) = \frac{4}{3}\pi r^3 \rho \quad \text{(mass of sphere, radius } r\text{, density } \rho\text{)}$$

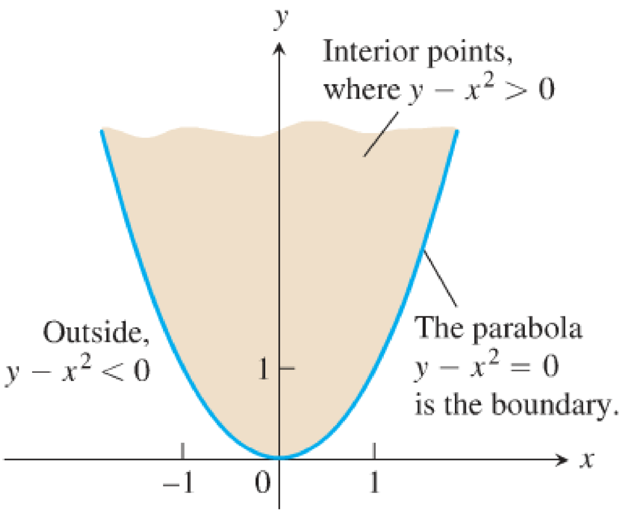
In the case of V the quantities r and h are the input (*independent*) variables and V is the *unique* output (*dependent*) variable.
If f is a function of two independent variables, x and y, the domain of f is a region in the x-y plane.

Example:
(Natural) domains and ranges for function of two variables

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire plane	$[-1, 1]$

Interior points, boundary points, open and closed sets are defined in higher dimensions in analogy to dealing with intervals on the real line.[1]

Example:
Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.
Since f is defined only where $y - x^2 \geq 0$, the domain is the *closed* (the set contains all boundary points), *unbounded* (why?) region shown below (shaded). The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior.



There are two ways to visualise a function $f(x, y)$:

Definition:
The set of all points (x, y, z) is called the **graph**, or **surface**, of $z = f(x, y)$.

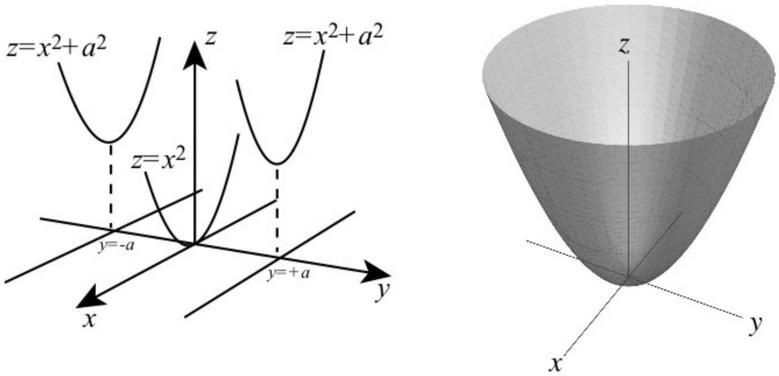
1. Sketch $z = f(x, y)$ in space.

Example:
Consider the function

$$f(x, y) = x^2 + y^2 .$$

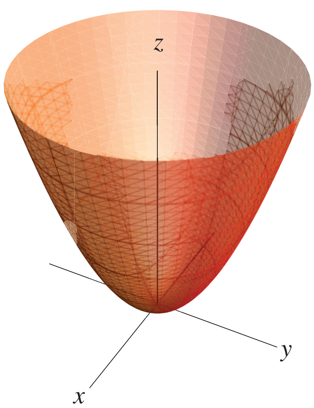
To visualise the surface, plot f for a fixed value of y, say $y = a$. In this case $z = x^2 + a^2$ and $z = z(x)$. The equation $z = x^2 + a^2$ defines a parabola in the plane $y = a$, perpendicular to the y-axis. Each different value of a gives a different parabola. For example, for $y = a = 0$ we have $z = x^2$. Therefore the required surface is made up of parabolas and forms a *paraboloid* as shown below.

[1] If you are not satisfied with this statement, please check out Thomas' Calculus p.760ff for details.

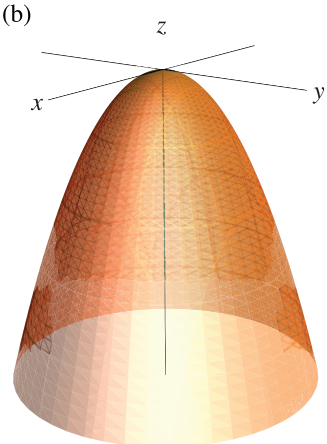


Examples of other surfaces are shown in the following figure. It displays the three dimensional surfaces defined by the functions (a) $f(x, y) = x^2 + y^2$, (b) $f(x, y) = -x^2 - y^2$, (c) $f(x, y) = x^2 + y^2 + 5$ and (d) $f(x, y) = y^2 - x^2$.

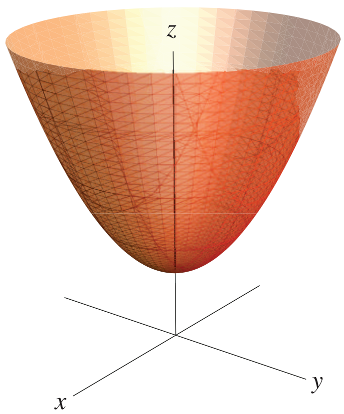
(a)



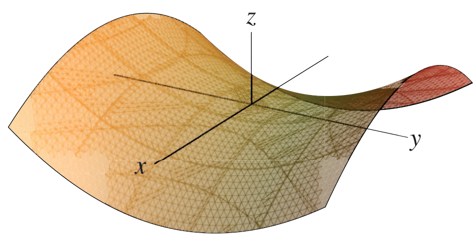
(b)



(c)



(d)



Definition:

The set of points in the domain where a function $f(x, y)$ has a constant value, $f(x, y) = c$, is called a **level curve** of f (cf. what is plotted in geographic maps, often called contour curves therein).

2. Draw and label **level curves**.

Example:

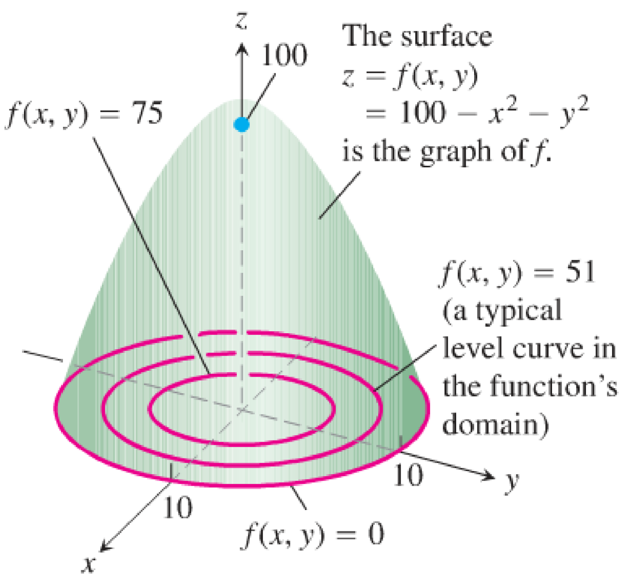
Graph the function $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$ and $f(x, y) = 75$ in the domain of f in the plane.

The domain is the entire x-y plane and the range is the set of real numbers ≤ 100. The graph is the paraboloid given by $z = 100 - x^2 - y^2$:

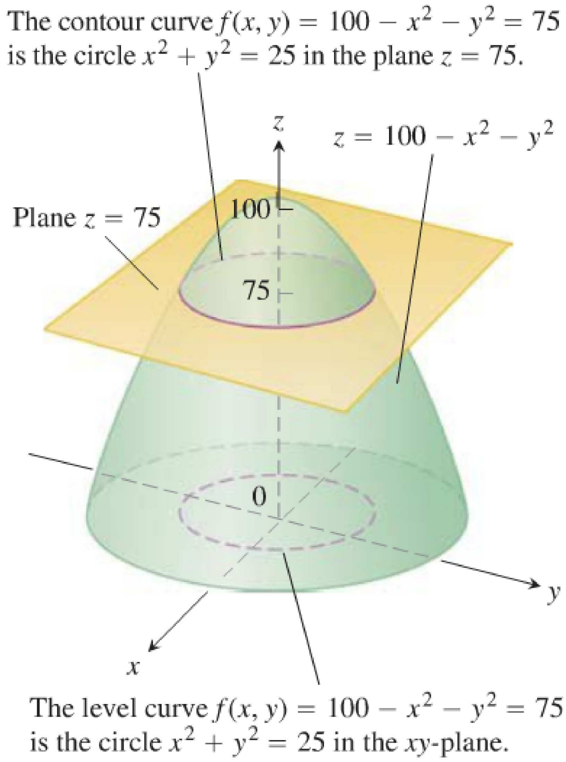
When $f(x, y) = 0$, we have $100 - x^2 - y^2 = 0$ or $x^2 + y^2 = 100$. This corresponds to a circle of radius 10.

When $f(x, y) = 51$, we have $100 - x^2 - y^2 = 51$ or $x^2 + y^2 = 49$. This corresponds to a circle of radius 7.

When $f(x, y) = 75$, we have $100 - x^2 - y^2 = 75$ or $x^2 + y^2 = 25$. This corresponds to a circle of radius 5.



The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is called the **contour curve** $f(x, y) = c$. The following figure shows the contour curve produced where the plane $z = 75$ intersects the surface $z = f(x, y) = 100 - x^2 - y^2$.

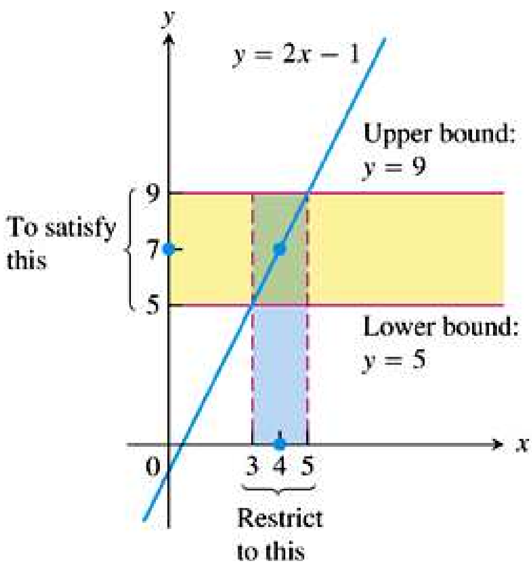


2.1.2 Limits and continuity in higher dimensions [Thomas' Calculus, Section 13.2]

Reminder: Limits
For functions of one variable we say that $f(x)$ approaches the **limit** L whenever $f(x)$ is *arbitrarily close* to L for all x *sufficiently close* to a, written as

$$\lim_{x \to a} f(x) = L \,.$$

Example:
$\lim_{x \to 4}(2x - 1) = 7.$



Analogously, if the values of $f(x, y)$ lie *arbitrarily close* to a fixed real number L for all points (x, y) *sufficiently close* to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). More rigorously:

DEFINITION Limit of a Function of Two Variables
We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

It can be shown that this definition leads to the following properties (you have seen an analogous theorem for functions of one variable in Calculus 1):

Theorem: *Properties of limits of functions of two variables*
If $L, M, k \in \mathbb{R}$, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x, y) = M$ then

1. $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x, y) \pm g(x, y)) = L \pm M$
2. $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
3. $\lim\limits_{(x,y)\to(x_0,y_0)} (kf(x, y)) = kL$
4. $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$, $M \neq 0$
5. If r and s are integers and $s \neq 0$ then
   $\lim\limits_{(x,y)\to(x_0,y_0)} (f(x, y))^{r/s} = L^{r/s}$ provided $L^{r/s}$ is a real number.

For polynomials and rational functions the limit as $(x, y) \to (x_0, y_0)$ can be calculated by evaluating the function at (x_0, y_0) (provided the rational function is defined at (x_0, y_0)).

End of Week 5

Examples:

(1)

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3\,.$$

(2) Find

$$\lim_{(x,y)\to(0,0)^+,x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}\,.$$

We need to avoid the whole path to the limit where $x = y$, hence the condition $x \neq y$. Accordingly, there is a problem with just setting $x = y = 0$ because $\sqrt{x} - \sqrt{y} \to 0$ as

$(x, y) \to (0,0)$. However, we can write

$$\begin{aligned}
\lim_{(x,y)\to(0,0)^+, x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)^+, x\neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\
&= \lim_{(x,y)\to(0,0)^+, x\neq y} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{(x-y)} \\
&= \lim_{(x,y)\to(0,0)^+, x\neq y} x(\sqrt{x} + \sqrt{y}) = 0\,.
\end{aligned}$$

Now we use limits to define continuity for a function of two variables.

Reminder: Continuity
For functions of one variable $f(x)$ is **continuous** at $x = a$ whenever $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists and the limit L equals $f(a)$, that is, $\lim_{x\to a} f(x) = f(a)$. Analogously:

DEFINITION Continuous Function of Two Variables
A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

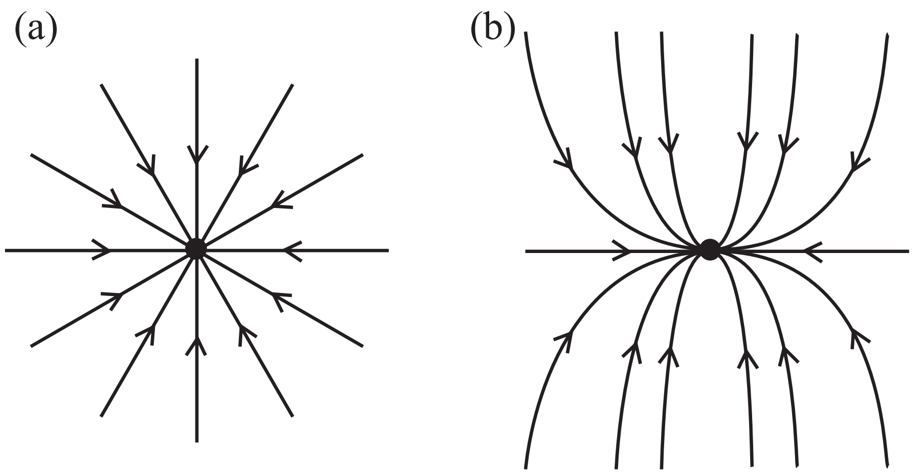
1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

It follows from the previous Theorem that polynomials and rational functions of two variables are continuous on their domains.
Recall that for functions of one variable both the left- and the right-sided limits had to have *the same value* for a limit to exist at a point. For functions of two (or more) variables, this translates into the **Two-Path Test for Nonexistence of a Limit**: It states that if a function $f(x, y)$ has different limits along two different paths as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.
The following figure gives examples of different paths approaching a point in radial and tangential directions:



To have a limit at a point we have to have *the same limit* as the point is approached from *all directions*, including (a) radial directions and (b) tangential directions.

Example:
Show that the function

$$f(x,y) = \frac{2x^2 y}{x^4 + y^2}$$

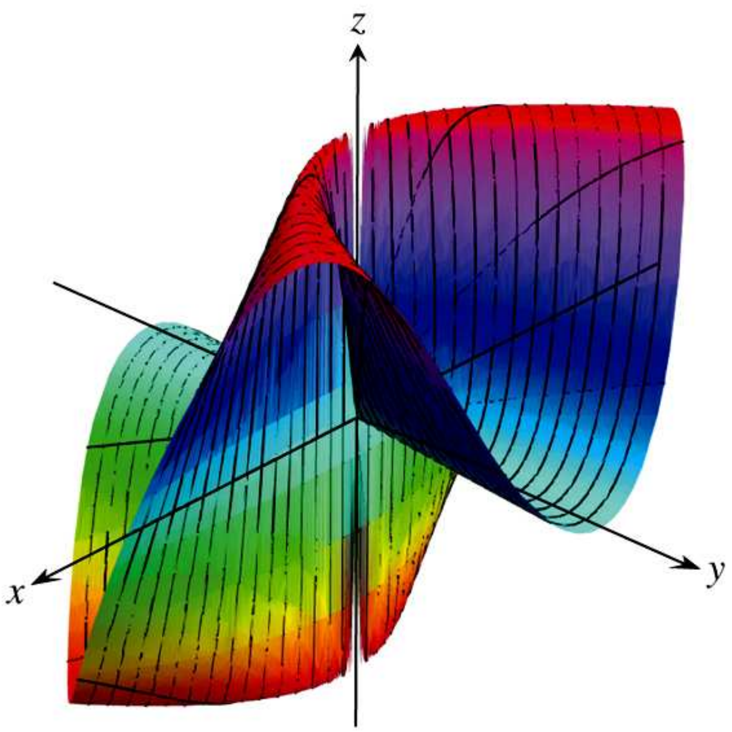
has *no* limit as $(x,y) \to (0,0)$.
We cannot use substitution as it leads to 0/0. However, we can consider what happens as we approach $(0,0)$ along a family of different curves. Remember, the choice of curves is up to us as the *Two-Path Test* does not specify what the path should be. You may wish to check, as an exercise, what happens for the family of paths $y = mx$ as $(x,y) \to (0,0)$. Here we consider the next more complicated case, which is the family of parabolas given by $y = kx^2$ $(x \neq 0)$. Along these curves the function is

$$f(x,y)\big|_{y=kx^2} = \left.\frac{2x^2 y}{x^4 + y^2}\right|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1+k^2}\,.$$

Therefore, as we approach $(0,0)$ along any curve $y = kx^2$, we have

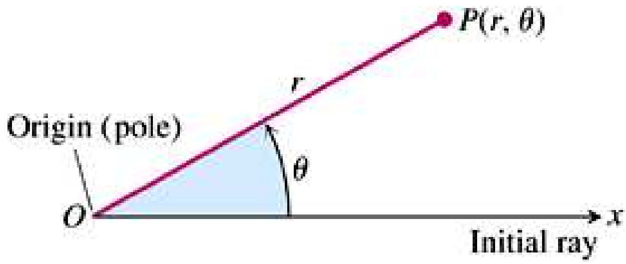
$$\lim_{(x,y)\to(0,0)} \left[f(x,y)\big|_{y=kx^2} \right] = \frac{2k}{1+k^2}\,.$$

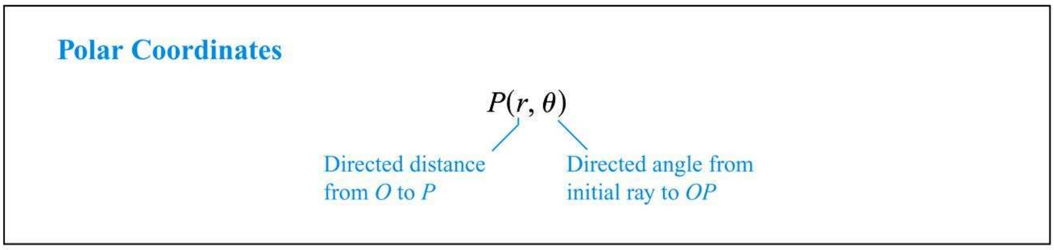
Consequently, the actual limit depends on which path of approach we take (i.e. which parabola we are on which is determined by the value of k). By the *Two-Path Test* there is hence *no limit* as $(x,y) \to (0,0)$. This is illustrated by looking at the surface of this function:



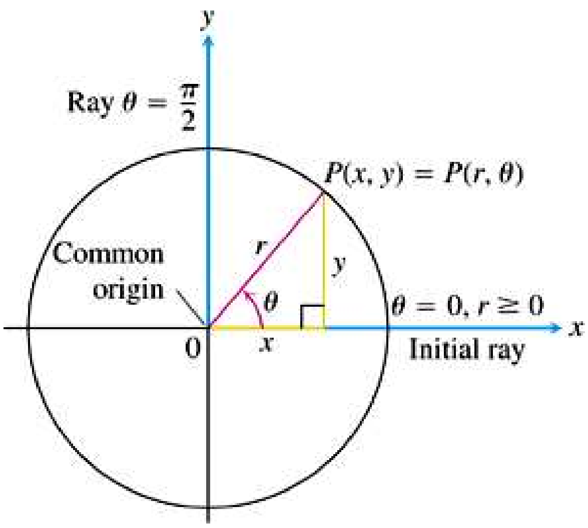
Sometimes it is useful to use polar coordinates.

Reminder: Polar coordinates
As an alternative to **Cartesian coordinates** (x, y), we can describe a point P in the plane by using **polar coordinates**:





These coordinates are particularly useful if a function, or a problem, has some circular symmetry. Typically, we restrict ourselves to $0 \leq r$ and $0 \leq \theta < 2\pi$ (why?). Polar and Cartesian coordinates can be converted into each other:



For the direction polar to Cartesian coordinates we easily derive

$$x = r \cos\theta \ , \quad y = r \sin\theta$$

That is, given (r, θ), we can compute (x, y). The direction Cartesian to polar coordinates is left to you as an exercise.[2]

[2]If you have not encountered polar coordinates before in sufficient detail, I highly recommend that you familiarize yourself with Thomas' Calculus, Section 10.3.

Example:
Determine the continuity of the function defined by

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

In polar coordinates, i.e., by using $x = r\cos\theta$, $y = r\sin\theta$, the function can be written as

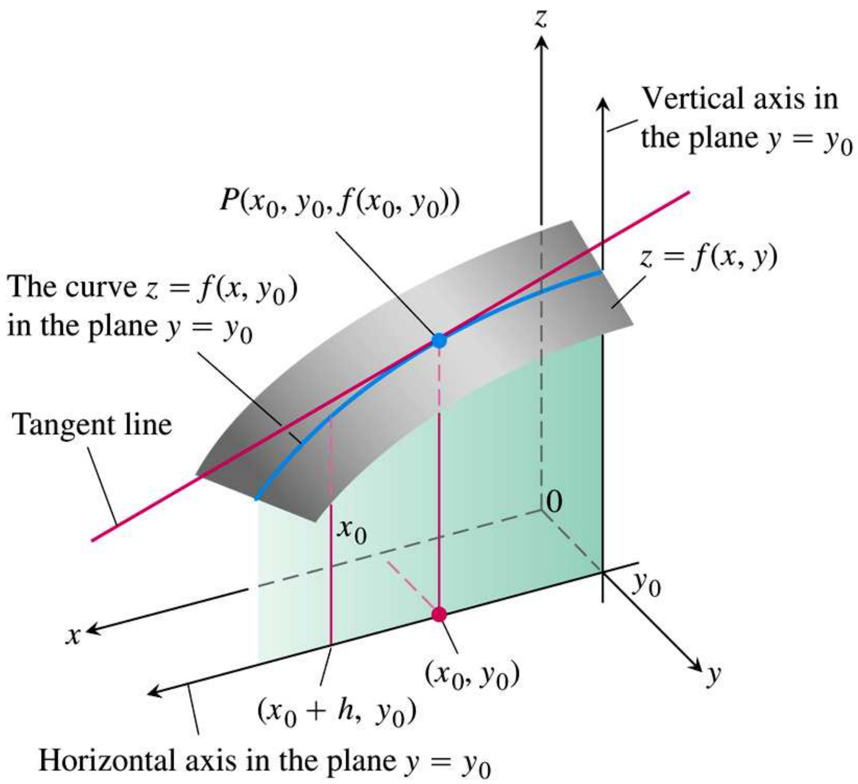
$$f(r,\theta) = \frac{2r^2\cos\theta\sin\theta}{r^2(\cos^2\theta + \sin^2\theta)} = \sin 2\theta$$

provided we are not at the origin (i.e. provided $r \neq 0$). Therefore, as $r \to 0$, the outcome depends on the angle θ. For example, along $\theta = \pi/4$, $f = \sin 2\theta = \sin\pi/2 = 1$ everywhere along the line. Therefore the function is not continuous.

2.1.3 Partial derivatives [Thomas' Calculus, Section 13.3]

Reminder: Derivative
For functions of one variable, $y = f(x)$, the *derivative* at a point is the slope of the tangent to the curve at that point.

But for functions of two variables, $z = f(x,y)$, an infinite number of tangents exist at a point. However, if we fix $y = y_0$ in $f(x,y)$ and let x vary, then $f(x,y_0)$ depends only on x:



That is, we can reduce the problem of the many-variable derivative effectively to the one-variable case by holding all but one of the independent variables constant.

Definition:
The **partial derivative** of $f(x,y)$ with respect to x at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = f_x(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)$$

provided the limit exists.

In complete analogy, the partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} = f_y(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0)$$

provided the limit exists.

For example, if $f(x,y)=x^2+y^2$ then $f_x=2x$, $f_y=2y$.

Note how we treat the other variables as constants when we do partial differentiation!

We can extend this to three (or more) dimensions. For example, if $f(x,y,z)=xy^2z^3$ then $f_x=y^2z^3$, $f_y=2xyz^3$, $f_z=3xy^2z^2$.

Example:
Find $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4,-5)$ for the function $f(x,y)=x^2+3xy+y-1$.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^2+3xy+y-1) = 2x+3y\\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2+3xy+y-1) = 3x+1\,.\end{aligned}$$

At the point $(4,-5)$ we have

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = -7\,, \qquad \left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 13\,.$$

Example:
Find $\partial z/\partial x$ if the equation $yz-\ln z = x+y$ (implicitly) defines $z=z(x,y)$.

$$\frac{\partial}{\partial x}(yz-\ln z) = \frac{\partial}{\partial x}(x+y)\,.$$

Hence

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1+0\,.$$

This gives

$$\left(y-\frac{1}{z}\right)\frac{\partial z}{\partial x} = 1\,; \qquad \Rightarrow \qquad \frac{\partial z}{\partial x} = \frac{z}{yz-1}\,.$$

We can also obtain higher order derivatives.

Example:
If $f(x,y) = x\cos y + y\,e^x$, find

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}\,, \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}\,, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}\,.$$

The first step is to find the first partial derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \cos y + y\,e^x \\ \frac{\partial f}{\partial y} &= -x\sin y + e^x\,. \end{aligned}$$

Now we take the partial derivatives of the first partial derivatives. This gives:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= y\,e^x \\ \frac{\partial^2 f}{\partial y \partial x} &= -\sin y + e^x \\ \frac{\partial^2 f}{\partial x \partial y} &= -\sin y + e^x \\ \frac{\partial^2 f}{\partial y^2} &= -x\cos y\,. \end{aligned}$$

This illustrates the following Theorem:

Theorem: *Mixed Derivative Theorem*
If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are *defined* throughout an open region containing a point (a,b) and are *all continuous* at (a,b) then

$$f_{xy}(a,b) = f_{yx}(a,b)\,.$$

(An example where $f_{xy}(a,b) \neq f_{yx}(a,b)$ is provided by the function discussed on p.34 of the lecture notes.)
The theorem can be extended to higher orders, provided the derivatives are continuous.

End of Week 6

Reminder:
For functions of a single variable it holds that if $y = f(x)$ is differentiable at $x = x_0$, then the change in the value of f that results from changing x from x_0 to $x_0 + \Delta x$ is given by the *differential approximation*

$$\Delta y = f'(x_0)\Delta x + \epsilon \Delta x$$

in which $\epsilon \to 0$ as $\Delta x \to 0$ (see Thomas' Calculus Section 3.9). For functions of two variables, the analogous property yields the *definition* of differentiability:

DEFINITION Differentiable Function

A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain.

Note in particular that for $z = f(x, y)$, *differentiability is more than the existence of the partial derivatives*, as becomes also clear from the following statement:

If f_x and f_y are *continuous* throughout an open region R, then f is *differentiable* at every point of R.

It also holds, in analogy to functions of a single variable:

If a function $f(x, y)$ is *differentiable* at a point (x_0, y_0) then f is *continuous* at (x_0, y_0).

If you are interested in the details underlying the above statements, like the *Increment Theorem*, please check out Thomas' Calculus p.785.

2.1.4 The chain rule [Thomas' Calculus, Section 13.4]

Reminder: Chain Rule for Functions of One Variable

If $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, then

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

Similarly:

Theorem: Chain Rule for Functions of Two Variables

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then $w = f(x(t), y(t))$ is a differentiable function of t and

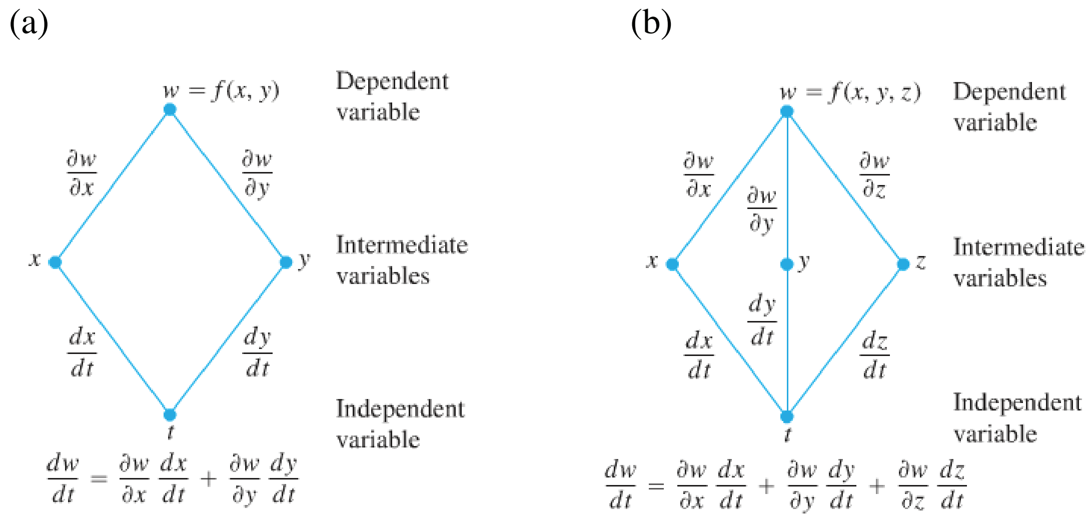
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\,.$$

This straightforwardly follows from the above definition of differentiability.

We can easily extend this theorem to functions $w = f(x, y, z)$ of three variables:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\,.$$

We can use **tree diagrams** to illustrate the application of the Chain Rule:



(a) To find $\mathrm{d}w/\mathrm{d}t$, start at w and read down each route to t, multiplying derivatives along the way; then add the products. (b) For functions of three variables there are three routes from w to t instead of two, but finding $\mathrm{d}w/\mathrm{d}t$ is still the same: read down each route, multiplying derivatives along the way; then add.

Example:
Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$.

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = y(-\sin t) + x(\cos t) = -\sin^2 t + \cos^2 t = \cos 2t\,.$$

Note that we could have done this more directly by noting that

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t\,; \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{2}\cdot 2\cos 2t = \cos 2t\,.$$

If $w = f(x,y)$ where $x = g(r,s)$ and $y = h(r,s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

and in analogy for functions $w = f(x,y,z)$. Also, if $w = f(x)$ and $x = g(r,s)$ then

$$\frac{\partial w}{\partial r} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\partial x}{\partial s}\,.$$

Example:
For $u = w(x,y,z)$, express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2\,, \quad x = \frac{r}{s}\,, \quad y = r^2 + \ln s\,, \quad z = 2r\,.$$

We have

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) = \frac{1}{s} + 12r\end{aligned}$$

and

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (1)\left(\frac{-r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}\,.\end{aligned}$$

Suppose that $w = F(x, y)$ is differentiable and that $F(x, y) = 0$ defines y (implicitly) as a differentiable function of x. Then

$$0 = \frac{dw}{dx} = F_x\frac{dx}{dx} + F_y\frac{dy}{dx} = F_x + F_y\frac{dy}{dx}\,.$$

Hence, at any point where $F_y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}\,.$$

This is the **Formula for Implicit Differentiation**.

Example:
Find $\mathrm{d}y/\mathrm{d}x$ if $y^2 - x^2 - \sin xy = 0$.

$$\begin{aligned}F(x, y) &= y^2 - x^2 - \sin xy \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{F_x}{F_y} = -\frac{(-2x - y\cos xy)}{(2y - x\cos xy)} = \frac{2x + y\cos xy}{2y - x\cos xy}\,.\end{aligned}$$

You may wish to compare this method with the one that you have learned in Calculus 1, i.e., differentiating the whole equation with respect to x and then solving for dy/dx.

2.2 Directional derivatives and extreme values

2.2.1 Directional derivatives and gradient vectors [Thomas' Calculus, Section 13.5]

We now investigate the derivative of a function $f(x, y)$ at a point *in a particular direction*:

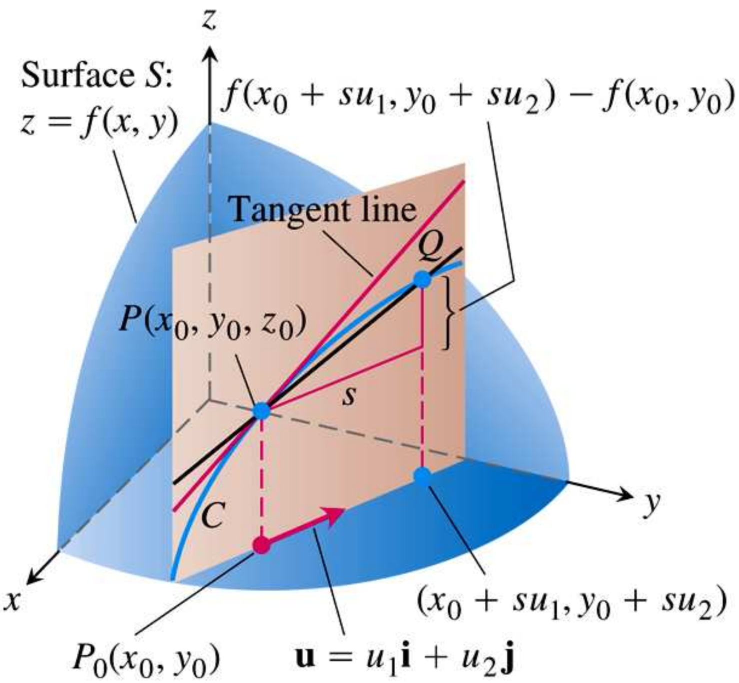
DEFINITION Directional Derivative

The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0}\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

It is also denoted by $(D_{\mathbf{u}}f)_{P_0}$ or $D_u f|_{P_0}$ as the derivative of f at the point P_0 in the direction of the unit vector $\mathbf{u}$. The meaning is illustrated in the following figure:



We can develop a more efficient formula for the directional derivative by considering the line

$$x = x_0 + su_1 \,, \qquad y = y_0 + su_2$$

through the point $P_0(x_0, y_0)$, parametrised with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then, as $f = f(x(s), y(s))$,

$$\begin{aligned}
\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)_{\mathbf{u},P_0} &= \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{\mathrm{d}x}{\mathrm{d}s} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{\mathrm{d}y}{\mathrm{d}s} \qquad \text{(via the Chain Rule)} \\
&= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2 \qquad \text{(use unit vector } \mathbf{u}\text{)} \\
&= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right] \cdot [u_1\mathbf{i} + u_2\mathbf{j}]
\end{aligned}$$

DEFINITION The **gradient vector** (or **gradient**) of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

The value of the gradient vector obtained by evaluating the partial derivatives at a point $P_0(x_0, y_0)$ is written

$$\nabla f|_{P_0} \qquad \text{or} \qquad \nabla f(x_0, y_0).$$

Note that for a function $f(x, y, z)$ we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \quad .$$

The expression $\nabla f = \text{grad } f$ is called "grad f", "gradient of f", "del f" or "nabla f". We can now write the directional derivative using the gradient:

THEOREM 9—The Directional Derivative Is a Dot Product
If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \nabla f|_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 with the vector $\mathbf{u}$. In brief, $D_\mathbf{u} f = \nabla f \cdot \mathbf{u}$.

Example:
Find the derivative of $f(x, y) = x\,e^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$. The unit vector is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\,.$$

Now

$$\begin{aligned} f_x(2,0) &= (e^y - y\sin(xy))|_{(2,0)} = e^0 - 0 = 1 \\ f_y(2,0) &= (xe^y - x\sin(xy))|_{(2,0)} = 2e^0 - 2\cdot 0 = 2\,. \end{aligned}$$

Hence

$$\nabla f|_{(2,0)} = f_x(2,0)\mathbf{i} + f_y(2,0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

and so

$$D_\mathbf{u} f|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1\,.$$

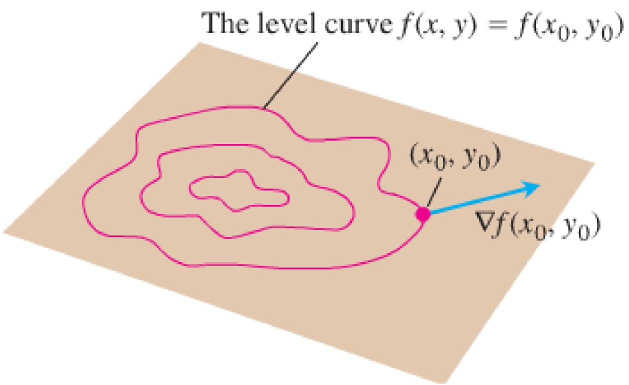
Note that

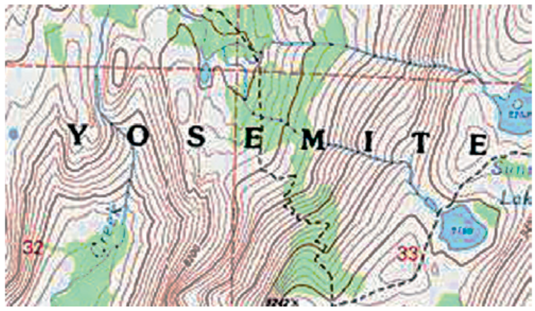
$$\boxed{D_\mathbf{u} f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta = |\nabla f|\cos\theta}$$

where θ is the angle between the vectors ∇f and $\mathbf{u}$. This implies the following:

1. f *increases most rapidly* when $\cos\theta = 1$ (i.e. $\mathbf{u}$ is parallel to ∇f)
2. f *decreases most rapidly* when $\cos\theta = -1$ (i.e. $\mathbf{u}$ is in opposite direction to ∇f)
3. f has *zero change* when $\cos\theta = 0$ (i.e. $\mathbf{u}$ is orthogonal to ∇f).

Point 1 implies (why?): ∇f points in the direction of *maximal increase* of f.
Point 3 implies (why?): At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$ the gradient of f is normal to the level curve through (x_0, y_0).
Point 2 is illustrated by the following geographical map.





End of Week 8

Tangent lines to level curves are always normal to the gradient. If (x, y) is a point on the tangent line through the point $P(x_0, y_0)$ then

$$\mathbf{T} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j}\,,$$

is a vector parallel to it. The *equation of the tangent* is then

$$\nabla f \cdot \mathbf{T} = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0\,.$$

An example illustrating the use of this equation will be discussed in the tutorials.

We can use the directional derivative for **estimating change in a specific direction**. Recall that for a differentiable function of one variable we can estimate the change $df = f(x_0 + dx) - f(x_0)$ along the increment dx by

$$df = f'(x_0)dx\,.$$

In higher dimensions we can analogously use the directional derivative:

Estimating the Change in f in a Direction u

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\text{Directional derivative}}\ \underbrace{ds}_{\text{Distance increment}}$$

2.2.2 Tangent planes and differentials [Thomas' Calculus, Section 13.6]

DEFINITIONS Tangent Plane, Normal Line

The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

It follows[3] that the equation of the tangent plane is

$$\nabla f|_{P_0} \cdot \vec{P_0P} = f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

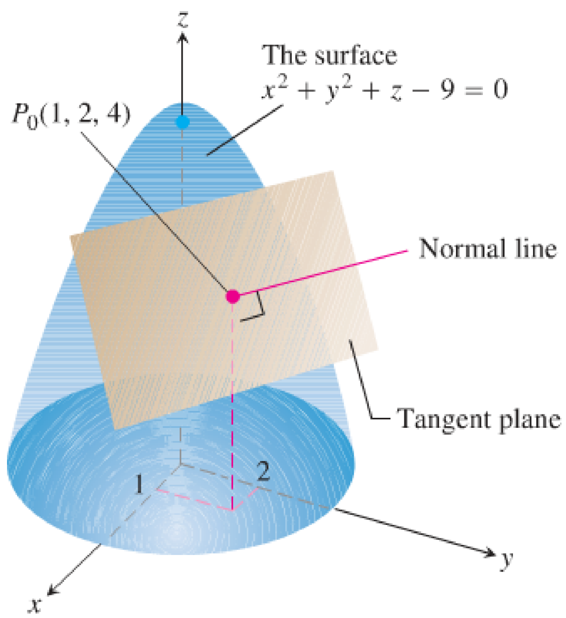
and the equation of the normal line is

$$x = x_0 + f_x(P_0)t\,, \qquad y = y_0 + f_y(P_0)t\,, \qquad z = z_0 + f_z(P_0)t\,.$$

Example:
Find the tangent plane and normal line of the (level) surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

(a circular paraboloid) at the point $P_0(1, 2, 4)$



$$\nabla f|_{P_0} = (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}$$

where at the point P_0 we have $f_x(P_0) = 2$, $f_y(P_0) = 4$ and $f_z(P_0) = 1$. Therefore the equation of the tangent plane is

$$2(x - 1) + 4(y - 2) + (z - 4) = 0$$

which simplifies to

$$2x + 4y + z = 14\,.$$

The normal line to the surface at P_0 is

$$x = 1 + 2t\,, \qquad y = 2 + 4t\,, \qquad z = 4 + t\,.$$

[3]See Section 11.5 in Thomas' Calculus for details if you are in trouble with this.

We remark that the gradient has the following algebraic properties:

$$\begin{aligned}
\nabla(kf) &= k\,\nabla f \qquad \text{for any number } k \\
\nabla(f \pm g) &= \nabla f \pm \nabla g \\
\nabla(fg) &= f\,\nabla g + g\,\nabla f \\
\nabla\left(\frac{f}{g}\right) &= \frac{g\,\nabla f - f\,\nabla g}{g^2}
\end{aligned}$$

(the proof is straightforward and left as an exercise)
Before we linearise a function of two variables, recall that a function $z = f(x,y)$ is *differentiable* at (x_0, y_0) if

$$\Delta z = f(x,y) - f(x_0,y_0) = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

with $\epsilon_1, \epsilon_2 \to 0$ $(\Delta x, \Delta y \to 0)$. Solve for $f(x,y)$ and approximate:

DEFINITIONS **Linearization, Standard Linear Approximation**

The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0). \qquad (5)$$

The approximation

$$f(x,y) \approx L(x,y)$$

is the **standard linear approximation** of f at (x_0, y_0).

Example:
Find the linearisation of

$$f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

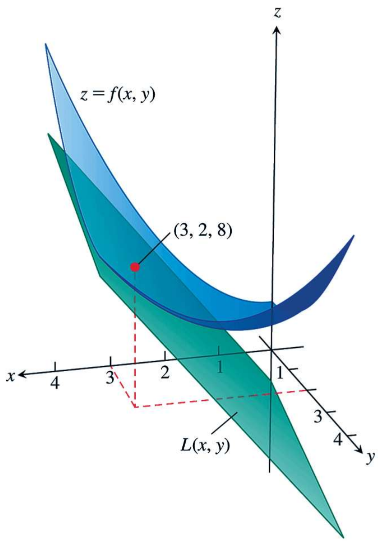
at the point $(3, 2)$.
We first evaluate f, f_x and f_y at the point $(x_0, y_0) = (3, 2)$:

$$\begin{aligned}
f(3,2) &= \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 8 \\
f_x(3,2) &= \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (2x - y)|_{(3,2)} = 4 \\
f_y(3,2) &= \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (-x + y)|_{(3,2)} = -1
\end{aligned}$$

giving

$$\begin{aligned}
L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \\
&= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2\,.
\end{aligned}$$

Hence the linearisation of f at $(3, 2)$ is $L(x, y) = 4x - y - 2$.



Recall that for $y = f(x)$ we have defined the *differential* $dy = f'(x)dx$.

DEFINITION Total Differential
If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

in the linearization of f is called the **total differential of f**.

Example:
A cylindrical can is designed to have a radius of 1 unit and a height of 5 units, but the radius is off by an amount of $dr = +0.03$ units and the height by $dh = -0.1$ units. Estimate the resulting absolute change in the volume of the can.

Using the above total differential we obtain

$$\Delta V \approx dV = V_r(r_0, h_0)dr + V_h(r_0, h_0)dh\,.$$

From $V = \pi r^2 h$ we obtain $V_r = 2\pi rh$ and $V_h = \pi r^2$. Hence,

$$dV = 2\pi rhdr + \pi r^2 dh = 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63\,.$$

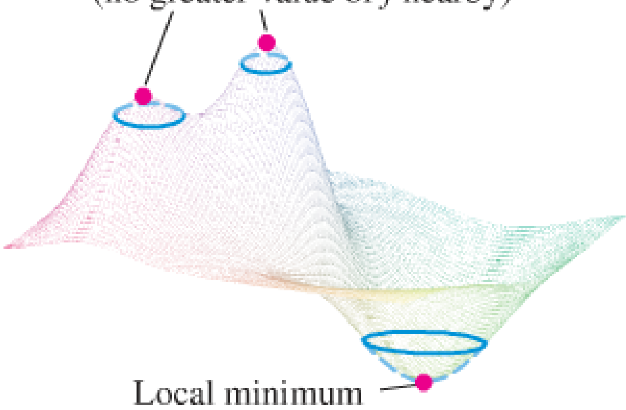
2.2.3 Extreme values and saddle points [Thomas' Calculus, Section 13.7]

When we investigated extreme values for functions of one variable we looked for points where the graph had a horizontal tangent line. For functions of two variables we look for points where the *surface* defined by $z = f(x, y)$ has a *horizontal tangent plane*. This leads to the following definition:

DEFINITIONS Local Maximum, Local Minimum
Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Local maxima correspond to "mountain peaks" on the surface $z = f(x, y)$ and local minima correspond to "valley bottoms":



Not too hard to show (with knowledge of Calculus I):

THEOREM 10—First Derivative Test for Local Extreme Values If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Define an important object (in complete analogy to Calculus I):

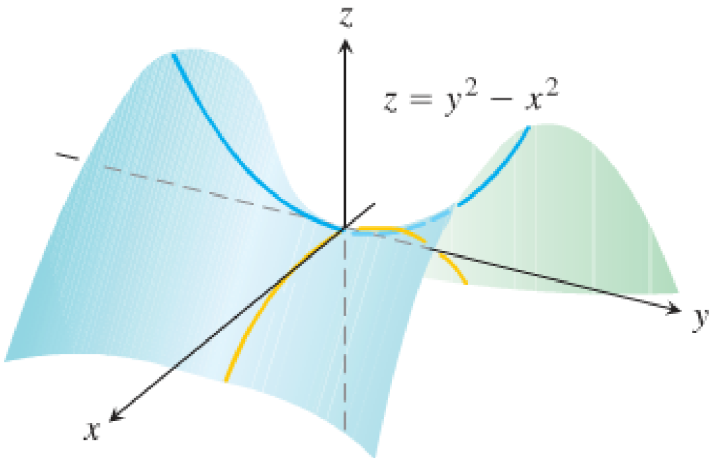
DEFINITION Critical Point
An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

Therefore local maxima and minima are critical points (why?) but critical points can also include **saddle points**:

DEFINITION Saddle Point

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.40).

An example of a saddle point is the origin in the following surface:



Therefore, *finding* critical points of a function is not sufficient to identify the *type* of critical point (local maximum, local minimum or saddle point). To do this we need to make use of second partial derivatives.

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The quantity $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of the function f. Note that

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix},$$

i.e., the Hessian is the *determinant* (cf. Vectors and Matrices) of the matrix of the second partial derivatives.[4]

Example:
Find the local extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$ and determine the nature of each.
$f(x,y)$ is defined and differentiable for all points in its domain. Hence, at extreme values f_x and f_y are simultaneously zero. This gives the two equations

$$f_x = y - 2x - 2 = 0\,; \qquad f_y = x - 2y - 2 = 0\,.$$

The solution of these equations is $x = y = -2$. Hence $(-2,-2)$ is the only point where f may take an extreme value. Now take the second derivatives:

$$f_{xx} = -2 < 0\,, \qquad f_{yy} = -2\,, \qquad f_{xy} = 1 (= f_{yx})\,.$$

At the point $(-2,-2)$,

$$f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - 1^2 = 3 > 0\,.$$

So $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$. Therefore f has a local maximum at $(-2,-2)$. The value of f at this point is $f(-2,-2) = 8$.

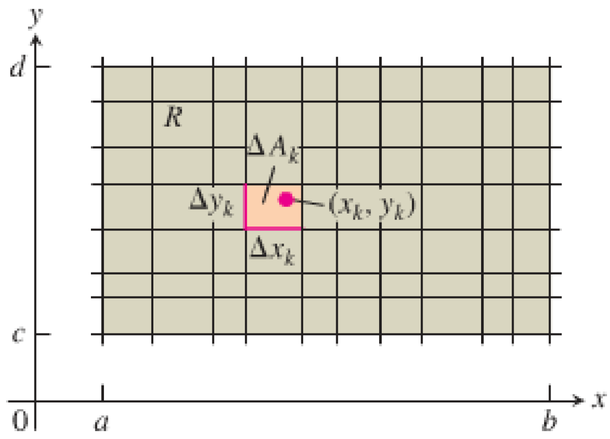
The previous theorems hold only for *interior points*. Note that functions defined on closed and bounded domains may also have local extreme values at *boundary points* (as in case of functions of one variable, see Calculus I). But we do not discuss this case in more detail.

[4] If you want to know why: check out Thomas' Calculus Section 13.9.

3 Multiple integrals

3.1 Double integrals [Thomas' Calculus, Section 14.1]

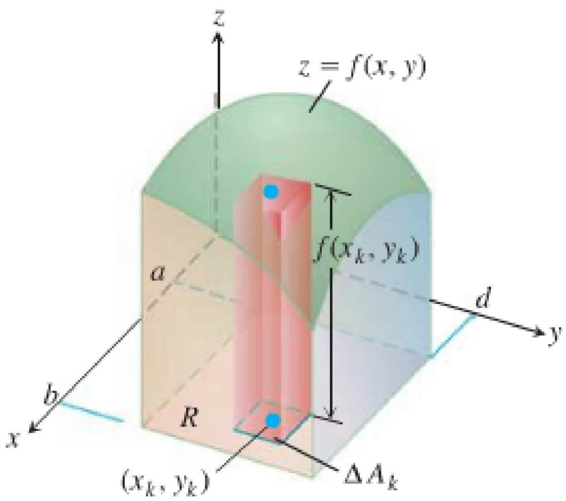
Consider a function $f(x, y)$ defined on a rectangular region $R: \; a \le x \le b, \; c \le y \le d$ partitioned into small rectangles A_k:



The area of a small rectangle with sides Δx_k and Δy_k is

$$\Delta A_k = \Delta x_k \, \Delta y_k \, .$$

Choose a point (x_k, y_k) in the (suitably numbered) kth rectangle with function value $f(x_k, y_k)$. We can consider $z = f(x, y)$ as defining the height z at the point (x, y). The product $f(x_k, y_k)\, \Delta A_k$ is then the *volume of a solid* with base area ΔA_k and height $f(x_k, y_k)$ (for which we assume that $f(x_k, y_k) > 0$):



The **Riemann sum** S_n of these solids over R is

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k \, .$$

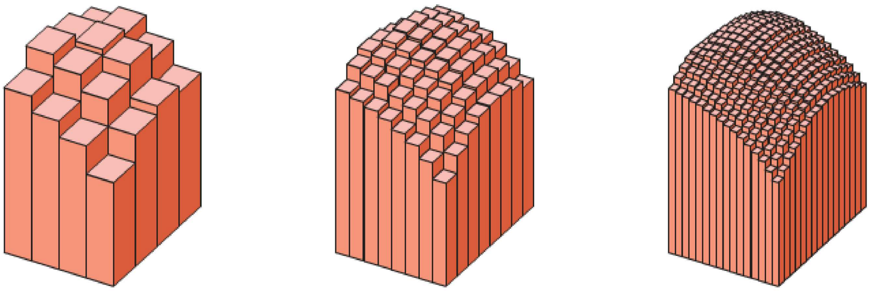
Now consider what happens as $\Delta A_k \to 0$ (as $n \to \infty$), i.e., we refine the partitioning. When the limit of these sums exists the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\int\!\!\int_R f(x, y)\, \mathrm{d}A \qquad \text{or} \qquad \int\!\!\int_R f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

The volume of the portion of the solid directly above the base ΔA_k is $f(x_k, y_k)\,\Delta A_k$. Hence the total volume above the region R is

$$\text{Volume} = \lim_{n\to\infty} S_n = \int_R\int f(x,y)\,\mathrm{d}A$$

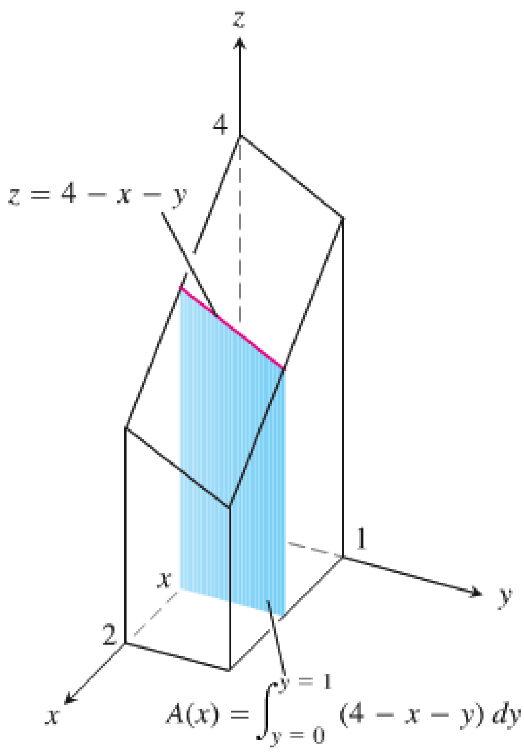
where $\Delta A_k \to 0$ as $n \to \infty$. The following figure shows how the Riemann sum approximations of the volume become more accurate as the number n of boxes increases:



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

End of Week 9

Consider the calculation of the volume under the plane $z = 4 - x - y$ over the rectangular region $R:\ 0 \le x \le 2$ and $0 \le y \le 1$ in the x-y plane.
First consider a slice perpendicular to the x-axis:



The volume under the plane is

$$\int_{x=0}^{x=2} A(x)\,\mathrm{d}x$$

where $A(x)$ is the cross-sectional area at x. For each value of x we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\,\mathrm{d}y$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x.

In calculating $A(x)$, x is held fixed and the integration takes place with respect to y.

Combining the above two equations we have

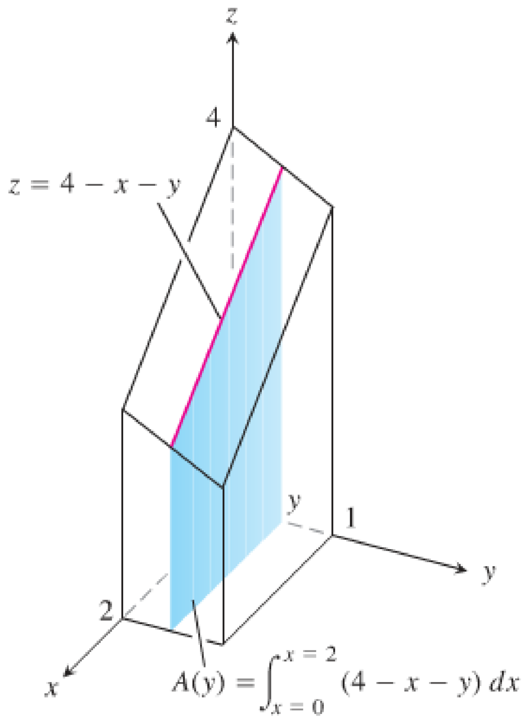
$$\begin{aligned}
\text{Volume} &= \int_{x=0}^{x=2} A(x)\,\mathrm{d}x \\
&= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} \mathrm{d}x = \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) \mathrm{d}x \\
&= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = \left(\frac{7}{2} \cdot 2 - \frac{2^2}{2} \right) - (0 - 0) = 5\,.
\end{aligned}$$

We can write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

This is an **iterated** or **repeated integral**. The expression states that we can get the volume under the plane by (i) integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then (ii) integrating the resulting expression in x from $x = 0$ to $x = 2$. In other words, *first do the* $\mathrm{d}y$ *integral and then do the* $\mathrm{d}x$ *integral.*

Now consider the plane perpendicular to the y-axis:



We have

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\,\mathrm{d}x = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} = 6 - 2y\,.$$

The volume is then

$$\text{Volume} \;=\; \int_{y=0}^{y=1} A(y)\,\mathrm{d}y = \int_{y=0}^{y=1} (6 - 2y)\,\mathrm{d}y = \left[6y - y^2\right]_0^1 = 5$$

as before.
This illustrates

THEOREM 1 Fubini's Theorem (First Form)
If $f(x, y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Example:
Calculate the volume V under $z = f(x, y) = x^2y$ over the rectangle R defined by $1 \le x \le 2$, $-3 \le y \le 4$.

$$\begin{aligned} V &= \int\!\!\int_R x^2y\,\mathrm{d}A = \int_{x=1}^{x=2} \left(\int_{y=-3}^{y=4} x^2y\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{x=1}^{x=2} \left[\frac{x^2y^2}{2}\right]_{y=-3}^{y=4} \mathrm{d}x = \int_{x=1}^{x=2} \frac{7x^2}{2}\,\mathrm{d}x = \left[\frac{7x^3}{6}\right]_{x=1}^{x=2} = \frac{49}{6}\,. \end{aligned}$$

Changing the order gives the same result:

$$\begin{aligned} V &= \iint_R x^2 y \, \mathrm{d}A = \int_{y=-3}^{y=4} \left(\int_{x=1}^{x=2} x^2 y \, \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{y=-3}^{y=4} \left[\frac{x^3 y}{3} \right]_{x=1}^{x=2} \mathrm{d}y = \int_{y=-3}^{y=4} \frac{7y}{3} \, \mathrm{d}y = \left[\frac{7y^2}{6} \right]_{y=-3}^{y=4} = \frac{49}{6} \, . \end{aligned}$$

In this example we could have separated the integrand into its x and y parts:

$$V = \int_{x=1}^{x=2} \left(\int_{y=-3}^{y=4} x^2 y \, \mathrm{d}y \right) \mathrm{d}x = \left(\int_{x=1}^{x=2} x^2 \, \mathrm{d}x \right) \left(\int_{y=-3}^{y=4} y \, \mathrm{d}y \right) = \frac{7}{3} \cdot \frac{7}{2} = \frac{49}{6} \, .$$

More generally, if $f(x, y) = g(x)\, h(y)$, (i.e. the function is **separable**) *and* the region is **rectangular** then

$$\begin{aligned} \iint_R g(x)\, h(y) \, \mathrm{d}A &= \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} g(x)\, h(y) \, \mathrm{d}y \right) \mathrm{d}x \\ &= \left(\int_{x=a}^{x=b} g(x) \, \mathrm{d}x \right) \left(\int_{y=c}^{y=d} h(y) \, \mathrm{d}y \right) . \end{aligned}$$

3.2 Double integrals over general regions and area

3.2.1 Double integrals over general regions [Thomas' Calculus, Section 14.2]

Now consider the case where the region R is *not rectangular*:[1]

THEOREM 2 Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

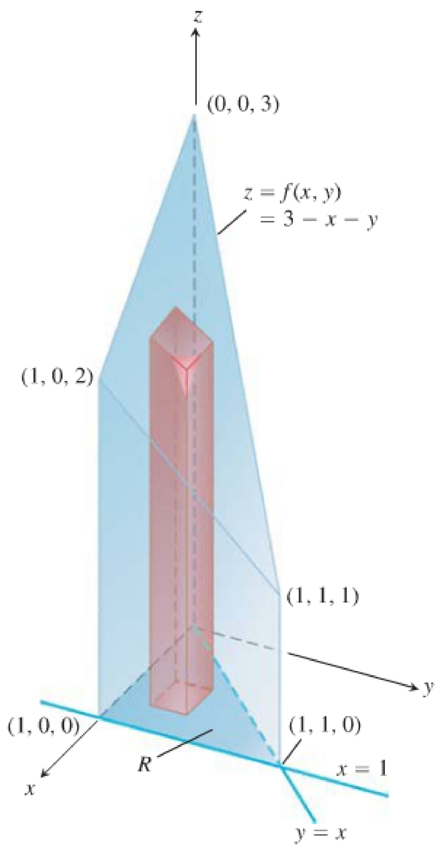
2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

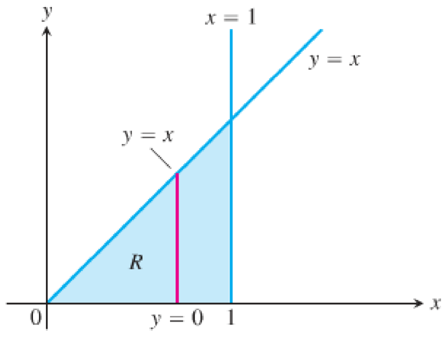
Example:

Find the volume of the prism $\iint_R (3 - x - y) \, \mathrm{d}A$ where R is the region bounded by the x-axis and the lines $x = 1$ and $y = x$.

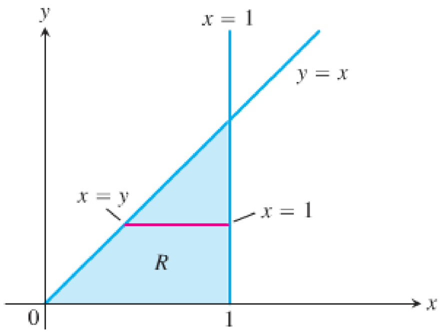
[1] See Thomas' Calculus, beginning of Section 15.2 for details underlying this theorem. Here we sweep under the rug that integrating over non-rectangular regions involves some further considerations.



(a)



(b)



(c)

The region of integration in the x-y plane and the volume defined by $z = 3 - x - y$ are shown in the figure. In order to do the double integral we will first consider the approach where we fix the value of x and do the y integral. We have

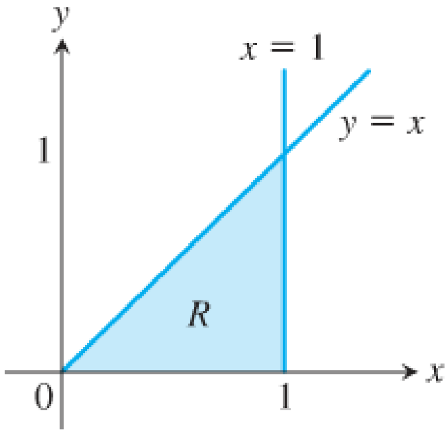
$$\begin{aligned}\int\!\!\int_R (3-x-y)\,\mathrm{d}A &= \int_{x=0}^{x=1}\int_{y=0}^{y=x}(3-x-y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x}\mathrm{d}x \\ &= \int_0^1\left(3x - \frac{3x^2}{2}\right)\,\mathrm{d}x = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1 = 1\,.\end{aligned}$$

We can also change the order of the integration where we fix the value of y and do the x integral. We have

$$\begin{aligned}\int\!\!\int_R (3-x-y)\,\mathrm{d}A &= \int_{y=0}^{y=1}\int_{x=y}^{x=1}(3-x-y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1}\mathrm{d}y \\ &= \int_0^1\left(\left(3 - \frac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right)\right)\,\mathrm{d}y \\ &= \int_0^1\left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right)\,\mathrm{d}y = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1\,.\end{aligned}$$

In some cases the order of integration can be crucial to solving the problem.

Example:
Calculate $\int\!\!\int_R (\sin x)/x\,\mathrm{d}A$ where R is the triangle in the x-y plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.



Taking vertical strips (i.e. keeping x fixed and allowing y to vary) gives

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left[y \frac{\sin x}{x} \right]_{y=0}^{y=x} \mathrm{d}x = \int_0^1 \sin x \, \mathrm{d}x$$
$$= [-\cos x]_0^1 = -\cos 1 + \cos 0 = 1 - \cos 1\,.$$

However, if we reverse the order of integration we get

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, \mathrm{d}x\, \mathrm{d}y$$

and $\int (\sin x)/x \, \mathrm{d}x$ cannot be expressed in terms of elementary functions making the integral difficult to do.

There are always two ways to do a double integral; choose the simpler because the other may be impossible!

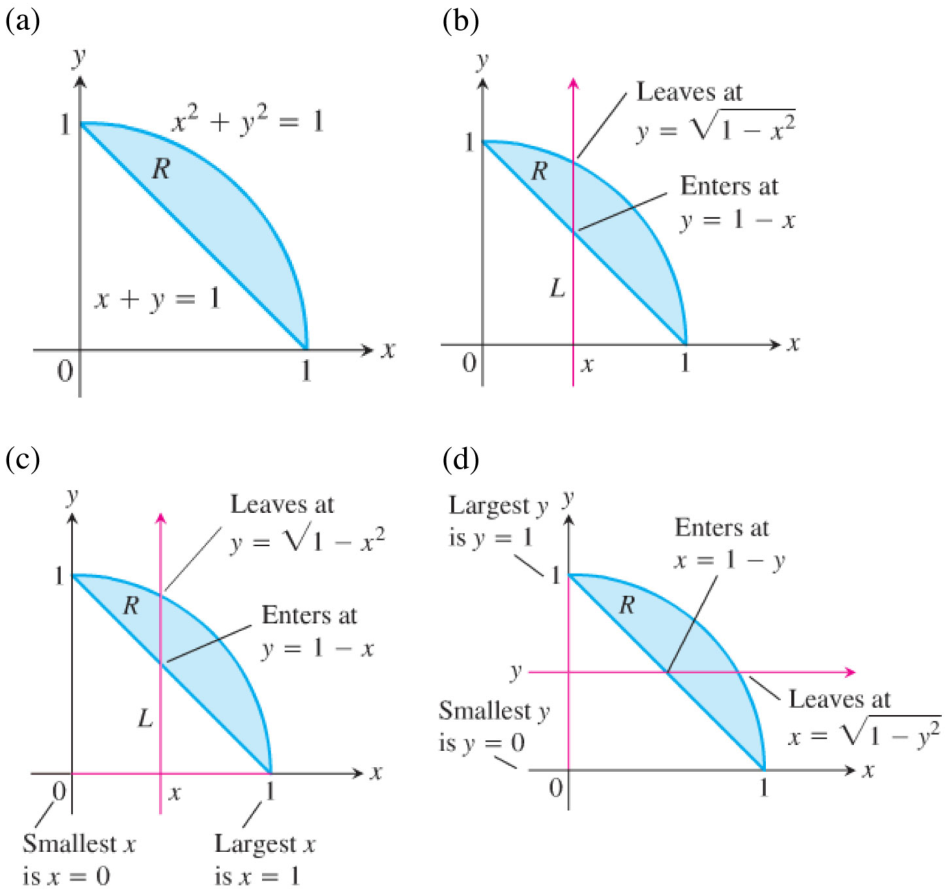
Summary: A key part of the process of double (and multiple) integration over a region is to find the **limits of the integration**. We illustrate the procedure by considering the double integral of a function over the region R given by the intersection of the line $x+y=1$ with the circle $x^2+y^2=1$ (see the picture next page).

(a) **Sketch the region of integration** and label its boundary curves.

(b) If we decide to use vertical cross-sections first: **Find the y-limits of integration**. Imagine a vertical line through the region, R, and mark the points where it enters and leaves R. In this case such a line would enter at $y = 1 - x$ and leave at $y = \sqrt{1-x^2}$.

(c) **Find the x-limits of integration**: Choose the x-limits that include all vertical lines through R. In this case the lower limit is $x = 0$ and the upper limit is $x = 1$.

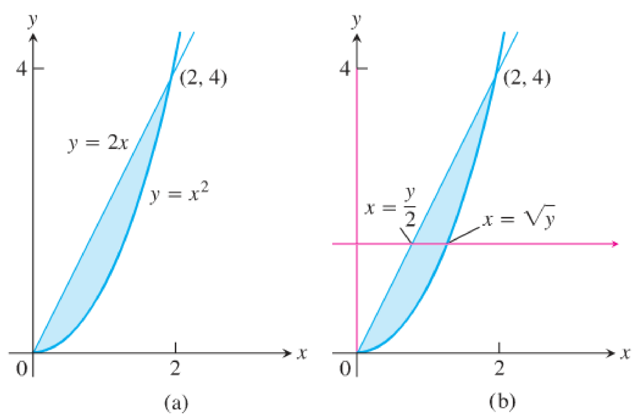
(d) This step may not be necessary: **Reversing the order of integration**. Then the x-limits would be from $x = 1 - y$ to $x = \sqrt{1-y^2}$ and the y-limits from $y = 0$ to $y = 1$.



Example:
Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\,\mathrm{d}y\,\mathrm{d}x$$

and write an equivalent integral with the order of integration reversed. Evaluate the integral.



As written, the order of integration would imply that we do the y-integral first, from $y = x^2$ to $y = 2x$, followed by the x-integral from $x = 0$ to $x = 2$. However, we are told to reverse the order of integration. This means we do the x-integration first, from $x = y/2$ to $x = \sqrt{y}$, followed by the y-integral from $y = 0$ to $y = 4$. In other words,

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,\mathrm{d}y\,\mathrm{d}x = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,\mathrm{d}x\,\mathrm{d}y$$

We can evaluate the integral using either ordering. Let us revert to the original:

$$\begin{aligned}\int_0^2 \int_{x^2}^{2x} (4x+2)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^2 [4xy+2y]_{x^2}^{2x}\,\mathrm{d}x = \int_0^2 \left(8x^2+4x-4x^3-2x^2\right)\,\mathrm{d}x \\ &= \int_0^2 \left(-4x^3+6x^2+4x\right)\,\mathrm{d}x = \left[-x^4+2x^3+2x^2\right]_0^2 \\ &= -16+16+8 = 8\,.\end{aligned}$$

Note that this example is *not* separable because it is a non-rectangular region (i.e. the limits on the x and y integrals now depend on the region of integration).

Double integrals can also be calculated over unbounded regions.

Example:
Evaluate the integral $\int_0^\infty \int_0^\infty x\,e^{-(x+2y)}\mathrm{d}x\,\mathrm{d}y$.
We have

$$\begin{aligned}\int_0^\infty \int_0^\infty x\,e^{-(x+2y)}\mathrm{d}x\,\mathrm{d}y &= \int_0^\infty \int_0^\infty e^{-2y} x\,e^{-x}\mathrm{d}x\,\mathrm{d}y \\ &\qquad \text{(integrate by parts with } u = x,\ \mathrm{d}v = e^{-x}\mathrm{d}x) \\ &= \int_0^\infty e^{-2y} \left\{ \left[-x\,e^{-x}\right]_0^\infty - \int_0^\infty \left(-e^{-x}\right)\,\mathrm{d}x \right\}\,\mathrm{d}y \\ &= \int_0^\infty e^{-2y} \left((0-0) + \left[-e^{-x}\right]_0^\infty\right)\,\mathrm{d}y \\ &= \left[-\frac{1}{2}e^{-2y}\right]_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}\,.\end{aligned}$$

Double integrals have the following properties:
Let $f(x,y), g(x,y)$ be continuous on the bounded region R. Then

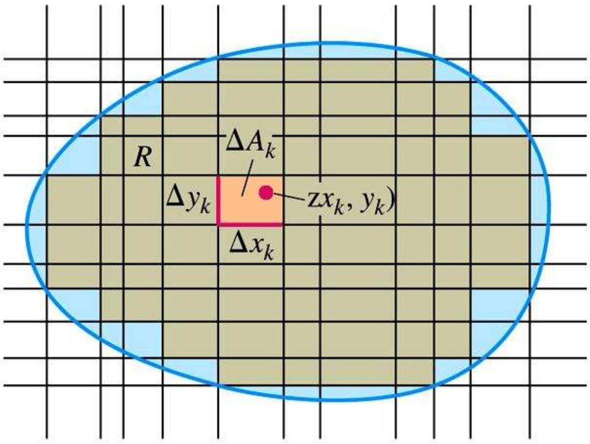
$$\begin{aligned}\iint_R c\,f(x,y)\,\mathrm{d}A &= c\iint_R f(x,y)\,\mathrm{d}A \quad \text{for any number } c \\ \iint_R \left(f(x,y) \pm g(x,y)\right)\,\mathrm{d}A &= \iint_R f(x,y)\,\mathrm{d}A \pm \iint_R g(x,y)\,\mathrm{d}A \\ \iint_R f(x,y)\,\mathrm{d}A &\geq 0 \quad \text{if} \quad f(x,y) \geq 0 \text{ on } R \\ \iint_R f(x,y)\,\mathrm{d}A &\geq \iint_R g(x,y)\,\mathrm{d}A \quad \text{if} \quad f(x,y) \geq g(x,y) \text{ on } R \\ \iint_R f(x,y)\,\mathrm{d}A &= \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A \\ &\qquad \text{if} \quad R = R_1 \cup R_2\,,\ R_1 \cap R_2 = \emptyset\end{aligned}$$

3.2.2 Area by double integration [Thomas' Calculus, Section 14.3]

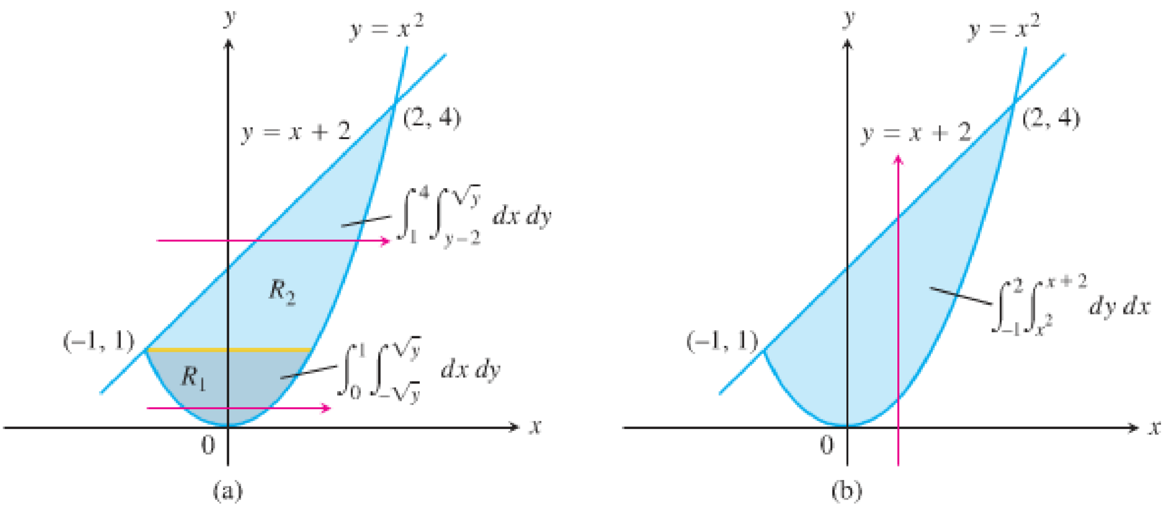
The **area** A of a closed, bounded plane region R is given by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k = \int\!\!\int_R \mathrm{d}A\,,$$

which is equivalent to calculating $\iint_R f(x,y)\,\mathrm{d}A$ with $f(x,y) = 1$.



Example:
Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$. Determining the points of intersection is essential to determining the limits on the integrations. We can find the points by setting $x^2 = x+2$ which gives $x^2 - x - 2 = (x+1)(x-2) = 0$, giving $x = -1$ and $x = 2$. The corresponding values of y are $y = 1$ and $y = 4$. So the points of intersection are $(-1, 1)$ and $(2, 4)$.



If we use vertical strips (i.e. fix x and vary y) for the first integral we will not have to split up the region of integration. From the diagram we see that the lower and upper limits for the first integration are therefore $y = x^2$ and $y = x + 2$. This gives

$$\begin{aligned} A &= \int_{-1}^{2}\int_{x^2}^{x+2} \mathrm{d}y\,\mathrm{d}x = \int_{-1}^{2} [y]_{x^2}^{x+2}\,\mathrm{d}x \\ &= \int_{-1}^{2} \left(x + 2 - x^2\right)\,\mathrm{d}x = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}\,. \end{aligned}$$

Double integrals can also be used to find the **average value** of the function $f(x, y)$ over the region R, which is defined to be

$$\langle f\rangle = \frac{1}{\text{area of } R}\iint_R f(x,y)\,\mathrm{d}A\,.$$

Example:
Find the average value of $f(x, y) = x\cos xy$ over the rectangle $R:\ 0 \le x \le \pi,\ 0 \le y \le 1$. The area of the region R is just π, the product of the length of the two sides of the rectangle. We just need to find $\iint_R f(x, y)\,\mathrm{d}A$ and then divide by π.

$$\begin{aligned} \int_0^{\pi}\int_0^{1} x\,\cos xy\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi} [\sin xy]_{y=0}^{y=1}\,\mathrm{d}x \\ &= \int_0^{\pi} (\sin x - 0)\,\mathrm{d}x = [-\cos x]_0^{\pi} = 1 + 1 = 2\,. \end{aligned}$$

Hence $\langle f\rangle = 2/\pi$.

End of Week 10

3.3 Substitution and triple integrals

3.3.1 Substitution in double Integrals [Thomas' Calculus, Sections 14.8 and 14.4]

For functions of one variable it is often useful to integrate by a change of variable, e.g. $x = x(u)$. Let us review **integration by substitution** in a slightly different way than you have learned in Calculus 1, namely *backwards*: Replace x by $x(u)$ and $\mathrm{d}x$ by $(\mathrm{d}x/\mathrm{d}u)\mathrm{d}u$.[2] Then alter the x-limits to the u-limits with $a < b$ and $u_1 < u_2$. First, assume that $x(u)$ increases with u giving $a = x(u_1)$ and $b = x(u_2)$. Then

$$I = \int_{x=a}^{x=b} f(x)\,\mathrm{d}x = \int_{u=u_1}^{u=u_2} f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\,.$$

If $x(u)$ decreases with u we have $a = x(u_2)$ and $b = x(u_1)$, and the u-limits are reversed. With $u_1 < u_2$ we therefore have a change of sign:

$$I = \int_{x=a}^{x=b} f(x)\,\mathrm{d}x = -\int_{u=u_1}^{u=u_2} f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\,.$$

[2]Note that here we interchange u and x compared to Calculus 1.

But $\mathrm{d}x/\mathrm{d}u < 0$ in this case, so we can combine both cases in one formula:

$$\int_{x=a}^{x=b} f(x)\,\mathrm{d}x = \int_{u=u_1}^{u=u_2} f(x(u)) \left|\frac{\mathrm{d}x}{\mathrm{d}u}\right| \mathrm{d}u\,.$$

Note that on the right-hand side of this equation the function $f(x)$ is expressed as $f(x(u))$. Also, the right-hand side of the equation includes a *scaling factor* $|\mathrm{d}x/\mathrm{d}u|$, multiplying the $\mathrm{d}u$; this comes from transforming from $\mathrm{d}x$ to $\mathrm{d}u$.
For functions of two variables one would similarly expect that the change in variables

$$x = x(u,v), \quad y = y(u,v)$$

(for example, for polar coordinates $u = r$ and $v = \theta$) would result in a change in the area dA by a *scaling factor* S such that

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = S\,\mathrm{d}u\,\mathrm{d}v\,.$$

As an example consider a *linear change* of coordinates:

$$x = x(u,v) = au + bv, \qquad y = y(u,v) = cu + dv$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

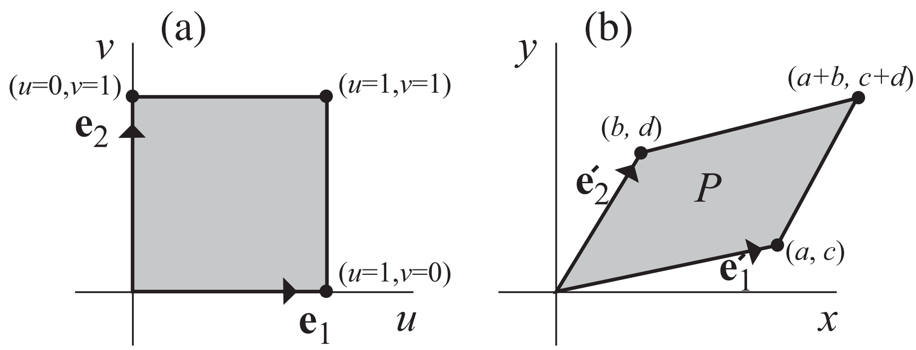
where a, b, c and d are constants.
Let us write $\mathbf{M}$ for the transformation matrix composed of a, b, c and d and recall that a unit square in (u,v) variables is spanned by the unit vectors

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e}_2$$

To see what happens to this unit square under the transformation $\mathbf{M}$, just apply $\mathbf{M}$. This gives

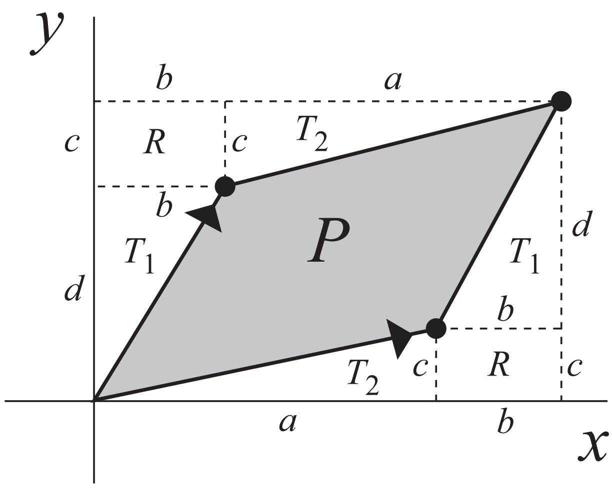
$$\begin{aligned} \mathbf{M}\,\mathbf{e}_1 &= \mathbf{e}_1' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \\ \mathbf{M}\,\mathbf{e}_2 &= \mathbf{e}_2' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \end{aligned}$$

where (a,c) and (b,d) represent the coordinates of the new corners in the (x,y) plane:



(note that the arrows are supposed to reach the respective points)

Therefore, under the transformation $\mathbf{M}$ we find that the unit square in (u, v) based on $\mathbf{e}_1$, $\mathbf{e}_2$ is transformed into the parallelogram in (x, y) based on $\mathbf{e}'_1$, $\mathbf{e}'_2$.
Note from the matrix and the diagram that the point $(1, 1)$ in (u, v) transforms to the point $(a + b, c + d)$ in (x, y).
Let us calculate the area of the parallelogram P:



We have

$$\begin{aligned}\text{Area } P &= [\text{Total area of rectangle}] \\ &\quad - [\text{Area of 2 pairs of equal triangles } T_1 \text{ and } T_2] \\ &\quad - [\text{Area of 2 rectangles } R]\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Area } P &= (a+b)(c+d) - 2 \cdot \frac{1}{2}ac - 2 \cdot \frac{1}{2}bd - 2bc \\ &= ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \mathbf{M}\end{aligned}$$

In view of the equation $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = S\,\mathrm{d}u\,\mathrm{d}v$ one may understand this result such that the unit square of area $\mathrm{d}u\,\mathrm{d}v$ gets multiplied by a factor of $S = \det \mathbf{M}$. The same argument shows that a small rectangle of sides du and dv with area $du\,dv$ also gets multiplied by $S = \det \mathbf{M}$. Therefore, for a linear change of variables a small rectangular area $\mathrm{d}u\,\mathrm{d}v$ in the (u, v) plane is transformed into the parallelogram area $dx\,dy = \det \mathbf{M}\,du\,dv$ in the (x, y) plane.
Now let us consider a *nonlinear change* of coordinates. We take the transformation to have the form

$$x = x(u, v), \qquad y = y(u, v)\,,$$

where according to the total differential the increments in x and y are given by

$$\begin{aligned}dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

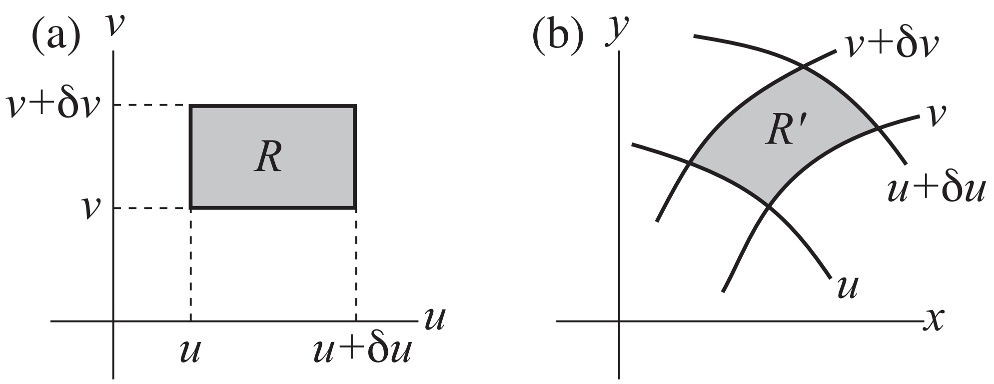
The **Jacobian matrix** is defined to be

$$\mathbf{M}(u,v) = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

and the **Jacobian determinant**, or **Jacobian**,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \mathbf{M}(u,v)\,.$$

This suggests that for a nonlinear change of variables we also have that a rectangular area $\mathrm{d}u\,\mathrm{d}v$ in the (u,v) plane) is transformed into the (deformed) 'parallelogram' area $\det \mathbf{M} du\, dv$ in the (x,y) plane.



(with $du = \delta u \;\; dv = \delta v$)

Therefore, the required **transformation formula for double integrals** under a change of variables is[3]

$$\int\!\!\int_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\!\!\int_{R'} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

where

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = |\det \mathbf{M}|$$

can be thought of as the scaling factor S.

Note that $|\cdot|$ denotes the absolute value of the determinant of the matrix, i.e., the modulus as in the one variable case. This may not be confused with the case of a matrix, where vertical lines on either side denote the determinant. For example, if we let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

[3]For a precise mathematical formulation of this result as a theorem see Thomas' Calculus p.907.

and

$$|\det \mathbf{A}| = |ad - bc| \;.$$

Example:
Evaluate the integral

$$I = \int\!\!\int_R (x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y$$

where R is a disk $x^2 + y^2 \leq a^2$, by changing to polar coordinates.
In polar coordinates we have

$$x = r\cos\theta, \qquad y = r\sin\theta\,.$$

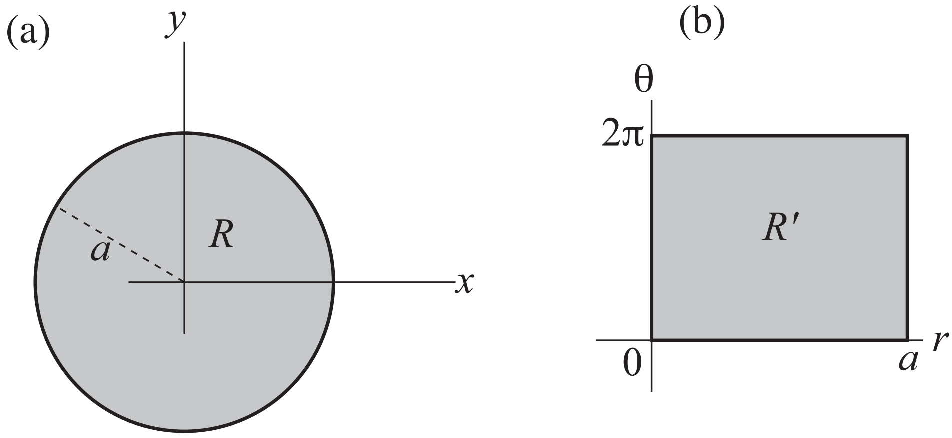
Therefore, taking $u = r$ and $v = \theta$, we can write the Jacobian matrix as

$$\mathbf{M} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and the Jacobian determinant is

$$\det \mathbf{M} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

where here and in the following we assume $r \geq 0$, so we do not need to take the absolute value. The original area R and the transformed area R' are shown below:



Note that the circle in the (x, y) plane transforms into a rectangle in the (r, θ) plane. Here R is the region given by $x^2 + y^2 \leq a^2$ and R' is the region given by $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$. Therefore

$$I = \int\!\!\int_R (x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y = \int\!\!\int_{R'} \left(r^2\right)(r)\; \mathrm{d}r\, \mathrm{d}\theta$$

where the r^2 on the right-hand integral comes from the transformed $x^2 + y^2$ and the $r\, \mathrm{d}r\, \mathrm{d}\theta$ is from the transformed $\mathrm{d}x\, \mathrm{d}y$ with r coming from the Jacobian determinant $\det \mathbf{M}$. Hence

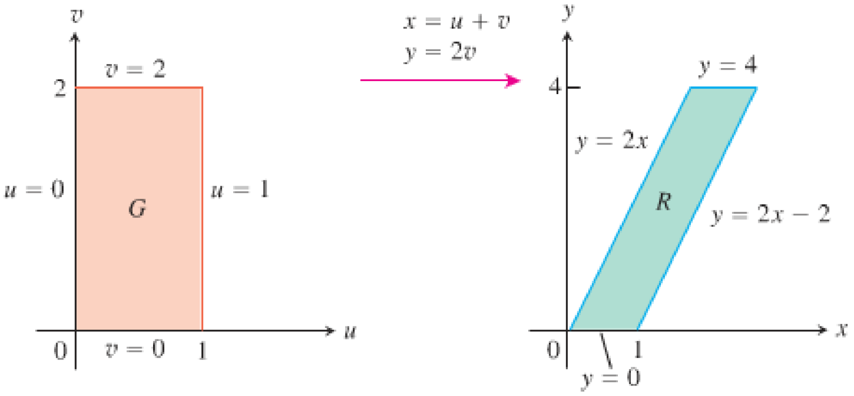
$$I = \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=2\pi} r^3\, \mathrm{d}r\, \mathrm{d}\theta \; = \left(\int_{r=0}^{r=a} r^3\, \mathrm{d}r\right)\left(\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta\right) = \frac{\pi a^4}{2}\,,$$

where we note that the integral is separable.

Example:
Evaluate the double integral

$$\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x-y}{2} \, dx \, dy$$

by applying the transformation $u = (2x-y)/2$, $v = y/2$ and integrating over an appropriate region of the u-v plane.
The region R in the x-y-plane looks as follows:



The corresponding region G in the u-v plane can be obtained by first writing x and y in terms of u and v as $x = u + v$ and $y = 2v$.
The boundaries of G are then found by substituting these equations for the boundaries of R:

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

The Jacobian of the transformation is

$$\begin{aligned} \det \mathbf{M}(u,v) &= \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \\ &= \begin{vmatrix} \partial(u+v)/\partial u & \partial(u+v)/\partial v \\ \partial(2v)/\partial u & \partial(2v)/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2\,. \end{aligned}$$

and we get

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx\, dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \left|\det \mathbf{M}(u,v)\right| \, du\, dv = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \cdot 2 \, du\, dv = 2$$

Note that for invertible transformations

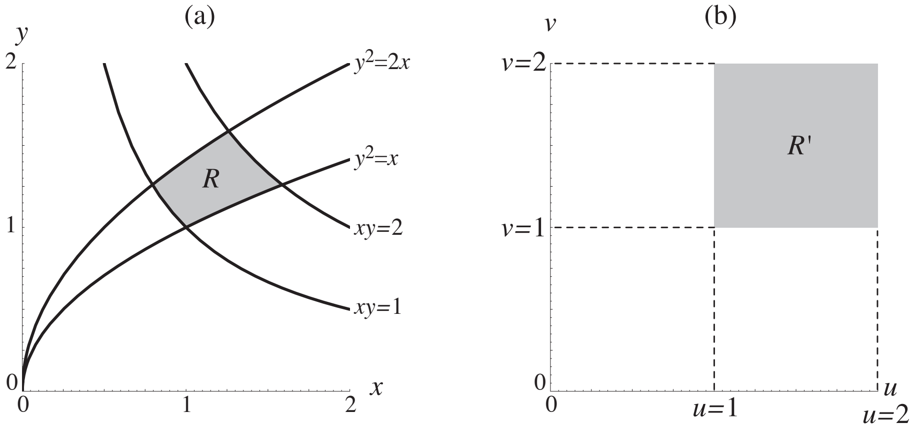
$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}, \tag{3.1}$$

as you have seen in Calculus 1 for a function of one variable. This can be useful in solving some problems.

Example:
Evaluate the integral

$$I = \int\int_R 1 \cdot \mathrm{d}x\,\mathrm{d}y$$

(i.e. the area of the region R) where R is enclosed by $y^2 = x$, $y^2 = 2x$, $xy = 1$ and $xy = 2$.



To solve the integral consider the change of variables defined by

$$u = y^2/x, \qquad v = xy\,.$$

Then we can write the four bounding curves as

$$y^2 = x \Leftrightarrow u = 1, \quad y^2 = 2x \Leftrightarrow u = 2, \quad xy = 1 \Leftrightarrow v = 1, \quad xy = 2 \Leftrightarrow v = 2\,.$$

So the region becomes a square (the region R' in part (b) of the above figure).
Now, for the Jacobian determinant it is easier to use Eq. (1) above. So, to calculate $\partial(x,y)/\partial(u,v)$ we first calculate $\partial(u,v)/$
$, p(x,y)$ and then take the inverse. Using $u = y^2/x$ and $v = xy$ we have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} -y^2/x^2 & 2y/x \\ y & x \end{vmatrix} = -3\frac{y^2}{x} = -3u\,.$$

Therefore, using Eq. (1),

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = -\frac{1}{3u}\,.$$

Hence

$$\begin{aligned} I &= \iint_R 1 \cdot \mathrm{d}x\,\mathrm{d}y = \iint_{R'} 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_{R'} \left|-\frac{1}{3u}\right| \mathrm{d}u\,\mathrm{d}v = \frac{1}{3}\int_{u=1}^{u=2}\int_{v=1}^{v=2} \frac{1}{u}\,\mathrm{d}v\,\mathrm{d}u \\ &= \frac{1}{3}\int_{u=1}^{u=2} \left[\frac{v}{u}\right]_{v=1}^{v=2} \mathrm{d}u \\ &= \frac{1}{3}\int_{u=1}^{u=2} \frac{1}{u}\,\mathrm{d}u = \frac{1}{3}\left[\ln u\right]_{u=1}^{u=2} = \frac{\ln 2}{3} \end{aligned}$$

End of Week 11

Example:
Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2} \mathrm{d}x \,.$$

If we call this integral I, we can write

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2} \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} \mathrm{d}y\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \mathrm{d}x \, \mathrm{d}y \,.$$

Now transform to polar coordinates with the limits $0 \le r < \infty$ and $-\pi \le \theta \le \pi$. This gives

$$\begin{aligned} I^2 &= \int_{-\pi}^{\pi}\int_{0}^{\infty} e^{-r^2/2} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}r \, \mathrm{d}\theta = \int_{-\pi}^{\pi}\int_{0}^{\infty} r\, e^{-r^2/2} \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} \left[-e^{-r^2/2}\right]_0^{\infty} \mathrm{d}\theta = \int_{-\pi}^{\pi} \left((0) - (-1)\right) \mathrm{d}\theta = \int_{-\pi}^{\pi} \mathrm{d}\theta = 2\pi \,. \end{aligned}$$

Hence $I = \sqrt{2\pi}$.
Note that the probability density function for a normal (or Gaussian) distribution is

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

for mean μ and standard deviation σ. If we write $t = (x-\mu)/\sigma$ (i.e. express the displacement from the mean in terms of the standard deviation) then the total probability is

$$\begin{aligned} P &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} \mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \sigma \, \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \, \mathrm{d}t = 1 \,. \quad \text{(by our previous result)} \end{aligned}$$

3.3.2 Triple integrals [Thomas' Calculus, Section 14.5]

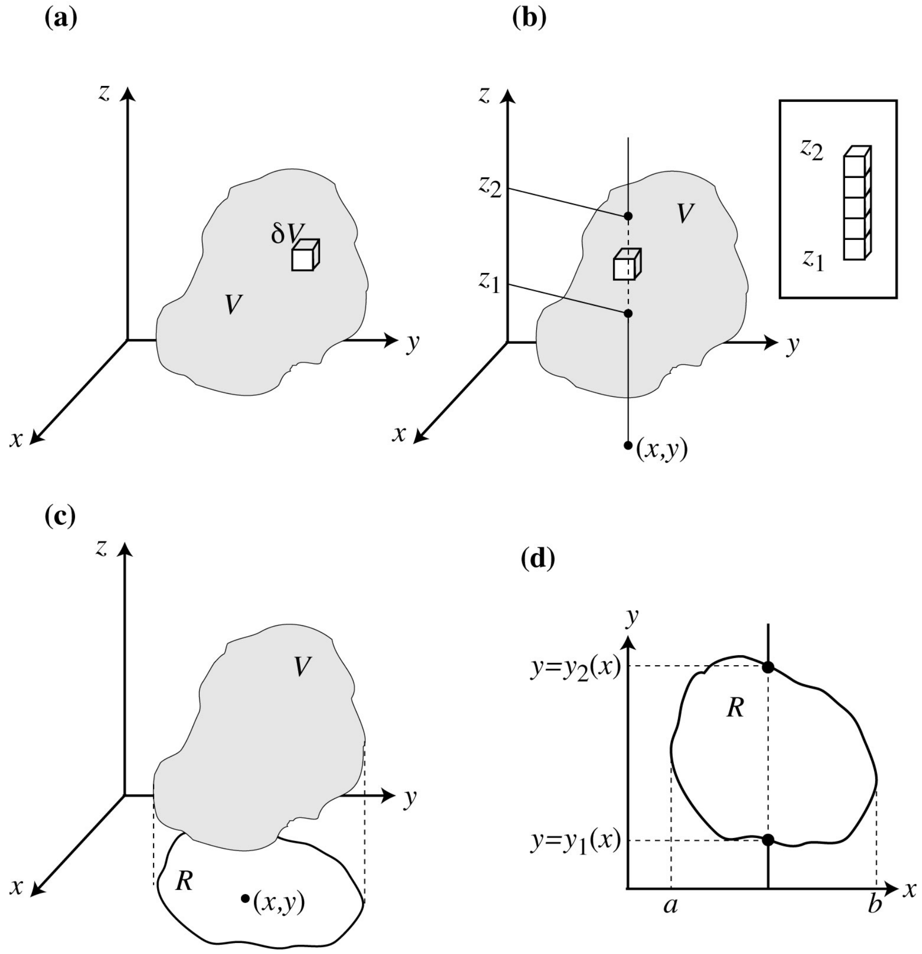
Triple integrals are integrations where the region of integration is a **volume**. The basic concepts are similar to those we introduced for two-dimensional (double) integrals, but now we have for the *Riemann sum*

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) \, \Delta V_k \,,$$

where $\Delta V_k = \Delta x_k \, \Delta y_k \, \Delta z_k$ are now small volumes at the point x_k, y_k, z_k, see (a) in the figure below (where it is $\Delta V_k = \delta V$).
The limit as the size of the volume element $\Delta V_k \to 0$ (as $n \to \infty$) is written as (if it exists)

$$\lim_{n\to\infty} S_n = \int\int\int_V f(x,y,z) \, \mathrm{d}V = \int\int\int_V f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,,$$

where V is the three-dimensional region being integrated over.
The integrals are, as in the two-dimensional case, evaluated by repeated integration where we integrate over one variable at a time. For example, we could start by integrating over z first, see (b) in the figure below. The procedure is as follows:



(a) **Sketch the region of integration** (if possible), see (a).

(b) **Choose a direction of integration and integrate:** For example, fix a point (x, y) and integrate over the allowed values of z in the region V. The z-integral limits are the small, filled circles at the bottom and the top of the dashed line with, say, $z = z_1(x, y)$ at the bottom and $z = z_2(x, y)$ at the top as shown in (b). Therefore we are summing vertically over the boxes shown in (b).

(c) This result depends on the choice of (x, y) and is defined in the region R of the (x, y) plane which is the projection of V onto this plane as shown in (c). This now **identifies the region in the (x, y) plane over which we must do the x and y integrations.**

(d) Now we can **take the double integral** of the result of the z-integration **over the region R in the (x, y) plane**, see (d), where here we first integrate along the y axis.

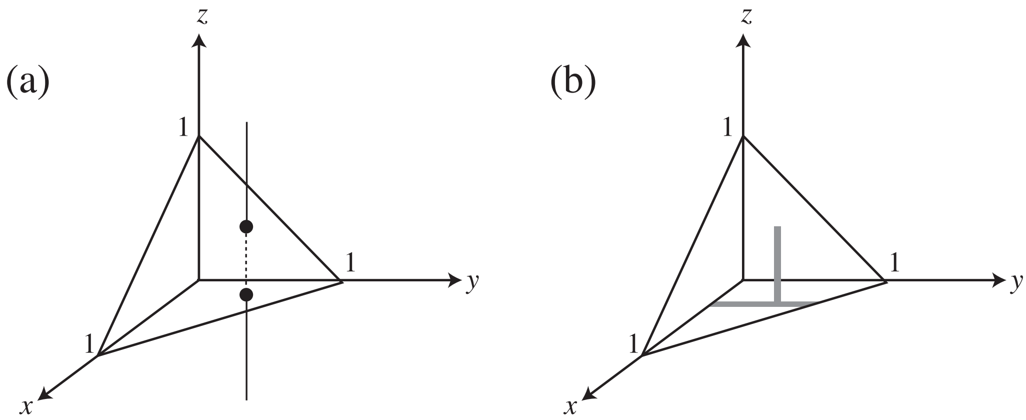
Therefore

$$\int\int\int_V f(x, y, z)\,\mathrm{d}V = \int_{x=a}^{x=b}\int_{y=y_1(x)}^{y=y_2(x)}\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.$$

Example:
Evaluate

$$\int\int\int_T f(x,y,z)\,\mathrm{d}V$$

over the tetrahedron T bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.
Note that the plane $x+y+z=1$ passes through $x=1$ (putting $y=z=0$) and similarly through $y=1$ and $z=1$ as shown below:



Now evidently for fixed (x,y) the z-limits are the heavy dots corresponding to $z=0$ at the bottom and $z=1-x-y$ at the top. This gives our z-limits.
The projection R of T onto the (x,y) plane is the triangle on which the tetrahedron rests, i.e. the triangle given by $x=0$, $y=0$ and $x+y=1$ (obtained by setting $z=0$). So

$$I=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.$$

For example, if $f(x,y,z)=1$ then

$$I=\iiint_T 1\cdot\mathrm{d}V=\iiint_T \mathrm{d}V=\textbf{volume of } T\,.$$

Therefore, in this case

$$\begin{aligned} I &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} [z]_{z=0}^{z=1-x-y}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} (1-x-y)\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{x=1}\left[y-xy-\frac{y^2}{2}\right]_{y=0}^{y=1-x}\mathrm{d}x \\ &= \int_{x=0}^{x=1}\frac{(1-x)^2}{2}\,\mathrm{d}x=\frac{1}{6} \end{aligned}$$

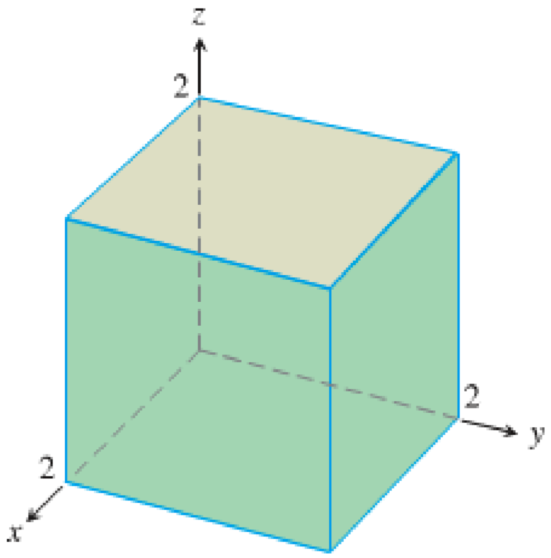
and this is the volume of the tetrahedron.

Triple integrals can be used to find the **average value of a function** $f(x,y,z)$ **over a volume** D defined as

$$\langle f(x,y,z)\rangle=\frac{1}{\text{volume of } D}\iiint_D f(x,y,z)\,\mathrm{d}V$$

Example:

Find the average value of $f(x,y,z) = xyz$ over the cube bounded by the planes $x = 2$, $y = 2$ and $z = 2$ in the first octant.



The volume of the cube is $2^3 = 8$. The integral is

$$\int_0^2\int_0^2\int_0^2 xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^2 x\,\mathrm{d}x \int_0^2 y\,\mathrm{d}y \int_0^2 z\,\mathrm{d}z = \left(\int_0^2 x\,\mathrm{d}x\right)^3 = \left(\left[\frac{x^2}{2}\right]_0^2\right)^3 = 8\,,$$

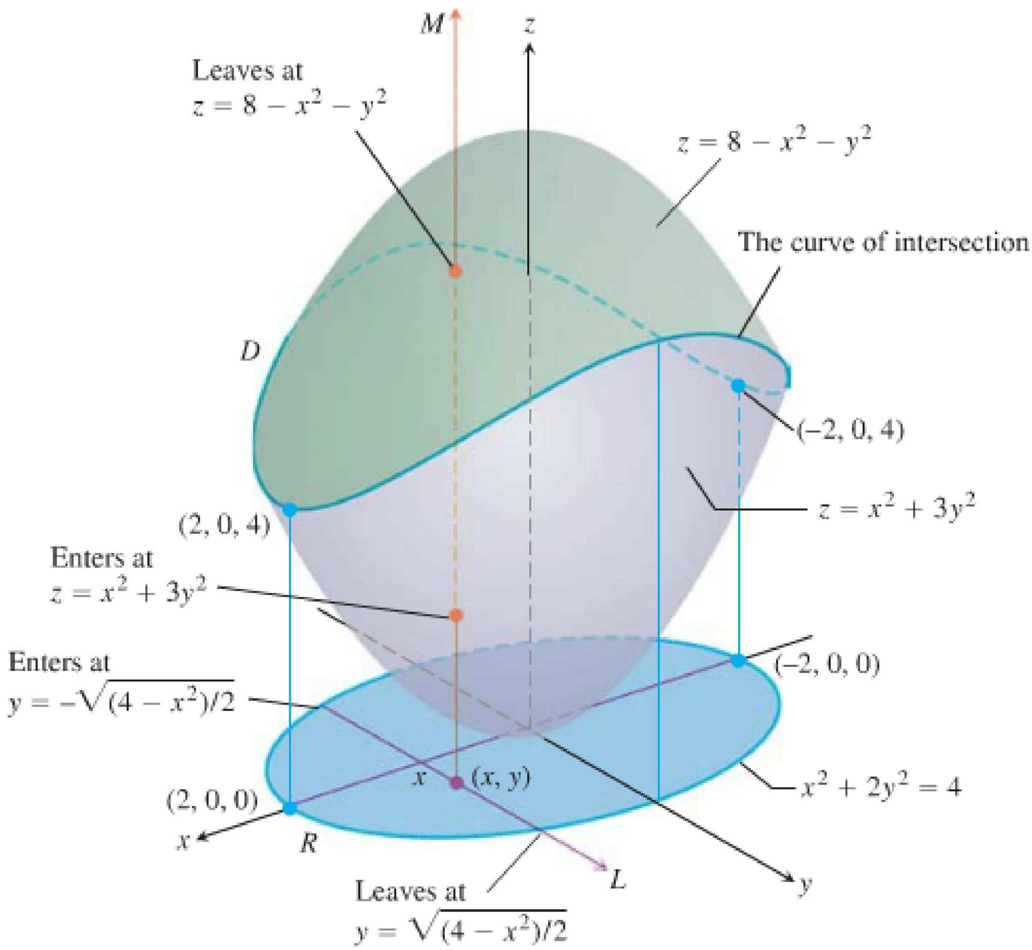
because the function is **separable** and the region is **cubic**. Therefore the average value of $f(x,y,z) = xyz$ over the cube is

$$\langle f(x,y,z)\rangle = \frac{1}{\text{volume of cube}}\iiint_{\text{cube}} xyz\,\mathrm{d}V = \frac{1}{8}\cdot 8 = 1\,.$$

Example:

Find the volume V of the region D enclosed by the surfaces $z = x^2+3y^2$ and $z = 8-x^2-y^2$.

The two surfaces intersect at $x^2 + 3y^2 = 8 - x^2 - y^2$. The equation $x^2 + 2y^2 = 4$ thus defines the boundary of the projection of D onto the x-y plane, which is the ellipse R:



We now have all the information necessary to do the integral:

$$
\begin{aligned}
V &= \iiint_D \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8-2x^2-4y^2\right)\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\left[(8-2x^2)y-\frac{4}{3}y^3\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\mathrm{d}x \\
&= \int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{(4-x^2)}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)\mathrm{d}x \\
&= \int_{-2}^{2}\left(8\left(\frac{4-x^2}{2}\right)^{3/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right)\mathrm{d}x \\
&= \frac{4\sqrt{2}}{3}\int_{-2}^{2}\left(4-x^2\right)^{3/2}\mathrm{d}x \quad [\text{since } (8-8/3)/(2^{3/2}) = 4\sqrt{2}/3] \\
&= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} 4^{3/2}\left(\cos^2\theta\right)^{3/2}\cdot 2\,\cos\theta\,\mathrm{d}\theta \quad [\text{using subst. } x = 2\sin\theta]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{4\sqrt{2}}{3} \cdot 16 \int_{-\pi/2}^{\pi/2} \cos^4\theta \,\mathrm{d}\theta = \frac{4\sqrt{2}}{3} \cdot 16 \int_{-\pi/2}^{\pi/2} \frac{1}{8} \left(3 + 4\cos 2\theta + \cos 4\theta\right) \mathrm{d}\theta \\
&= \frac{4\sqrt{2}}{3} \cdot 2 \left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/2}^{\pi/2} \\
&= \frac{4\sqrt{2}}{3} \cdot 2 \cdot 3 \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 8\sqrt{2}\,\pi\,.
\end{aligned}
$$

3.3.3 Substitution in triple Integrals [Thomas' Calculus, Section 14.8]

Changing variables in triple integrals is similar to the procedure used for double integrals. Suppose

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w)\,.$$

We define the **Jacobian matrix** for change of variables from (x, y, z) to (u, v, w) to be

$$\mathbf{M}(u, v, w) = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{pmatrix}$$

and the corresponding **Jacobian determinant** as

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \mathbf{M}$$

such that the transformation for volume is

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,.$$

As before, for invertible transformations we have

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \left(\frac{\partial(u, v, w)}{\partial(x, y, z)}\right)^{-1}\,.$$

The integral under change of variables becomes

$$
\begin{aligned}
&\iiint_V f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \\
&\quad \iiint_{V'} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w\,,
\end{aligned}
$$

where V' is the transformed volume in (u, v, w) coordinates.

Example:
A volume V in the first octant is bounded by the six surfaces $xy = 1$, $xy = 2$, $yz = 1$, $yz = 2$, $xz = 1$ and $xz = 2$. Using the change of variables

$$r = xy, \qquad s = yz, \qquad t = xz$$

and by assuming that this tranformation is invertible on V, evaluate the integral

$$\iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

The new limits are $r = 1$ to $r = 2$, $s = 1$ to $s = 2$ and $t = 1$ to $t = 2$. The Jacobian determinant is

$$\begin{aligned} \frac{\partial(r,s,t)}{\partial(x,y,z)} &= \begin{vmatrix} \partial r/\partial x & \partial r/\partial y & \partial r/\partial z \\ \partial s/\partial x & \partial s/\partial y & \partial s/\partial z \\ \partial t/\partial x & \partial t/\partial y & \partial t/\partial z \end{vmatrix} = \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix} \\ &= y \begin{vmatrix} z & y \\ 0 & x \end{vmatrix} - x \begin{vmatrix} 0 & y \\ z & x \end{vmatrix} \\ &= y(xz) + x(yz) = 2xyz\,. \end{aligned}$$

But

$$\frac{\partial(x,y,z)}{\partial(r,s,t)} = \left(\frac{\partial(r,s,t)}{\partial(x,y,z)}\right)^{-1} = \frac{1}{2xyz}$$

and so

$$\begin{aligned} \iiint_V xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iiint_{V'} xyz\left|\frac{1}{2xyz}\right|\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t = \int_{t=1}^{t=2}\int_{s=1}^{s=2}\int_{r=1}^{r=2}\frac{1}{2}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{2}\,[r]_1^2\,[s]_1^2\,[t]_1^2 = \frac{1}{2}\cdot 1\cdot 1\cdot 1 = \frac{1}{2}\,. \end{aligned}$$

THE END