Mathematical problems of General Relativity
Assessment problems

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The problems are to be handed in by email (scanned!) by the 8th March 2021.

1. Let n_a denote the unit normal to a spacelike 3-dimensional hypersurface $\mathcal{S}$. Show that the 3-dimensional covariant derivative D_a defined by

$$D_a v^b = h_a^c h_d{}^b \nabla_c v^d$$

is compatible with the spatial metric $h_{ab} = g_{ab} + n_a n_b$. Moreover, show that D_a is torsion free.

2. Show that in the non-vacuum case

$$R_{ab} - \tfrac{1}{2} R g_{ab} = T_{ab},$$

the constraint equations are given by

$$\begin{aligned} & r + K^2 - K_{ab}K^{ab} = 2\rho, \\ & D^b K_{ab} - D_a K = J_a, \end{aligned}$$

where

$$\rho \equiv T_{ab} n^a n^b, \qquad J_a \equiv -h_a{}^b n^c T_{bc}$$

denote, respectively, the energy density and momentum density vector with respect to the normal observer n^a.

3. In spherical *Painlevé-Gullstrand* coordinates, the line element of the Schwarzschild spacetime is given by

$$g = -\mathrm{d}t^2 + \left(\sqrt{\frac{2m}{r}} \mathrm{d}t + \mathrm{d}r \right)^2 + r^2 (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2).$$

Identify the lapse, shift and the 3-metric for this form of the spacetime metric with respect to the foliation given by hypersurfaces of constant t. Compute the extrinsic curvature.

4. Show that in a static spacetime, and using adapted coordinates such that $\xi^a \partial_a = \partial_t$, the Killing vector condition $\mathcal{L}_\xi g_{ab} = 0$ together with the definitions of h_{ij} and K_{ij} imply that

$$\partial_t h_{ij} = \partial_t K_{ij} = 0.$$

From the above conditions deduce the static equations

$$\begin{aligned} & D_i D_j \alpha = r_{ij}, \\ & r = 0. \end{aligned}$$

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