# Mathematical problems of General Relativity
## *Problem sheet 4*

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January 29, 2021

1. Verify that given the conformal rescaling $h_{ij} = \vartheta^4 \bar{h}_{ij}$, the associated Christoffels symbols are related to each other via

$$\Gamma^i{}_{jk} = \bar{\Gamma}^i{}_{jk} + 2(\delta_j{}^i \partial_k \ln \vartheta + \delta_k{}^i \partial_j \ln \vartheta - \bar{h}_{jk}\bar{h}^{il}\partial_l \ln \vartheta).$$

2. Verify that from the transformation rule for the Ricci tensor under conformal rescalings

$$r_{ij} = \bar{r}_{ij} - 2(\bar{D}_i\bar{D}_j \ln \vartheta + \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l\bar{D}_m \ln \vartheta) + 4(\bar{D}_i \ln \vartheta \bar{D}_j \ln \vartheta - \bar{h}_{ij}\bar{h}^{lm}\bar{D}_l \ln \vartheta \bar{D}_m \ln \vartheta).$$

   it follows that the Ricci scalar transforms as

$$r = \vartheta^{-4}\bar{r} - 8\bar{\theta}^{-5}\bar{D}_k\bar{D}^k\vartheta.$$

3. Verify that under the conformal rescaling $h_{ij} = \vartheta^4 \bar{h}_{ij}$ the time symmetric Hamiltonian constraint $r = 0$ implies the Yamabe equation

$$\bar{D}_k\bar{D}^k\vartheta - \tfrac{1}{8}\bar{r}\vartheta = 0$$

   What is the corresponding expression in the case of non-vanishing extrinsic curvature?

4. Verify that the conformal factor of the Brill-Lindquist initial data is indeed a solution to the Yamabe equation.

5. Let $n_a$ denote the unit normal of an hypersurface $\mathcal{S}$, and let $\xi^a$ denote a Killing vector on the spacetime $(\mathcal{M}, g_{ab})$. If the Killing vector is orthogonal to $\mathcal{S}$ (i.e. $\xi^a$ is proportional to $n^a$) show that the Frobenius condition

$$\xi_{[a}\nabla_b\xi_{c]} = 0$$

   is satisfied.

6. Show that in a stationary spacetime, and using adapted coordinates such that $\xi^a\partial_a = \partial_t$, the Killing vector condition $\mathcal{L}_\xi g_{ab} = 0$ together with the definitions of $h_{ij}$ and $K_{ij}$ imply that

$$\partial_t h_{ij} = \partial_t K_{ij} = 0.$$

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